SOLUTION OF THE QUADRATICALLY HYPONORMAL COMPLETION PROBLEM

RAÚL E. CURTO and WOO YOUNG LEE

Abstract. For \( m \geq 1 \), let \( \alpha : \alpha_0 < \cdots < \alpha_m \) be a collection of \((m + 1)\) positive weights. The Quadratically Hyponormal Completion Problem seeks necessary and sufficient conditions on \( \alpha \) to guarantee the existence of a quadratically hyponormal unilateral weighted shift \( W \) with \( \alpha \) as initial segment of weights. We prove that \( \alpha \) admits a quadratically hyponormal completion if and only if the self-adjoint \( m \times m \) matrix

\[
D_{m-1}(s) := \begin{pmatrix}
q_0 & r_0 & 0 & \ldots & 0 & 0 \\
r_0 & q_1 & r_1 & \ldots & 0 & 0 \\
0 & r_1 & q_2 & \ldots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & q_{m-2} & r_{m-2} \\
0 & 0 & 0 & \ldots & r_{m-2} & q_{m-1}
\end{pmatrix}
\]

is positive and invertible, where \( q_k := u_k + |s|^2 r_k \), \( r_k := s \sqrt{w_k} \), \( u_k := \alpha_k^2 - \alpha_{k-1}^2 \), \( v_k := \alpha_k^2 \alpha_{k+1}^2 - \alpha_k^2 \alpha_{k-1}^2 \), \( w_k := \alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)^2 \), and, for notational convenience, \( \alpha_{-2} = \alpha_{-1} = 0 \). As a particular case, this result shows that a collection of four positive numbers \( \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 \) always admits a quadratically hyponormal completion. This provides a new qualitative criterion to distinguish quadratic hyponormality from 2-hyponormality.

1. Introduction

Let \( \mathcal{H} \) be a complex Hilbert space and let \( \mathcal{L}(\mathcal{H}) \) be the set of bounded linear operators on \( \mathcal{H} \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be normal if \( T^* T = TT^* \), hyponormal if \( T^* T \geq TT^* \), and subnormal if \( T = N|_{\mathcal{H}} \), where \( N \) is normal on some Hilbert space \( \mathcal{K} \supseteq \mathcal{H} \). If \( T \) is subnormal then \( T \) is also hyponormal. The Bram-Halmos criterion for subnormality states that an operator \( T \) is subnormal if and only if \( \sum_{i,j} (T^i x_j, T^j x_i) \geq 0 \) for all finite collections \( x_0, x_1, \ldots, x_k \in \mathcal{H} \) \([2],[5, II.1.9]\). It is easy to see that this is equivalent to the following positivity test:

\[
\begin{pmatrix}
I & T & \cdots & T^k \\
T & T^* T & \cdots & T^{k} T^* T \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^* T^k & \cdots & T^{k} T^{k} T^k
\end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).
\]

1991 Mathematics Subject Classification. Primary 47B20, 47B35, 47B37; Secondary 47-04, 47A20, 47A57.

Key words and phrases. Weighted shifts, propagations, subnormal, \( k \)-hyponormal, quadratically hyponormal, completions.

The work of the first-named author was partially supported by NSF research grants DMS-9800931 and DMS-0099357.

The work of the second-named author was partially supported by the Brain Korea 21 Project.

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Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (1.1) for all $k$. Let $[A, B] := AB - BA$ denote the commutator of two operators $A$ and $B$, and define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix

\[(1.2) \quad M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k\]

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k + 1) \times (k + 1)$ operator matrix in (1.1). The Bram-Halmos criterion can be then rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([17]).

Recall ([1],[6],[17]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be weakly $k$-hyponormal if

\[LS(T, T^2, \ldots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k \right\}\]

consists of hyponormal operators, or equivalently, $M_k(T)$ is weakly positive, i.e.,

\[(1.3) \quad (M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, (M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \quad \text{for} \ x \in \mathcal{H} \text{ and } \lambda_0, \ldots, \lambda_k \in \mathbb{C} \ (17)\].

If $k = 2$ then $T$ is said to be quadratically hyponormal, and if $k = 3$ then $T$ is said to be cubically hyponormal. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be polynomially hyponormal if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that $k$-hyponormal $\Rightarrow$ weakly $k$-hyponormal, but the converse is not true in general.

The classes of (weakly) $k$-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([8],[9],[11],[12],[13],[15],[17],[20],[26]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle; in fact, even subnormality for Toeplitz operators has not yet been characterized (cf.[7],[22]). For weighted shifts, positive results appear in [8] and [13], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [18] and [19]).

Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \ldots$ (called weights), the (unilateral) weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2$. It is straightforward to check that $W_\alpha$ can never be normal, and that $W_\alpha$ is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. Weighted shifts provide a fertile ground to analyze the relative position of subnormality, polynomial hyponormality, $k$-hyponormality and weak $k$-hyponormality. In [13], the first named author and L. Fialkow applied the solution of the Subnormal Completion Problem [12] to build the first example of a one-parameter family of non-2-hyponormal quadratically hyponormal weighted shifts of recursive type. Additional examples were later found in [4],[14],[15] and [25], in some cases providing qualitative indicators to separate quadratically hyponormal from 2-hyponormality.

In this paper we obtain a new qualitative indicator, as a corollary to the Quadratically Hyponormal Completion Criterion. Our main result is a complete solution of the Quadratically Hyponormal Completion Problem:
Given \( \alpha : \alpha_0 < \alpha_1 < \cdots < \alpha_m \), find necessary and sufficient conditions for the existence of a quadratically hyponormal weighted shift whose first \((m+1)\) weights are those in \( \alpha \).

As a special case, we show that given four weights \( \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 \) there always exists a quadratically hyponormal completion, thus providing a new criterion to distinguish between quadratic hyponormality and 2-hyponormality.

**Definition 1.1.** A unilateral weighted shift \( W_\alpha \) is flat (or briefly, \( \alpha \) is flat) if \( \alpha_1 = \alpha_2 = \alpha_3 = \cdots \).

J. Stampfli [28] showed that for subnormal weighted shifts \( W_\alpha \), a propagation phenomenon occurs which forces the flatness of \( W_\alpha \) whenever two equal weights are present. Later, A. Joshi proved in [24] that the shift with weights \( \alpha_0 = \alpha_1 = a, \alpha_2 = \alpha_3 = \cdots = b, \ 0 < a < b \), is not quadratically hyponormal, and P. Fan [21] established that for \( a = 1, b = 2, \) and \( 0 < s < \sqrt{5}/5, \) \( W_\alpha + s W_\alpha^2 \) is not hyponormal.

On the other hand, it was shown in [8, Theorem 2] that a hyponormal weighted shift with three equal weights cannot be quadratically hyponormal without being flat: If \( W_\alpha \) is quadratically hyponormal and \( \alpha_n = \alpha_{n+1} = \alpha_{n+2} \) for some \( n \geq 0 \), then \( \alpha_1 = \alpha_2 = \alpha_3 = \cdots \), i.e., \( W_\alpha \) is subnormal. Furthermore, in [8, Proposition 11] it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force flatness, thereby subnormality.

**Theorem 1.2 (Propagation).** Let \( W_\alpha \) be a weighted shift with weight sequence \( \{\alpha_n\}_{n=0}^\infty \).

(i) ([28, Theorem 6]) Let \( W_\alpha \) be subnormal. If \( \alpha_n = \alpha_{n+1} \) for some \( n \geq 0 \), then \( \alpha \) is flat, i.e., \( \alpha_1 = \alpha_2 = \alpha_3 = \cdots \).

(ii) ([8, Corollary 6]) Let \( W_\alpha \) be 2-hyponormal. If \( \alpha_n = \alpha_{n+1} \) for some \( n \geq 0 \), then \( \alpha \) is flat.

(iii) ([3, Theorem 1]) Let \( W_\alpha \) be quadratically hyponormal. If \( \alpha_n = \alpha_{n+1} \) for some \( n \geq 1 \), then \( \alpha \) is flat.

An immediate consequence of Theorem 1.2 is that, for purposes of testing quadratic hyponormality, we can decompose all nonsubnormal weighted shifts into two classes: those with \( \alpha_0 = \alpha_1 \) and those with strictly increasing weight sequences.

To study quadratic hyponormality, we consider the selfcommutator \([W_\alpha + s W_\alpha^2]^*, W_\alpha + s W_\alpha^2\]. For \( s \in \mathbb{C} \), we write \( D(s) := [(W_\alpha + s W_\alpha^2)^*, W_\alpha + s W_\alpha^2] \) and we let

\[
D_n(s) := P_n[(W_\alpha + s W_\alpha^2)^*, W_\alpha + s W_\alpha^2] P_n = \begin{pmatrix}
q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\
\bar{r}_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\
0 & r_1 & q_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\
0 & 0 & 0 & \cdots & r_{n-1} & q_n
\end{pmatrix},
\]

where \( P_n \) is the orthogonal projection onto the subspace generated by \( \{e_0, \cdots, e_n\} \),

\[
\begin{align*}
q_n &:= u_n + |s|^2 v_n \\
r_n &:= s \sqrt{w_n} \\
u_n &:= \alpha_n^2 - \alpha_{n-1}^2 \\
v_n &:= \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_{n-2}^2 \\
w_n &:= \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2)^2,
\end{align*}
\]
and, for notational convenience, $\alpha_{-2} = \alpha_{-1} := 0$. Clearly, $W_\alpha$ is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then $d_n$ satisfies the following 2-step recursive formula:

\begin{equation}
(1.6) \quad d_0 = q_0, \quad d_1 = q_0 q_1 - |r_0|^2, \quad d_{n+2} = q_n d_{n+1} - |r_{n+1}|^2 d_n.
\end{equation}

