SOLUTION OF THE QUADRATICALLY HYPONORMAL COMPLETION PROBLEM

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Abstract. For $m \geq 1$, let $\alpha : \alpha_0 < \cdots < \alpha_m$ be a collection of $(m+1)$ positive weights. The *Quadratically Hyponormal Completion Problem* seeks necessary and sufficient conditions on α to guarantee the existence of a quadratically hyponormal unilateral weighted shift W with α as initial segment of weights. We prove that α admits a quadratically hyponormal completion if and only if the self-adjoint $m \times m$ matrix

$$D_{m-1}(s) := \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{m-2} & \bar{r}_{m-2} \\ 0 & 0 & 0 & \dots & r_{m-2} & q_{m-1} \end{pmatrix}$$

is positive and invertible, where $q_k := u_k + |s|^2 v_k$, $r_k := s\sqrt{w_k}$, $u_k := \alpha_k^2 - \alpha_{k-1}^2$, $v_k := \alpha_k^2 \alpha_{k+1}^2 - \alpha_{k-1}^2 \alpha_{k-2}^2$, $w_k := \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2$, and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. As a particular case, this result shows that a collection of *four* positive numbers $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ *always* admits a quadratically hyponormal completion. This provides a new qualitative criterion to distinguish quadratic hyponormality from 2-hyponormality.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2],[5, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1). \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 47B20, 47B35, 47B37; Secondary 47-04, 47A20, 47A57.

Key words and phrases. Weighted shifts, propagations, subnormal, k-hyponormal, quadratically hyponormal, completions.

The work of the first-named author was partially supported by NSF research grants DMS-9800931 and DMS-0099357.

The work of the second-named author was partially supported by the Brain Korea 21 Project.

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Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{1.2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k + 1) \times (k + 1)$ operator matrix in (1.1). The Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([17]).

Recall ([1],[6],[17]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e.,

$$(M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C} \text{ ([17]).} \tag{1.3}$$

If $k = 2$ then T is said to be *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general.

The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([8],[9],[11],[12],[13],[15], [17],[20],[26]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle; in fact, even subnormality for Toeplitz operators has not yet been characterized (cf.[7],[22]). For weighted shifts, positive results appear in [8] and [13], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [18] and [19]).

Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. Weighted shifts provide a fertile ground to analyze the relative position of subnormality, polynomial hyponormality, k-hyponormality and weak k-hyponormality. In [13], the first named author and L. Fialkow applied the solution of the Subnormal Completion Problem [12] to build the first example of a one-parameter family of non-2-hyponormal quadratically hyponormal weighted shifts of recursive type. Additional examples were later found in [4],[14],[15] and [25], in some cases providing qualitative indicators to separate quadratically hyponormality from 2-hyponormality.

In this paper we obtain a new qualitative indicator, as a corollary to the Quadratically Hyponormal Completion Criterion. Our main result is a complete solution of the Quadratically Hyponormal Completion Problem:

Given $\alpha : \alpha_0 < \alpha_1 < \cdots < \alpha_m$, find necessary and sufficient conditions for the existence of a quadratically hyponormal weighted shift whose first $(m+1)$ weights are those in α.

As a special case, we show that given four weights $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$ there always exists a quadratically hyponormal completion, thus providing a new criterion to distinguish between quadratic hyponormality and 2-hyponormality.

Definition 1.1. A unilateral weighted shift W_α is *flat* (or briefly, α is flat) if $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.

J. Stampfli [28] showed that for subnormal weighted shifts W_α, a *propagation* phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. Later, A. Joshi proved in [24] that the shift with weights $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = \cdots = b$, $0 < a < b$, is *not* quadratically hyponormal, and P. Fan [21] established that for $a = 1$, $b = 2$, and $0 < s < \sqrt{5}/5$, $W_\alpha + s\,W_\alpha^2$ is *not* hyponormal. On the other hand, it was shown in [8, Theorem 2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: *If W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.* Furthermore, in [8, Proposition 11] it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force flatness, thereby subnormality.

