The Beurling-Lax-Halmos Theorem for Infinite Multiplicity

Raúl E. Curto, In Sung Hwang and Woo Young Lee

Abstract. In this paper we consider several questions emerging from the Beurling-Lax-Halmos Theorem, which characterizes the shift-invariant subspaces of vector-valued Hardy spaces. The Beurling-Lax-Halmos Theorem states that a backward shift-invariant subspace is a model space $\mathcal{H}(\Delta) \equiv H^2_E \ominus \Delta H^2_E$, for some inner function Δ. Our first question calls for a description of the set F in H^2_E such that $\mathcal{H}(\Delta) = E^*_F$, where E^*_F denotes the smallest backward shift-invariant subspace containing the set F. In our pursuit of a general solution to this question, we are naturally led to take into account a canonical decomposition of operator-valued strong L^2-functions. Next, we ask: Is every shift-invariant subspace the kernel of a (possibly unbounded) Hankel operator ? Consideration of the question on the structure of shift-invariant subspaces leads us to study and coin a new notion of "Beurling degree" for an inner function. We then establish a deep connection between the spectral multiplicity of the model operator (the truncated backward shift) and the Beurling degree of the corresponding characteristic function. At the same time, we consider the notion of meromorphic pseudo-continuations of bounded type for operator-valued functions, and then use this notion to study the spectral multiplicity of model operators between separable complex Hilbert spaces. In particular, we consider the case of multiplicity-free: more precisely, for which characteristic function Δ of the model operator T does it follow that T is multiplicity-free, i.e., T has multiplicity 1 ? We show that if Δ has a meromorphic pseudo-continuation of bounded type in the complement of the closed unit disk and the adjoint of the flip of Δ is an outer function, then T is multiplicity-free.

Contents

1 Introduction

The celebrated Beurling Theorem [Beu] characterizes the shift-invariant subspaces of the Hardy space. P.D. Lax [Lax] extended the Beurling Theorem to the case of finite multiplicity, and proved the so-called Beurling-Lax Theorem. Subsequently, P.R. Halmos [Ha1] gave a beautiful proof for the case of infinite multiplicity, and thus established the so-called Beurling-Lax-Halmos Theorem. Since then, the Beurling-Lax-Halmos Theorem has been extended to various settings and extensively applied in connection with model theory, system theory and the interpolation problem by many authors (cf. [Sa1], [AD], [Car] for multiply-connected domains, [Pop] for the free-noncommutative full Fock space, [MT] and [ADR] for the Drury-Arveson space, [Hed], [ARS], [Shi], [BaB] for the Bergman space, [Ric] for the Dirichlet space, [AS] for the complex and quaternionic setting, [BH1], [BH2], [BH4], [BH3], [BR] for the linear groups setting, and [dBR] for the de Branges-Rovnyak space).

In this paper, we will focus on a detailed analysis of the Beurling-Lax-Halmos Theorem for infinite multiplicity. We obtain answers to several questions emerging from the Beurling-Lax-Halmos Theorem and establish some new and exciting results, including: (i) a canonical decomposition for operator-valued L^2-functions (in fact, for a much bigger

---

2010 *Mathematics Subject Classification.* Primary 46E40, 47B35, 30H10, 30J05; Secondary 43A15, 47A15

Key words. The Beurling-Lax-Halmos Theorem, strong L^2-functions, a complementary factor of an inner function, the Beurling degree.

class of functions), (ii) the introduction of the Beurling degree of an inner function, and (iii) the study of the spectral multiplicity of a model operator.

Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$. Throughout this paper, whenever we deal with operator-valued functions Φ on $\mathbb{T}$, we assume that $\Phi(z)$ is a bounded linear operator between separable complex Hilbert spaces for almost all $z \in \mathbb{T}$. For a separable complex Hilbert space E, let S_E be the shift operator on the E-valued Hardy space H^2_E, i.e.,

$$(S_E f)(z) := zf(z) \quad \text{for each } f \in H^2_E.$$

The Beurling-Lax-Halmos Theorem states that every subspace M invariant under S_E (i.e., a closed subspace of H^2_E such that $S_E f \in M$ for all $f \in M$) is of the form $\Delta H^2_{E'}$, where E' is a closed subspace of E and Δ is an *inner* function. As usual, Δ is an inner function if $\Delta(z)$ is an isometric operator from E' into E for almost all $z \in \mathbb{T}$, i.e., $\Delta^*\Delta = I_{E'}$ a.e. on $\mathbb{T}$. If, in addition, $\Delta\Delta^* = I_E$ a.e. on $\mathbb{T}$, then Δ is called a *two-sided* inner function.