If we let $t := |s|^2$, we observe that $d_n$ is a polynomial in $t$ of degree $n + 1$, and if we write $d_n \equiv \sum_{i=0}^{n+1} c(n,i) t^i$, then the coefficients $c(n,i)$ satisfy a double-indexed recursive formula, namely

\begin{equation}
(1.7) \quad c(n+2,i) = u_{n+2} c(n+1,i) + v_{n+2} c(n+1,i-1) - w_{n+1} c(n,i-1),
\end{equation}

\[ c(n,0) = u_0 \cdots u_n, \quad c(n,n+1) = v_0 \cdots v_n, \quad c(1,1) = u_1 v_0 + v_1 u_0 - w_0 \]

($n \geq 0$, $i \geq 1$). We say that $W_\alpha$ is positively quadratically hyponormal if $c(n,i) \geq 0$ for every $n \geq 0$, $0 \leq i \leq n+1$ (cf. [10]). Positive quadratic hyponormality implies quadratic hyponormality, but the converse is false (cf. [4]).

The idea of the proof of Theorem 1.2 (iii) is based on the following observation: if $W_\alpha$ is quadratically hyponormal with $\alpha_1 = \alpha_2 = 1$, then a straightforward calculation shows that

\[ d_4(t) = \alpha_0^2 \alpha_3^2 (\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 t^2 + c(4,3) t^3 + c(4,4) t^4 + c(4,5) t^5, \]

so

\[ \lim_{t \to 0^+} \frac{d_4(t)}{t^2} = \alpha_0^2 \alpha_3^2 (\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 \geq 0, \]

which forces $\alpha_0 = 1$ or $\alpha_3 = 1$, so that three equal weights are present and hence by [8, Theorem 2], flatness occurs.

Note that in Theorem 1.2 (iii) the condition “$n \geq 1$” cannot be relaxed to “$n \geq 0$”. For example, if

\begin{equation}
(1.8) \quad \alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n \geq 2),
\end{equation}

then $W_\alpha$ is quadratically hyponormal (cf. [8, Proposition 7]) but not cubically hyponormal (and hence not subnormal); indeed, if we let

\[ C_5(t) := \det \left( P_5 \left[ (W_\alpha + t W_\alpha^2 + t^2 W_\alpha^3)^*, (W_\alpha + t W_\alpha^2 + t^2 W_\alpha^3) \right] P_5 \right) \]

then

\[ \lim_{t \to 0^+} \frac{C_5(t)}{t^8} = -\frac{1}{2041200} < 0. \]

We conclude this section by recalling that when $\alpha_0 = \alpha_1 = 1$, quadratic hyponormality implies

\begin{equation}
(1.9) \quad \alpha_2 < \sqrt{2} \quad \text{and} \quad \alpha_3 \geq (2 - \alpha_2^2)^{-2} \quad (\text{cf. [10, p.78]}).\]

2. Recursively Generated Shifts in the Study of Completions

If \( W_\alpha \) is a weighted shift with weight sequence \( \alpha = \{\alpha_n\}_{n=0}^{\infty} \), then the moments of \( W_\alpha \) are usually defined by \( \beta_0 := 1, \beta_{n+1} := \alpha_n \beta_n \) \( (n \geq 0) \) [27]; however, we prefer to reserve this term for the sequence \( \gamma_n := \beta_n^2 \) \( (n \geq 0) \). A criterion for \( k \)-hyponormality can be given in terms of these moments ([8, Theorem 4]): if we build a \( (k+1) \times (k+1) \) Hankel matrix \( A(n; k) \) by

\[
A(n; k) := \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\
\gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k}
\end{pmatrix} \quad (n \geq 0),
\]