Theorem 1.2 (Propagation). *Let W_α be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$.*

(i) ([28, Theorem 6]) *Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.*

(ii) ([8, Corollary 6]) *Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*

(iii) ([3, Theorem 1]) *Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.*

An immediate consequence of Theorem 1.2 is that, for purposes of testing quadratic hyponormality, we can decompose all nonsubnormal weighted shifts into two classes: those with $\alpha_0 = \alpha_1$ and those with strictly increasing weight sequences.

To study quadratic hyponormality, we consider the selfcommutator $[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$. For $s \in \mathbb{C}$, we write $D(s) := [(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$ and we let

(1.4)
$$D_n(s) := P_n[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]P_n = \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix},$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \cdots, e_n\}$,

$$(1.5) \qquad \begin{cases} q_n := u_n + |s|^2 v_n \\ r_n := s\sqrt{w_n} \\ u_n := \alpha_n^2 - \alpha_{n-1}^2 \\ v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2 \\ w_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} := 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then d_n satisfies the following 2–step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0 q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n. \tag{1.6}$$

If we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$, and if we write $d_n \equiv \sum_{i=0}^{n+1} c(n,i) t^i$, then the coefficients $c(n,i)$ satisfy a double-indexed recursive formula, namely

(1.7)

$$c(n+2,i) = u_{n+2}\, c(n+1,i) + v_{n+2}\, c(n+1,i-1) - w_{n+1}\, c(n,i-1),$$
$$c(n,0) = u_0 \cdots u_n, \quad c(n,n+1) = v_0 \cdots v_n, \quad c(1,1) = u_1 v_0 + v_1 u_0 - w_0$$

($n \geq 0$, $i \geq 1$). We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for every $n \geq 0$, $0 \leq i \leq n+1$ (cf. [10]). Positive quadratic hyponormality implies quadratic hyponormality, but the converse is false (cf. [4]).

The idea of the proof of Theorem 1.2 (iii) is based on the following observation: if W_α is quadratically hyponormal with $\alpha_1 = \alpha_2 = 1$, then a straightforward calculation shows that

$$d_4(t) = \alpha_0^2 \alpha_4^2 (\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 t^2 + c(4,3)t^3 + c(4,4)t^4 + c(4,5)t^5,$$

so

$$\lim_{t \to 0+} \frac{d_4(t)}{t^2} = \alpha_0^2 \alpha_4^2 (\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 \ \geq 0,$$

which forces $\alpha_0 = 1$ or $\alpha_3 = 1$, so that three equal weights are present and hence by [8, Theorem 2], flatness occurs.

Note that in Theorem 1.2 (iii) the condition "$n \geq 1$" cannot be relaxed to "$n \geq 0$". For example, if

$$\alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}} \ (n \geq 2), \tag{1.8}$$

then W_α is quadratically hyponormal (cf. [8, Proposition 7]) but not cubically hyponormal (and hence not subnormal); indeed, if we let

$$C_5(t) := \det\left(P_5 \left[(W_\alpha + t\, W_\alpha^2 + t^2\, W_\alpha^3)^*,\ (W_\alpha + t\, W_\alpha^2 + t^2\, W_\alpha^3) \right] P_5 \right)$$

then

$$\lim_{t \to 0+} \frac{C_5(t)}{t^8} = -\frac{1}{2041200} < 0.$$

We conclude this section by recalling that when $\alpha_0 = \alpha_1 = 1$, quadratic hyponormality implies

$$\alpha_2 < \sqrt{2} \quad \text{and} \quad \alpha_3 \geq (2 - \alpha_2^2)^{-2} \quad \text{(cf. [10, p.78])}. \tag{1.9}$$