There exists an equivalent description of a closed subspace M of H^2_E which is invariant under the backward shift operator S^*_E; that is, $M = \mathcal{H}(\Delta) := H^2_E \ominus \Delta H^2_{E'}$ for some inner function Δ. The space $\mathcal{H}(\Delta)$ is often called a model space or a de Branges-Rovnyak space [dBR], [Sa2], [SFBK]. Thus, for a subset F of H^2_E, if E^*_F denotes the smallest S^*_E-invariant subspace containing F, i.e.,

$$E^*_F := \bigvee \{S^{*n}_E F : n \geq 0\},$$

(where $\bigvee$ denotes the closed linear span), then $E^*_F = \mathcal{H}(\Delta)$ for some inner function Δ.

Now, given a backward shift-invariant subspace $\mathcal{H}(\Delta)$, we may ask:

Question 1.1. (i) What is the smallest number of vectors in F satisfying $\mathcal{H}(\Delta) = E^*_F$?
(ii) More generally, we are interested in the problem of describing the set F in H^2_E such that $\mathcal{H}(\Delta) = E^*_F$.

To examine Question 1.1 we need to consider (bounded linear) operator-valued functions (defined on the unit circle) constructed by arranging the vectors in F as column vectors. In other words, in what follows we will encounter (bounded linear) operator-valued functions whose "column" vectors are L^2-functions. This approach naturally leads to the notion of (operator-valued) strong L^2-function. This notion seems to have been introduced by V. Peller [Pel, Appendix 2.3] for the purpose of defining general symbols of vectorial Hankel operators. However, Peller's book gives only the definition of a strong L^2-function, and does not describe the properties of such functions. Besides Peller's book, we have not found any other references in the literature to strong L^2-functions. In Appendix A we study strong L^2-functions (including operator-valued L^2- and L^∞-functions) and then derive some basic properties.

Let $\mathcal{B}(D,E)$ denote the set of all bounded linear operators between separable complex Hilbert spaces D and E. A *strong L^2-function* Φ is a $\mathcal{B}(D,E)$-valued function defined almost everywhere on the unit circle $\mathbb{T}$ such that $\Phi(\cdot)x \in L^2_E$ for each $x \in D$. We can easily see that every operator-valued L^p-function ($p \geq 1$) is a strong L^p-function (cf. p. 59). Following V. Peller [Pel], we write $L^2_s(\mathcal{B}(D,E))$ for the set of strong L^2-functions with values in $\mathcal{B}(D,E)$.

The set $L^2_s(\mathcal{B}(D,E))$ constitutes a nice collection of general symbols of vectorial Hankel operators (see [Pel]). Similarly, we write $H^2_s(\mathcal{B}(D,E))$ for the set of strong L^2-functions with values in $\mathcal{B}(D,E)$ such that $\Phi(\cdot)x \in H^2_E$ for each $x \in D$. Of course, $H^2_s(\mathcal{B}(D,E))$ contains all $\mathcal{B}(D,E)$-valued H^2-functions. In Appendix A, we study operator-valued Hardy classes as well as strong L^2-functions as a groundwork of this paper.

Question 1.1 is closely related to a canonical decomposition of strong L^2-functions. We first observe that if Φ is an operator-valued L^∞-function, then the kernel of the Hankel operator H_{Φ^*} is shift-invariant. Thus by the Beurling-Lax-Halmos Theorem, the kernel of the Hankel operator H_{Φ^*} is of the form $\Delta H^2_{E'}$ for some inner function Δ. If the kernel of the Hankel operator H_{Φ^*} is trivial, take $E' = \{0\}$. Of course, Δ need not be a two-sided inner function. In fact, we can show that if Φ is an operator-valued L^∞-function and Δ is a two-sided inner function, then the kernel of the Hankel operator H_{Φ^*} is $\Delta H^2_{E'}$ if and only if Φ is expressed in the form

$$\Phi = \Delta A^*, \tag{1}$$

where A is an operator-valued H^∞-function such that Δ and A are right coprime (see Lemma 2.4). The expression (1) is called the (canonical) *Douglas-Shapiro-Shields factorization* of an operator-valued L^∞-function Φ (see [DSS], [FB], [Fu2]; in particular, [Fu2] contains many important applications of the Douglas-Shapiro-Shields factorization to linear system theory).