then \( W_\alpha \) is \( k \)-hyponormal if and only if \( A(n; k) \geq 0 \) \( (n \geq 0) \). C. Berger’s characterization of subnormality for unilateral weighted shifts (cf. [23], [5, III.8.16]) states that \( W_\alpha \) is subnormal if and only if there exists a Borel probability measure \( \mu \) supported in \([0, \|W_\alpha\|^2]\), with \( \|W_\alpha\|^2 \in \text{supp} \mu \), such that \( \gamma_n = \int t^n d\mu(t) \) for all \( n \geq 0 \). Given an initial segment of weights \( \alpha : \alpha_0, \cdots, \alpha_m \), a sequence \( \hat{\alpha} \in \ell^\infty(\mathbb{Z}_+) \) such that \( \hat{\alpha}_i = \alpha_i \) \( (i = 0, \cdots, m) \) is said to be recursively generated by \( \alpha \) if there exist \( r \geq 1 \) and \( \varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R} \) such that \( \gamma_{n+r} = \varphi_0 \gamma_n + \cdots + \varphi_{r-1} \gamma_{n+r-1} \) \( (n \geq 0) \), where \( \gamma_0 := 1, \gamma_n := \hat{\alpha}_0^2 \cdots \hat{\alpha}_{n-1}^2 \) \( (n \geq 1) \); in this case \( W_{\hat{\alpha}} \) is said to be recursively generated. If we let \( g(t) := t^r - (\varphi_{r-1} t^{r-1} + \cdots + \varphi_0) \), then \( g \) has \( r \) distinct real roots \( 0 \leq s_0 < \cdots < s_{r-1} \) ([12, Theorem 3.9]). Let

\[
V := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
s_0 & s_1 & \cdots & s_{r-1} \\
\vdots & \vdots & \ddots & \vdots \\
s_{r-1} & s_{r-2} & \cdots & s_0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\rho_0 \\
\vdots \\
\rho_{r-1}
\end{pmatrix} := V^{-1} \begin{pmatrix}
\gamma_0 \\
\vdots \\
\gamma_{r-1}
\end{pmatrix}.
\]

If the associated recursively generated weighted shift \( W_{\hat{\alpha}} \) is subnormal, then its Berger measure is of the form \( \mu := \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{s_{r-1}} \). Let \( \alpha : \alpha_0, \cdots, \alpha_m \) \( (m \geq 0) \) be an initial segment of positive weights and let \( \omega = \{\omega_n\}_{n=0}^{\infty} \) be a bounded sequence of positive numbers. We say that \( W_\omega \) is a completion of \( \alpha \) if \( \omega_n = \alpha_n \) \( (0 \leq n \leq m) \), and we write \( \alpha \subset \omega \). The completion problem for a property \( (P) \) entails finding necessary and sufficient conditions on \( \alpha \) to ensure the existence of a weight sequence \( \omega \supseteq \alpha \) such that \( W_\omega \) satisfies \( (P) \). In 1966, Stampfli [28] showed that for arbitrary \( \alpha_0 < \alpha_1 < \alpha_2 \) there always exists a subnormal weighted shift \( W_\alpha \) whose first three weights are \( \alpha_0, \alpha_1, \alpha_2 \); he also proved that given four or more weights, it may not be possible to find a subnormal completion. In [12, Theorem 3.5], the following criterion was established.

Subnormal Completion Criterion. If \( \alpha : \alpha_0, \cdots, \alpha_m \) \( (m \geq 0) \) is an initial segment of positive weights then the following are equivalent:

(i) \( \alpha \) has a subnormal completion;

(ii) \( \alpha \) has a recursively generated subnormal completion;

(iii) \( \alpha \) has an \( (\left\lceil \frac{m}{2} \right\rceil + 1) \)-hyponormal completion;

(iv) the Hankel matrices

\[
A(k) := \begin{pmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_k \\
\gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k}
\end{pmatrix} \quad \text{and} \quad B(\ell - 1) := \begin{pmatrix}
\gamma_1 & \gamma_2 & \cdots & \gamma_\ell \\
\gamma_2 & \gamma_3 & \cdots & \gamma_{\ell+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_\ell & \gamma_{\ell+1} & \cdots & \gamma_{2\ell-1}
\end{pmatrix}
\]
are both positive \((k := \lceil \frac{m+1}{2} \rceil\) and \(\ell := \lceil \frac{m}{2} \rceil + 1\)) and the vector
\[
\begin{pmatrix}
g_{k+1} \\
\vdots \\
g_{2k+1}
\end{pmatrix}
\text{ (resp. } \begin{pmatrix}
g_{k+1} \\
\vdots \\
g_{2k}
\end{pmatrix}
\)

is in the range of \(A(k)\) (resp. \(B(\ell - 1)\)) when \(m\) is even (resp. odd).

Thus the Subnormal Completion Problem reduces to the Recursive Completion Problem, which entails finding necessary and sufficient conditions on \(\alpha\) to ensure that the recursively generated weight sequence \(\hat{\alpha}\) is well-defined and bounded.

Also in [12, Proposition 3.19], the following criterion on \(k\)-hyponormal completions was established.