2. Recursively Generated Shifts in the Study of Completions

If W_α is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the *moments* of W_α are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ $(n \geq 0)$ [27]; however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2$ $(n \geq 0)$. A criterion for k-hyponormality can be given in terms of these moments ([8, Theorem 4]): if we build a $(k+1) \times (k+1)$ Hankel matrix $A(n;k)$ by

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0),$$

then W_α is k-hyponormal if and only if $A(n;k) \geq 0$ $(n \geq 0)$. C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [23], [5, III.8.16]) states that W_α is subnormal if and only if there exists a Borel probability measure μ supported in $[0, ||W_\alpha||^2]$, with $||W_\alpha||^2 \in \operatorname{supp} \mu$, such that $\gamma_n = \int t^n d\mu(t)$ for all $n \geq 0$. Given an initial segment of weights $\alpha : \alpha_0, \cdots \alpha_m$, a sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ $(i = 0, \cdots, m)$ is said to be *recursively generated* by α if there exist $r \geq 1$ and $\varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R}$ such that $\gamma_{n+r} = \varphi_0 \gamma_n + \cdots + \varphi_{r-1} \gamma_{n+r-1}$ (all $n \geq 0$), where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$; in this case $W_{\hat{\alpha}}$ is said to be *recursively generated*. If we let $g(t) := t^r - \left(\varphi_{r-1} t^{r-1} + \cdots + \varphi_0\right)$, then g has r distinct real roots $0 \leq s_0 < \cdots < s_{r-1}$ ([12, Theorem 3.9]). Let

$$V := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ s_0 & s_1 & \ldots & s_{r-1} \\ \vdots & \vdots & & \vdots \\ s_0^{r-1} & s_1^{r-1} & \ldots & s_{r-1}^{r-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix} := V^{-1} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

If the associated recursively generated weighted shift $W_{\hat{\alpha}}$ is subnormal, then its Berger measure is of the form $\mu := \rho_0 \delta_{s_0} + \cdots + \rho_{r-1} \delta_{r-1}$. Let $\alpha : \alpha_0, \cdots, \alpha_m$ $(m \geq 0)$ be an initial segment of positive weights and let $\omega = \{\omega_n\}_{n=0}^\infty$ be a bounded sequence of positive numbers. We say that W_ω is a *completion* of α if $\omega_n = \alpha_n$ $(0 \leq n \leq m)$, and we write $\alpha \subset \omega$. The *completion problem* for a property (P) entails finding necessary and sufficient conditions on α to ensure the existence of a weight sequence $\omega \supset \alpha$ such that W_ω satisfies (P). In 1966, Stampfli [28] showed that for arbitrary $\alpha_0 < \alpha_1 < \alpha_2$ there always exists a subnormal weighted shift W_α whose first *three* weights are $\alpha_0, \alpha_1, \alpha_2$; he also proved that given *four* or more weights, it may not be possible to find a subnormal completion. In [12, Theorem 3.5], the following criterion was established.

Subnormal Completion Criterion. *If $\alpha : \alpha_0, \cdots, \alpha_m$ $(m \geq 0)$ is an initial segment of positive weights then the following are equivalent:*

(i) *α has a subnormal completion;*
(ii) *α has a recursively generated subnormal completion;*
(iii) *α has an $([\frac{m}{2}]+1)$–hyponormal completion;*
(iv) *the Hankel matrices*

$$A(k) := \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{pmatrix} \quad \textit{and} \quad B(\ell-1) := \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_\ell \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{\ell+1} \\ \vdots & \vdots & & \vdots \\ \gamma_\ell & \gamma_{\ell+1} & \cdots & \gamma_{2\ell-1} \end{pmatrix}$$

are both positive ($k := [\frac{m+1}{2}]$ *and* $\ell := [\frac{m}{2}] + 1$) *and the vector*

$$\begin{pmatrix} \gamma_{k+1} \\ \vdots \\ \gamma_{2k+1} \end{pmatrix} \quad (\textit{resp.} \begin{pmatrix} \gamma_{k+1} \\ \vdots \\ \gamma_{2k} \end{pmatrix})$$

is in the range of $A(k)$ *(resp.* $B(\ell - 1)$*) when* m *is even (resp. odd).*

Thus the Subnormal Completion Problem reduces to the *Recursive Completion Problem*, which entails finding necessary and sufficient conditions on α to ensure that the recursively generated weight sequence $\hat{\alpha}$ is well-defined and bounded.