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We recall that a meromorphic function $\varphi : \mathbb{D} \to \mathbb{C}$ is said to be *of bounded type* (or *in the Nevanlinna class*) if it is a quotient of two bounded analytic functions. A matrix function of bounded type is defined by a matrix-valued function whose entries are all of bounded type. Very recently, a systematic study on matrix-valued functions of bounded type was undertaken in the research monograph [CHL3]. It is also known that every matrix-valued L^∞-function whose adjoint is of bounded type satisfies (1) (cf. [GHR]). In fact, if we extend the notion of "bounded type" for operator-valued L^∞-functions (as we will do in Definition 2.23 for a bigger class), then we may say that the expression (1) characterizes the class of L^∞-functions whose flips are of bounded type, where the flip $\breve{\Phi}$ of Φ is defined by $\breve{\Phi}(z) := \Phi(\overline{z})$. From this viewpoint, we may ask whether there exists an appropriate decomposition corresponding to general L^∞-functions, more generally, to strong L^2-functions. An answer to the following question is the first objective of this paper,

Question 1.2. What is a canonical decomposition of strong L^2-functions ?

To establish a canonical decomposition of strong L^2-functions, we need to introduce new notions; this will be done in Section 2. First of all, we coin the notion of "complementary factor" denoted by Δ_c, of an inner function Δ with values in $\mathcal{B}(D,E)$. This notion is defined by using the kernel of Δ^*, denoted by $\ker \Delta^*$, which is defined by the set of vectors f in H^2_E such that $\Delta^* f = 0$ a.e. on $\mathbb{T}$. Moreover, the kernel of H_{Δ^*} can be represented by orthogonally adding the complementary factor Δ_c to Δ (see Lemma 2.7). We also employ a notion of "degree of non-cyclicity" on the set of all subsets (or vectors) of H^2_E, which is a complementary notion of "degree of cyclicity" due to V.I. Vasyunin and N.K. Nikolskii [VN]. The *degree of non-cyclicity*, denoted by $\mathrm{nc}(F)$, of subsets $F \subseteq H^2_E$, is defined by the number

$$\mathrm{nc}(F) := \sup_{\zeta \in \mathbb{D}} \dim \bigl\{ g(\zeta) : g \in H^2_E \ominus E^*_F \bigr\}. \tag{2}$$

Thus, in comparison with the degree of cyclicity, the degree of non-cyclicity admits ∞, which is often beneficial when trying to understand the Beurling-Lax-Halmos Theorem. Now, for a canonical decomposition of strong L^2-functions Φ, we are tempted to guess that Φ can be factored as ΔA^* (where Δ is a possibly one-sided inner function) as in the Douglas-Shapiro-Shields factorization, in which Δ is two-sided inner. But this is not the case. In fact, we can see that a canonical decomposition is actually affected by the kernel of Δ^* through some examples (see p. 30). Upon reflection, we recognize that this is not an accident. This is accomplished in Section 3.

Theorem 3.1 realizes the idea inside those examples: if Φ is a strong L^2-function with values in $\mathcal{B}(D, E)$, then Φ can be expressed in the form

$$\Phi = \Delta A^* + B, \tag{3}$$

where Δ is an inner function with values in $\mathcal{B}(E', E)$, Δ and A are right coprime, $\Delta^* B = 0$, and $\mathrm{nc}\{\Phi_+\} \leq \dim E'$ (where $\{\Phi_+\}$ denotes the set of all "column" vectors of the analytic part of Φ. Loosely defined, $\Phi_+(\cdot)x = P_+(\Phi(\cdot)x)$, where P_+ is the orthogonal projection from L^2 onto H^2; a precise definition is given on p.11. In particular, if $\dim E' < \infty$ (for instance, if $\dim E < \infty$), then the expression (3) is unique (up to a unitary constant right factor) (see Theorem 3.1, p. 32). The expression (3) will be called *a canonical decomposition* of a strong L^2-function Φ. The proof of Theorem 3.1 shows that the inner function Δ in the canonical decomposition (3) of a strong L^2-function Φ can be obtained from the equation

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$$

which is guaranteed by the Beurling-Lax-Halmos Theorem (see Corollary 2.6). In this case, the expression (3) will be called the *BLH-canonical decomposition* of Φ, recalling that Δ comes from the Beurling-Lax-Halmos Theorem. However, if $\dim E' = \infty$ (even in the case when $\dim D < \infty$), then it is possible to get another inner function Θ of a canonical decomposition (3) for the same function: in this case, $\ker H^*_{\breve{\Phi}} \neq \Theta H^2_{E''}$. Therefore the canonical decomposition of a strong L^2-function is not unique in general (see Remark 3.2). But the second assertion of Theorem 3.1 says that if the codomain of $\Phi(z)$ is finite-dimensional (in particular, if Φ is a matrix-valued L^2-function), then the canonical decomposition (3) of Φ is unique; in other words, the inner function Δ in (3) should be obtained from the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$. Thus the unique canonical decomposition (3) of matrix-valued L^2-functions is precisely the BLH-canonical decomposition.