\textbf{\(k\)-Hyponormal Completion Criterion.} If \(\alpha: \alpha_0, \cdots, \alpha_{2m} (m \geq 1)\) is an initial segment of positive weights then for \(1 \leq k \leq m\) the following are equivalent:

(i) \(\alpha\) has a \(k\)-hyponormal completion;

(ii) the Hankel matrix

\[
A(j, k) := \begin{pmatrix}
g_j & \cdots & g_{j+k} \\
\vdots & & \vdots \\
g_{j+k} & \cdots & g_{j+2k}
\end{pmatrix}
\]

is positive for all \(j, 0 \leq j \leq 2m - 2k + 1\), and the vector

\[
\begin{pmatrix}
g_{2m-k+2} \\
\vdots \\
g_{2m+1}
\end{pmatrix}
\]

is in the range of \(A(2m - 2k + 2, k - 1)\).

If \(\alpha\) admits a \(k\)-hyponormal completion, then it admits a recursively generated one.

3. The Quadratically Hyponormal Completion Criterion

We now formulate and solve the corresponding problem for \(quadratic\) hyponormality.

\textbf{Quadratically Hyponormal Completion Problem.} Given \(\alpha: \alpha_0 < \alpha_1 < \cdots < \alpha_m\), find necessary and sufficient conditions for the existence of a quadratically hyponormal weighted shift whose first \((m + 1)\) weights are given by \(\alpha\).

We pause to recall that, for a square matrix \(A\), the notation \(A > 0\) means \(A \geq 0\) and \(A\) invertible.

\textbf{Theorem 3.1 (Quadratically Hyponormal Completion Criterion).} Let \(m \geq 2\) and let \(\alpha: \alpha_0 < \alpha_1 < \cdots < \alpha_m\) be an initial segment of positive weights. Then the following statements are equivalent:

(i) \(\alpha\) has a quadratically hyponormal completion;

(ii) \(D_{m-1}(t) > 0\) for all \(t \geq 0\).
Moreover, a quadratically hyponormal completion $\omega$ of $\alpha$ can be obtained in the following recursively generated fashion:

$$
\omega : \alpha_0, \ldots, \alpha_{m-2}, (\alpha_{m-1}, \alpha_m, \alpha_{m+1}),
$$

where $\alpha_{m+1}$ is chosen so that $\alpha_{m+1}^2 > \max\{\alpha_m^2, \frac{\alpha_{m-1}^2}{\alpha_m} [M (\alpha_m^2 - \alpha_{m-2}^2) + \alpha_{m-2}^2]\}$

$(M := \max_{t \in [0, \infty)} \frac{d_{m-2}(t)}{d_{m-1}(t)})$.

Proof. We will use the notation in Section 1. First of all, note that $D_{m-1}(t) > 0$ for all $t \geq 0$ if and only if $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \ldots, m-1$; this follows from the Nested Determinants Test (see [12, Remark 2.4]) or Choleski’s Algorithm (see [12, Proposition 2.3]). A straightforward calculation gives

$$
d_0(t) = \alpha_0^2 + \alpha_0^2 \alpha_1^2 t
$$

$$
d_1(t) = \alpha_0^2 (\alpha_1^2 - \alpha_0^2) + \alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_0^2) t + \alpha_0^2 \alpha_1^2 \alpha_2^2 t^2
$$

$$
d_2(t) = \alpha_0^2 (\alpha_1^2 - \alpha_0^2) (\alpha_2^2 - \alpha_1^2) + \alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_0^2) (\alpha_3^2 - \alpha_1^2) t
$$

$$
+ \alpha_0^2 \alpha_1^2 \alpha_2^2 \left\{ \alpha_0^2 (\alpha_2^2 - \alpha_0^2) - \alpha_1^2 (\alpha_1^2 - \alpha_0^2) \right\} t^2 + \alpha_0^2 \alpha_1^2 \alpha_2^2 (\alpha_2^2 \alpha_3^2 - \alpha_1^2 \alpha_0^2) t^3,
$$

which shows that all coefficients of $d_i$ ($i = 0, 1, 2$) are positive, so that $d_i(t) > 0$ for all $t \geq 0$ and $i = 0, 1, 2$.

Now suppose $\alpha$ has a quadratically hyponormal completion. Then evidently, $d_n(t) \geq 0$ for all $t \geq 0$ and all $n \geq 0$. In view of Theorem 1.2 (iii), $\{\alpha_n\}_{n=m}^\infty$ is strictly increasing. Thus $d_n(0) = u_0 \cdots u_n = \prod_{i=0}^n (\alpha_i^2 - \alpha_{i-1}^2) \geq 0$ for all $n \geq 0$. If $d_{n_0}(t_0) = 0$ for some $t_0 > 0$ and the first such $n_0 > 0$ ($3 \leq n_0 \leq m-1$), then (1.6) implies that $0 \leq d_{n_0+1}(t_0) = -|r_{n_0}(t_0)|^2 d_{n_0-1}(t_0) \leq 0$, which forces $r_{n_0}(t_0) = 0$, so that $\alpha_{n_0+1} = \alpha_{n_0-1}$, a contradiction. Therefore $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \ldots, m-1$. This proves the implication (i) $\Rightarrow$ (ii).