Also in [12, Proposition 3.19], the following criterion on k-hyponormal completions was established.

k-Hyponormal Completion Criterion. *If* $\alpha : \alpha_0, \cdots, \alpha_{2m}$ ($m \geq 1$) *is an initial segment of positive weights then for* $1 \leq k \leq m$ *the following are equivalent:*

(i) α *has a* k*-hyponormal completion;*

(ii) *the Hankel matrix*

$$A(j,k) := \begin{pmatrix} \gamma_j & \cdots & \gamma_{j+k} \\ \vdots & & \vdots \\ \gamma_{j+k} & \cdots & \gamma_{j+2k} \end{pmatrix}$$

is positive for all j, $0 \leq j \leq 2m - 2k + 1$, *and the vector*

$$\begin{pmatrix} \gamma_{2m-k+2} \\ \vdots \\ \gamma_{2m+1} \end{pmatrix}$$

is in the range of $A(2m - 2k + 2, k - 1)$.

If α *admits a* k*-hyponormal completion, then it admits a recursively generated one.*

3. The Quadratically Hyponormal Completion Criterion

We now formulate and solve the corresponding problem for *quadratic hyponormality.*

Quadratically Hyponormal Completion Problem. *Given* $\alpha : \alpha_0 < \alpha_1 < \cdots < \alpha_m$, *find necessary and sufficient conditions for the existence of a quadratically hyponormal weighted shift whose first* $(m + 1)$ *weights are given by* α.

We pause to recall that, for a square matrix A, the notation $A > 0$ means $A \geq 0$ and A invertible.

Theorem 3.1 (Quadratically Hyponormal Completion Criterion). *Let* $m \geq 2$ *and let* $\alpha : \alpha_0 < \alpha_1 < \cdots < \alpha_m$ *be an initial segment of positive weights. Then the following statements are equivalent:*

(i) α *has a quadratically hyponormal completion;*

(ii) $D_{m-1}(t) > 0$ *for all* $t \geq 0$.

Moreover, a quadratically hyponormal completion ω of α can be obtained in the following recursively generated fashion:

$$\omega: \ \alpha_0, \cdots, \alpha_{m-2}, (\alpha_{m-1}, \alpha_m, \alpha_{m+1})^\wedge,$$

where α_{m+1} is chosen so that $\alpha_{m+1}^2 > \max\{\alpha_m^2, \ \frac{\alpha_{m-1}^2}{\alpha_m^2}\left[M\,(\alpha_m^2 - \alpha_{m-2}^2)^2 + \alpha_{m-2}^2\right]\}$ ($M := \max_{t\in[0,\infty)} \frac{d_{m-2}(t)}{d_{m-1}(t)}$).

Proof. We will use the notation in Section 1. First of all, note that $D_{m-1}(t) > 0$ for all $t \geq 0$ if and only if $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \cdots, m-1$; this follows from the Nested Determinants Test (see [12, Remark 2.4]) or Choleski's Algorithm (see [12, Proposition 2.3]). A straightforward calculation gives