For the reverse implication, we must find a bounded sequence $\{\alpha_n\}_{n=m+1}^\infty$ such that $\alpha_n(t) > 0$ for all $t \geq 0$ and all $n \geq 0$. Suppose $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \ldots, m-1$. We now claim that there exists a constant $M_k > 0$ for which

$$
(3.1) \quad \frac{d_{k-1}(t)}{d_k(t)} \leq M_k \quad \text{for all } t \geq 0 \text{ and for } k = 1, \ldots, m-1.
$$

Indeed, since $\frac{d_{k-1}(t)}{d_k(t)}$ is a continuous function of $t$ on $[0, \infty)$, and $\deg(d_{k-1}) < \deg(d_k)$, it follows that

$$
\max_{t \in [0, \infty)} \frac{d_{k-1}(t)}{d_k(t)} \leq \max \left\{ 1, \max_{t \in [0, \xi]} \frac{d_{k-1}(t)}{d_k(t)} \right\} =: M_k,
$$

where $\xi$ is the largest root of the equation $d_{k-1}(t) = d_k(t)$. This gives (3.1). Now a straightforward calculation shows that

$$
d_m(t) = q_m(t)d_{m-1}(t) - |r_{m-1}(t)|^2 d_{m-2}(t)
$$

$$
= \left[ u_m + \left( v_m - w_{m-1} \frac{d_{m-2}(t)}{d_{m-1}(t)} \right) t \right] d_{m-1}(t).
$$
So if we write \( e_m(t) := v_m - w_{m-1} \frac{d_{m-2}(t)}{d_{m-1}(t)} \), then by (3.1), \( e_m(t) \geq v_m - w_{m-1}M_{m-1} \).

Now choose \( \alpha_{m+1} \) so that \( v_m - w_{m-1}M_{m-1} > 0 \), i.e.,

\[
\alpha_{m+1}^2 > \max \left\{ \alpha_m^2, \frac{\alpha_{m-1}^2}{\alpha_m^2} \left[ M(\alpha_m^2 - \alpha_{m-2}^2) + \alpha_{m-2}^2 \right] \right\},
\]

where \( M := \max_{t \in [0, \infty)} \frac{d_{m-2}(t)}{d_{m-1}(t)} \). Then \( e_m(t) \geq 0 \) for all \( t \geq 0 \), so that

\[
d_m(t) = (u_m + e_m(t))d_{m-1}(t) \geq u_md_{m-1}(t).
\]

Therefore, \( d_{m-1}(t) \leq \frac{d_m(t)}{u_m} \). With \( \alpha_{m+2} \) to be chosen later, we now consider \( d_{m+1} \).

We have

\[
d_{m+1}(t) = q_{m+1}(t)d_m(t) - |r_{m+1}(t)|^2d_{m-1}(t)
\geq \frac{1}{u_m} \left[ u_mq_{m+1}(t) - |r_{m+1}(t)|^2 \right] d_m(t)
= \frac{1}{u_m} \left[ u_mu_{m+1} + (u_mv_{m+1} - w_m)t \right] d_m(t)
= u_{m+1}d_m(t) + \frac{t}{u_m}(u_mv_{m+1} - w_m)d_m(t).
\]

Write \( f_{m+1} := u_mv_{m+1} - w_m \). If we choose \( \alpha_{m+2} \) such that \( f_{m+1} \geq 0 \), then \( d_{m+1}(t) \geq 0 \) for all \( t > 0 \). In particular we can choose \( \alpha_{m+2} \) so that \( f_{m+1} = 0 \), i.e.,

\[
u_mv_{m+1} = w_m,
\]

or

\[
\alpha_{m+2}^2 := \frac{\alpha_m^2(\alpha_{m+1}^2 - \alpha_{m-1}^2) + \alpha_{m-1}^2(\alpha_m^2 - \alpha_{m-1}^2)}{\alpha_{m+1}^2(\alpha_m^2 - \alpha_{m-1}^2)},
\]

or equivalently,

\[
\alpha_{m+2}^2 := \alpha_{m+1}^2 + \frac{\alpha_{m-1}^2(\alpha_{m+1}^2 - \alpha_m^2)^2}{\alpha_{m+1}^2(\alpha_m^2 - \alpha_{m-1}^2)}.
\]