$$\begin{aligned}
d_0(t) &= \alpha_0^2 + \alpha_0^2\alpha_1^2\, t\\
d_1(t) &= \alpha_0^2(\alpha_1^2 - \alpha_0^2) + \alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_0^2)\, t \, + \, \alpha_0^2\alpha_1^4\alpha_2^2\, t^2\\
d_2(t) &= \alpha_0^2(\alpha_1^2 - \alpha_0^2)(\alpha_2^2 - \alpha_1^2) \, + \, \alpha_0^2\alpha_2^2(\alpha_1^2 - \alpha_0^2)(\alpha_3^2 - \alpha_1^2)\, t\\
&\quad + \alpha_0^2\alpha_1^2\alpha_2^2\left\{\alpha_3^2(\alpha_2^2 - \alpha_0^2) - \alpha_1^2(\alpha_1^2 - \alpha_0^2)\right\} t^2 + \alpha_0^2\alpha_1^4\alpha_2^2(\alpha_2^2\alpha_3^2 - \alpha_1^2\alpha_0^2)\, t^3,
\end{aligned}$$

which shows that all coefficients of d_i ($i = 0, 1, 2$) are positive, so that $d_i(t) > 0$ for all $t \geq 0$ and $i = 0, 1, 2$.

Now suppose α has a quadratically hyponormal completion. Then evidently, $d_n(t) \geq 0$ for all $t \geq 0$ and all $n \geq 0$. In view of Theorem 1.2 (iii), $\{\alpha_n\}_{n=m}^\infty$ is strictly increasing. Thus $d_n(0) = u_0 \cdots u_n = \prod_{i=0}^n (\alpha_i^2 - \alpha_{i-1}^2) > 0$ for all $n \geq 0$. If $d_{n_0}(t_0) = 0$ for some $t_0 > 0$ and the first such $n_0 > 0$ ($3 \leq n_0 \leq m-1$), then (1.6) implies that $0 \leq d_{n_0+1}(t_0) = -|r_{n_0}(t_0)|^2 d_{n_0-1}(t_0) \leq 0$, which forces $r_{n_0}(t_0) = 0$, so that $\alpha_{n_0+1} = \alpha_{n_0-1}$, a contradiction. Therefore $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \cdots, m-1$. This proves the implication (i) $\Rightarrow$ (ii).

For the reverse implication, we must find a bounded sequence $\{\alpha_n\}_{n=m+1}^\infty$ such that $d_n(t) \geq 0$ for all $t \geq 0$ and all $n \geq 0$. Suppose $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, \cdots, m-1$. We now claim that there exists a constant $M_k > 0$ for which

$$\text{(3.1)} \qquad \frac{d_{k-1}(t)}{d_k(t)} \leq M_k \qquad \text{for all } t \geq 0 \text{ and for } k = 1, \cdots, m-1.$$

Indeed, since $\frac{d_{k-1}(t)}{d_k(t)}$ is a continuous function of t on $[0, \infty)$, and $\deg(d_{k-1}) < \deg(d_k)$, it follows that

$$\max_{t\in[0,\,\infty)} \frac{d_{k-1}(t)}{d_k(t)} \ \leq \ \max\left\{1, \ \max_{t\in[0,\,\xi]} \frac{d_{k-1}(t)}{d_k(t)}\right\} =: M_k,$$

where ξ is the largest root of the equation $d_{k-1}(t) = d_k(t)$. This gives (3.1). Now a straightforward calculation shows that

$$\begin{aligned}
d_m(t) &= q_m(t) d_{m-1}(t) - |r_{m-1}(t)|^2 d_{m-2}(t)\\
&= \left[u_m + \left(v_m - w_{m-1}\frac{d_{m-2}(t)}{d_{m-1}(t)}\right) t\right] d_{m-1}(t)\,.
\end{aligned}$$

So if we write $e_m(t) := v_m - w_{m-1}\frac{d_{m-2}(t)}{d_{m-1}(t)}$, then by (3.1), $e_m(t) \geq v_m - w_{m-1}M_{m-1}$. Now choose α_{m+1} so that $v_m - w_{m-1}M_{m-1} > 0$, i.e.,

$$\alpha_{m+1}^2 > \max\left\{\alpha_m^2,\ \frac{\alpha_{m-1}^2}{\alpha_m^2}\left[M\left(\alpha_m^2 - \alpha_{m-2}^2\right)^2 + \alpha_{m-2}^2\right]\right\},$$

where $M := \max_{t\in[0,\infty)} \frac{d_{m-2}(t)}{d_{m-1}(t)}$. Then $e_m(t) \geq 0$ for all $t \geq 0$, so that

$$d_m(t) = (u_m + e_m(t)\,t)d_{m-1}(t) \geq u_m d_{m-1}(t) > 0.$$

Therefore, $d_{m-1}(t) \leq \frac{d_m(t)}{u_m}$. With α_{m+2} to be chosen later, we now consider d_{m+1}. We have

$$\begin{aligned} d_{m+1}(t) &= q_{m+1}(t)d_m(t) - |r_m(t)|^2 d_{m-1}(t) \\ &\geq \frac{1}{u_m}\left[u_m q_{m+1}(t) - |r_m(t)|^2\right] d_m(t) \\ &= \frac{1}{u_m}\left[u_m u_{m+1} + (u_m v_{m+1} - w_m)t\right] d_m(t) \\ &= u_{m+1}d_m(t) + \frac{t}{u_m}(u_m v_{m+1} - w_m)\, d_m(t). \end{aligned}$$

Write $f_{m+1} := u_m v_{m+1} - w_m$. If we choose α_{m+2} such that $f_{m+1} \geq 0$, then $d_{m+1}(t) \geq 0$ for all $t > 0$. In particular we can choose α_{m+2} so that $f_{m+1} = 0$. i.e., $u_m v_{m+1} = w_m$, or

$$\alpha_{m+2}^2 := \frac{\alpha_m^2(\alpha_{m+1}^2 - \alpha_{m-1}^2)^2 + \alpha_{m-1}^2\alpha_m^2(\alpha_m^2 - \alpha_{m-1}^2)}{\alpha_{m+1}^2(\alpha_m^2 - \alpha_{m-1}^2)},$$

or equivalently,

$$\alpha_{m+2}^2 := \alpha_{m+1}^2 \ + \ \frac{\alpha_{m-1}^2(\alpha_{m+1}^2 - \alpha_m^2)^2}{\alpha_{m+1}^2(\alpha_m^2 - \alpha_{m-1}^2)}.$$

In this case, $d_{m+1}(t) \geq u_{m+1}\, d_m(t) \geq 0$. Repeating the argument (with α_{m+3} to be chosen later), we obtain

$$\begin{aligned} d_{m+2}(t) &= q_{m+2}(t)d_{m+1}(t) - |r_{m+1}(t)|^2 d_m(t) \\ &\geq \frac{1}{u_{m+1}}\left[u_{m+1}q_{m+2}(t) - |r_{m+1}(t)|^2\right] d_{m+1}(t) \\ &= \frac{1}{u_{m+1}}\left[u_{m+1}u_{m+2} + (u_{m+1}v_{m+2} - w_{m+1})\,t\right] d_{m+1}(t) \\ &= u_{m+2}\, d_{m+1}(t) + \frac{t}{u_{m+1}}(u_{m+1}v_{m+2} - w_{m+1})\, d_{m+1}(t). \end{aligned}$$

Write $f_{m+2} := u_{m+1}v_{m+2} - w_{m+1}$. If we choose α_{m+3} such that $f_{m+2} = 0$, i.e.,

$$\alpha_{m+3}^2 := \alpha_{m+2}^2 \ + \ \frac{\alpha_m^2(\alpha_{m+2}^2 - \alpha_{m+1}^2)^2}{\alpha_{m+2}^2(\alpha_{m+1}^2 - \alpha_m^2)},$$

then $d_{m+2}(t) \geq u_{m+2}\, d_{m+1}(t) \geq 0$. Continuing this process with the sequence $\{\alpha_n\}_{n=m+2}^{\infty}$ defined recursively by