In this case, \( d_{m+1}(t) \geq u_{m+1}d_m(t) \geq 0 \). Repeating the argument (with \( \alpha_{m+3} \) to be chosen later), we obtain

\[
d_{m+2}(t) = q_{m+2}(t)d_{m+1}(t) - |r_{m+1}(t)|^2d_m(t)
\geq \frac{1}{u_{m+1}} \left[ u_{m+1}q_{m+2}(t) - |r_{m+1}(t)|^2 \right] d_{m+1}(t)
= \frac{1}{u_{m+1}} \left[ u_{m+1}u_{m+2} + (u_{m+1}v_{m+2} - w_{m+1})t \right] d_{m+1}(t)
= u_{m+2}d_{m+1}(t) + \frac{t}{u_{m+1}}(u_{m+1}v_{m+2} - w_{m+1})d_{m+1}(t).
\]

Write \( f_{m+2} := u_{m+1}v_{m+2} - w_{m+1} \). If we choose \( \alpha_{m+3} \) such that \( f_{m+2} = 0 \), i.e.,

\[
\alpha_{m+3}^2 := \alpha_{m+2}^2 + \frac{\alpha_m^2(\alpha_{m+2}^2 - \alpha_{m+1}^2)^2}{\alpha_{m+2}^2(\alpha_{m+1}^2 - \alpha_m^2)},
\]
then \( d_{m+2}(t) \geq u_{m+2}d_{m+1}(t) \geq 0 \). Continuing this process with the sequence \( \{\alpha_n\}_{n=m+2}^{\infty} \) defined recursively by

\[
\varphi_1 := \frac{\alpha_m^2 (\alpha_m^2 - \alpha_{m-1}^2)}{\alpha_m^2 - \alpha_{m-1}^2}, \quad \varphi_0 := -\frac{\alpha_{m-1}^2 \alpha_m^2 (\alpha_m^2 - \alpha_m^2)}{\alpha_m^2 - \alpha_{m-1}^2}
\]

and

\[
(3.2) \quad \alpha_{n+1}^2 := \varphi_1 + \frac{\varphi_0}{\alpha_n^2} \quad (n \geq m + 1),
\]

we obtain that \( d_n(t) \geq 0 \) for all \( t > 0 \) and all \( n \geq m + 2 \). On the other hand, by an argument of Stampfli [28, Theorem 5], the sequence \( \{\alpha_n\}_{n=m+2}^{\infty} \) is bounded. Therefore, a quadratically hyponormal completion \( \{\alpha_n\}_{n=0}^{\infty} \) is obtained. The recursive relation (3.2) shows that the sequence \( \{\alpha_n\}_{n=m+2}^{\infty} \) is obtained recursively from \( \alpha_{m-1}, \alpha_m \) and \( \alpha_{m+1} \), that is, \( \{\alpha_n\}_{n=m-1}^{\infty} = (\alpha_{m-1}, \alpha_m, \alpha_{m+1})^\wedge \) (see [13], [28]). This completes the proof.

Given four weights \( \alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 \), it may not be possible to find a 2-hyponormal completion. In fact, by the preceding criterion for subnormal and \( k \)-hyponormal completions, the following statements are equivalent:

(i) \( \alpha \) has a subnormal completion;
(ii) \( \alpha \) has a 2-hyponormal completion;
(iii) \( \det \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix} \geq 0. \)

By contrast, a quadratically hyponormal completion always exists for four weights.

**Corollary 3.2.** *For arbitrary \( \alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 \), there always exists a quadratically hyponormal completion \( \omega \) of \( \alpha \).*

**Proof.** In the proof of Theorem 3.1, we showed that \( d_n(t) > 0 \) for all \( t \geq 0 \) and for \( n = 0, 1, 2 \). Thus the result immediately follows from Theorem 3.1. \( \square \)

**Remark 3.3.** To discuss the hypothesis \( \alpha_0 < \alpha_1 < \cdots < \alpha_m \) in Theorem 3.1, we consider the case where \( \alpha : \alpha_0, \alpha_1, \cdots, \alpha_m \) admits equal weights:

(i) If \( \alpha_0 < \alpha_1 = \cdots = \alpha_m \) then there exists a trivial quadratically hyponormal completion (in fact, a subnormal completion) \( \omega : \alpha_0 < \alpha_1 = \cdots = \alpha_n = \alpha_{n+1} = \cdots \).
(ii) If \( \{\alpha_n\}_{n=0}^{m} \) is such that \( \alpha_j = \alpha_{j+1} \) for some \( j = 1, 2, \cdots, m-1 \), and \( \alpha_j \neq \alpha_k \) for some \( 1 \leq j, k \leq m \), then in view of Theorem 1.2 (iii), there does not exist any quadratically hyponormal completion of \( \alpha \).
(iii) If \( \alpha_0 = \alpha_1 \), the conclusion of Theorem 3.1 may fail: for example, if \( \alpha : 1, 1, 2, 3 \) then \( d_n(t) > 0 \) for all \( t \geq 0 \) and for \( n = 0, 1, 2 \), whereas \( \alpha \) admits no quadratically hyponormal completion because by (1.9) we must have \( \alpha_2^2 < 2 \).