$$\varphi_1 := \frac{\alpha_m^2(\alpha_{m+1}^2 - \alpha_{m-1}^2)}{\alpha_m^2 - \alpha_{m-1}^2}, \quad \varphi_0 := -\frac{\alpha_{m-1}^2\alpha_m^2(\alpha_{m+1}^2 - \alpha_m^2)}{\alpha_m^2 - \alpha_{m-1}^2}$$

and

$$\alpha_{n+1}^2 := \varphi_1 + \frac{\varphi_0}{\alpha_n^2} \quad (n \geq m+1), \tag{3.2}$$

we obtain that $d_n(t) \geq 0$ for all $t > 0$ and all $n \geq m+2$. On the other hand, by an argument of Stampfli [28, Theorem 5], the sequence $\{\alpha_n\}_{n=m+2}^{\infty}$ is bounded. Therefore, a quadratically hyponormal completion $\{\alpha_n\}_{n=0}^{\infty}$ is obtained. The recursive relation (3.2) shows that the sequence $\{\alpha_n\}_{n=m+2}^{\infty}$ is obtained recursively from α_{m-1}, α_m and α_{m+1}, that is, $\{\alpha_n\}_{n=m-1}^{\infty} = (\alpha_{m-1}, \alpha_m, \alpha_{m+1})^\wedge$ (see [13], [28]). This completes the proof. □

Given *four* weights $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, it may not be possible to find a 2-hyponormal completion. In fact, by the preceding criterion for subnormal and k-hyponormal completions, the following statements are equivalent:

(i) α has a subnormal completion;
(ii) α has a 2-hyponormal completion;
(iii) $\det \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix} \geq 0.$

By contrast, a quadratically hyponormal completion *always* exists for *four* weights.

Corollary 3.2. *For arbitrary $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3$, there always exists a quadratically hyponormal completion ω of α.*

Proof. In the proof of Theorem 3.1, we showed that $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, 1, 2$. Thus the result immediately follows from Theorem 3.1. □

Remark 3.3. To discuss the hypothesis $\alpha_0 < \alpha_1 < \cdots < \alpha_m$ in Theorem 3.1, we consider the case where $\alpha : \alpha_0, \alpha_1, \cdots, \alpha_m$ admits equal weights:

(i) If $\alpha_0 < \alpha_1 = \cdots = \alpha_m$ then there exists a trivial quadratically hyponormal completion (in fact, a subnormal completion) $\omega : \alpha_0 < \alpha_1 = \cdots = \alpha_n = \alpha_{n+1} = \cdots$.
(ii) If $\{\alpha_n\}_{n=0}^{m}$ is such that $\alpha_j = \alpha_{j+1}$ for some $j = 1, 2, \cdots, m-1$, and $\alpha_j \neq \alpha_k$ for some $1 \leq j, k \leq m$, then in view of Theorem 1.2 (iii), there does not exist any quadratically hyponormal completion of α.
(iii) If $\alpha_0 = \alpha_1$, the conclusion of Theorem 3.1 may fail: for example, if α : $1, 1, 2, 3$ then $d_n(t) > 0$ for all $t \geq 0$ and for $n = 0, 1, 2$, whereas α admits no quadratically hyponormal completion because by (1.9) we must have $\alpha_2^2 < 2$.

Problem 3.4. *Given $\alpha : \alpha_0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m$, find necessary and sufficient conditions for the existence of a quadratically hyponormal completion ω of α.*

In [14], related to Problem 3.4, weighted shifts of the form $1, (1, \sqrt{b}, \sqrt{c})^\wedge$ have been studied and their quadratic hyponormality completely characterized in terms of b and c.

Remark 3.5. In Theorem 3.1, the recursively quadratically hyponormal completion requires a sufficiently large α_{m+1}. One might conjecture that if the quadratically hyponormal completion of $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$ exists, then

$$\omega : \alpha_0, \cdots, \alpha_{m-3}, (\alpha_{m-2}, \alpha_{m-1}, \alpha_m)^\wedge$$

is such a completion. However, if $\alpha : \sqrt{\frac{9}{10}}, \sqrt{1}, \sqrt{2}, \sqrt{3}$ then $\omega : \sqrt{\frac{9}{10}}, (\sqrt{1}, \sqrt{2}, \sqrt{3})^\wedge$ is not quadratically hyponormal (by [13, Theorem 4.3], [25, Theorem 4.6]), even though by Corollary 3.2 a quadratically hyponormal completion does exist.

We conclude this section by establishing that for *five* or more weights, the gap between 2-hyponormal and quadratically hyponormal completions can be extremal.

Proposition 3.6. *For $a < b < c$, let $\eta : (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ be a recursively generated weight sequence, and consider $\alpha(x) : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{x}, \eta_4$ (five weights). Then*

(i) *α has a subnormal completion $\Longleftrightarrow x = \eta_3$;*
(ii) *α has a 2-hyponormal completion $\Longleftrightarrow x = \eta_3$;*
(iii) *α has a quadratically hyponormal completion $\Longleftrightarrow c < x < \eta_4^2$.*

Proof. Assertions (i) and (ii) follow from the argument used in the proof of [16, Theorem 3.2]. For assertion (iii), observe that by Theorem 3.1, α has a quadratically hyponormal completion if and only if $d_3(t) > 0$ for all $t \geq 0$. Without loss of generality, we write $a = 1$, $b = 1 + r$, $c = 1 + r + s$, and $x = 1 + r + s + u$ ($r > 0$, $s > 0$, $u > 0$). A straightforward calculation using *Mathematica* shows that the Maclaurin coefficients $c(3, i)$ of $d_3(t)$ are given by

$$c(3,0) = rsu;$$
$$c(3,1) = s^3(r+s)(1+r+s+u)(r+r^2+2rs+s^2)^{-1};$$
$$\begin{aligned} c(3,2) = {} & (1+r+s)(s^4+rsu+4r^2su+5r^3su+2r^4su+2rs^2u+7r^2s^2u+5r^3s^2u \\ & +2s^3u+4rs^3u+4r^2s^3u+s^4u+rs^4u+r^2u^2+2r^3u^2+r^4u^2+3r^2su^2 \\ & +3r^3su^2+2rs^2u^2+3r^2s^2u^2+s^3u^2+rs^3u^2)(r+r^2+2rs+s^2)^{-1}; \end{aligned}$$
$$\begin{aligned} c(3,3) = {} & (1+r)(r+s)(1+r+s)(1+r+s+u)(r^2s^2+r^3s^2+s^3+2rs^3 \\ & +2r^2s^3+s^4+rs^4+r^2u+2r^3u+r^4u+3r^2su+3r^3su+2rs^2u+3r^2s^2u \\ & +s^3u+rs^3u)(r^2+r^3+2r^2s+rs^2)^{-1}; \text{ and} \end{aligned}$$
$$\begin{aligned} c(3,4) = {} & (1+r)^2(1+r+s)(r+r^2+2s+2rs+s^2+u+ru+su)(r^2s+2r^3s+r^4s \\ & +rs^2+4r^2s^2+3r^3s^2+s^3+3rs^3+3r^2s^3+s^4+rs^4+r^2u+2r^3u+r^4u \\ & +3r^2su+3r^3su+2rs^2u+3r^2s^2u+s^3u+rs^3u)(r^2+r^3+2r^2s+rs^2)^{-1}. \end{aligned}$$

This readily shows that for $c < x < \alpha_4^2$, all Maclaurin coefficients of $d_3(t)$ are positive, so that $d_3(t) > 0$ for all $t \geq 0$. Moreover if $x = c$ or α_4^2 then Theorem 1.2 shows that no quadratically hyponormal completion exists. This proves assertion (iii). ∎

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