**Problem 3.4.** *Given \( \alpha : \alpha_0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m \), find necessary and sufficient conditions for the existence of a quadratically hyponormal completion \( \omega \) of \( \alpha \).*

In [14], related to Problem 3.4, weighted shifts of the form \( 1, (1, \sqrt{b}, \sqrt{c})^\wedge \) have been studied and their quadratic hyponormality completely characterized in terms of \( b \) and \( c \).
Remark 3.5. In Theorem 3.1, the recursively quadratically hyponormal completion requires a sufficiently large $\alpha_{m+1}$. One might conjecture that if the quadratically hyponormal completion of $\alpha: \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$ exists, then

$$\omega: \alpha_0, \cdots, \alpha_{m-3}, (\alpha_{m-2}, \alpha_{m-1}, \alpha_m)^\omega$$

is such a completion. However, if $\alpha: \sqrt[10]{10}, \sqrt[3]{1}, \sqrt[2]{2}, \sqrt[3]{3}$ then $\omega: \sqrt[10]{10}, (\sqrt[2]{1}, \sqrt[2]{2}, \sqrt[3]{3})^\omega$ is not quadratically hyponormal (by [13, Theorem 4.3], [25, Theorem 4.6]), even though by Corollary 3.2 a quadratically hyponormal completion does exist.

We conclude this section by establishing that for five or more weights, the gap between 2-hyponormal and quadratically hyponormal completions can be extremal.

Proposition 3.6. For $a < b < c$, let $\eta: (\sqrt[10]{a}, \sqrt[5]{b}, \sqrt[3]{c})^\omega$ be a recursively generated weight sequence, and consider $\alpha(x): \sqrt[10]{a}, \sqrt[5]{b}, \sqrt[3]{c}, x, \eta$ (five weights). Then

(i) $\alpha$ has a subnormal completion $\iff x = \eta_3$;

(ii) $\alpha$ has a 2-hyponormal completion $\iff x = \eta_3$;

(iii) $\alpha$ has a quadratically hyponormal completion $\iff c < x < \eta_1^2$.

Proof. Assertions (i) and (ii) follow from the argument used in the proof of [16, Theorem 3.2]. For assertion (iii), observe that by Theorem 3.1, $\alpha$ has a quadratically hyponormal completion if and only if $d_3(t) > 0$ for all $t \geq 0$. Without loss of generality, we write $a = 1$, $b = 1 + r$, $c = 1 + r + s$, and $x = 1 + r + s + u$ ($r > 0$, $s > 0$, $u > 0$). A straightforward calculation using Mathematica shows that the Maclaurin coefficients $c(3,i)$ of $d_3(t)$ are given by

$$c(3,0) = rsu;$$
$$c(3,1) = s^3(r + s)(1 + r + s + u)(r + r^2 + 2rs + s^2)^{-1};$$
$$c(3,2) = (1 + r + s)(s^4 + rsu + 4r^2su + 5r^3su + 2r^4su + 2rs^2u + 7r^2s^2u + 5r^3s^2u + 2s^3u + 4rs^3u + 4r^2s^3u + s^4u + rs^4u + r^3u^2 + 2r^3u^2 + r^4u^2 + 3r^2su^2 + 3r^3su^2 + 2rs^2u^2 + s^3u^2 + rs^3u^2)(r + r^2 + 2rs + s^2)^{-1};$$
$$c(3,3) = (1 + r)(r + s)(1 + r + s + u)(r^2s^2 + r^3s^2 + s^3 + 2rs^3 + 2r^2s^3 + s^4 + rs^4 + r^2u + 2r^3u + r^4u + 3r^2su + 3r^3su + 2rs^2u + 3r^2s^2u + s^3u + rs^3u)(r^2 + r^3 + 2rs^2 + s^2)^{-1};$$
$$c(3,4) = (1 + r)^2(1 + r + s)(r + r^2 + 2s + 2rs + s^2 + u + ru + su)(r^2s^2 + 2rs^3 + r^4s + rs^2 + 4r^2s^2 + 3rs^2 + s^3 + 3rs^3 + 3r^2s^3 + s^4 + rs^4 + r^2u + 2r^3u + r^4u + 3r^2su + 3r^3su + 2rs^2u + 3r^2s^2u + s^3u + rs^3u)(r^2 + r^3 + 2rs^2 + s^2)^{-1}.$$ 

This readily shows that for $c < x < \alpha_1^2$, all Maclaurin coefficients of $d_3(t)$ are positive, so that $d_3(t) > 0$ for all $t \geq 0$. Moreover if $x = c$ or $\alpha_1^2$ then Theorem 1.2 shows that no quadratically hyponormal completion exists. This proves assertion (iii).

References