Further, if the flip $\breve{\Phi}$ of Φ is of bounded type then B turns to be a zero function, so that the decomposition (3) reduces to the Douglas-Shapiro-Shields factorization. In fact, the Douglas-Shapiro-Shields factorization was given for L^∞-functions, but the case $B = 0$ in (3) is available for strong L^2-functions. Moreover, the notion of "bounded type" for matrix-valued functions is not appropriate for operator-valued functions, i.e., the statement "each entry of the matrix is of bounded type" does not produce a natural extension to operator-valued functions even though it has a meaning for infinite matrices (remember that we deal with operators between separable Hilbert spaces).

Thus we need to introduce an appropriate notion of "bounded type" for operator-valued functions. We will do this in Subsection 2.5. Moreover, to guarantee the statement "each entry is of bounded type," we adopt the notion of "meromorphic pseudo-continuation of bounded type" in $\mathbb{D}^e := \{z : 1 < |z| \leq \infty\}$, which coincides with the notion of "bounded type" for matrix-valued functions (cf. [Fu1]): This will be done in Subsection 2.6.

On the other hand, we recall that the *spectral multiplicity* for a bounded linear operator T acting on a separable complex Hilbert space E is defined by the number μ_T:

$$\mu_T := \inf \dim F,$$

where $F \subseteq E$, the infimum being taken over all generating subspaces F, i.e., subspaces such that $M_F \equiv \bigvee\{T^n F : n \geq 0\} = E$. In the definition of the spectral multiplicity, F may be taken as a subset rather than a subspace. In this case, we may regard μ_T as the quantity $\inf \dim \bigvee\{f : f \in F\}$ such that $M_F = E$. Unless this leads to ambiguity, we will deal with M_F for subsets $F \subseteq E$. If S_E is the shift operator on H^2_E, then it is known that $\mu_{S_E} = \dim E$. By contrast, if S_E^* is the backward shift operator on H^2_E, then S_E^* has a cyclic vector, i.e., $\mu_{S_E^*} = 1$. Moreover, the cyclic vectors of S_E^* form a dense subset of H^2_E (see [Ha4], [Ni1], [Wog]). We here observe that Question 1.1(i) is identical to the problem of finding the spectral multiplicity of the truncated backward shift operator $S_E^*|_{\mathcal{H}(\Delta)}$, i.e., the restriction of S_E^* to its invariant subspace $\mathcal{H}(\Delta)$. The second objective of this paper is to show that this problem has a deep connection with a canonical decomposition of strong L^2-functions involved with the inner function Δ.

To understand the smallest S_E^*-invariant subspace containing a subset $F \subseteq H^2_E$, we need to consider the kernels of the adjoints of unbounded Hankel operators with strong L^2-symbols involved with F. Thus we will deal with unbounded Hankel operators H_Φ with strong L^2-symbols Φ. However, the adjoint of the unbounded Hankel operator need not be a Hankel operator. But if Φ is an L^∞-function then $H_{\Phi^*} = H^*_{\breve{\Phi}}$, where $\breve{\Phi}$ is the flip of Φ. Thus for a bounded symbol Φ, we may use the notations H_{Φ^*} and $H^*_{\breve{\Phi}}$ interchangeably. By contrast, for a strong L^2-function Φ, H_{Φ^*} may not be equal to $H^*_{\breve{\Phi}}$ even though Φ^* is a strong L^2-function. In particular, the kernel of an unbounded Hankel operator H_{Φ^*} is likely to be trivial because it is defined on the dense subset of polynomials. From this viewpoint, to avoid potential technical issues in our arguments, we will deal with the operator $H^*_{\breve{\Phi}}$ in place of H_{Φ^*}. In spite of this, and since the kernel of the adjoint of an unbounded operator is always closed, we can show that via the Beurling-Lax-Halmos Theorem, the kernel of $H^*_{\breve{\Phi}}$ with strong L^2-symbol Φ is still of the form $\Delta H^2_{E'}$ (see Corollary 2.6).

We now consider several questions, which are of independent interest. This will be done in Section 2. The next question arises naturally from the Beurling-Lax-Halmos Theorem.

Question 1.3. Since the kernel of the Hankel operator $H^*_{\breve{\Phi}}$ is of the form $\Theta H^2_{E'}$, which property of Φ determines the dimension of the space E' ? In particular, if Φ is an $n \times m$ matrix-valued L^2-function and $\dim E' = r$, which property of Φ determines the number r ?

To answer Question 1.3, we employ the notion of degree of non-cyclicity (2). Indeed, we can show that if the kernel of the adjoint of the Hankel operator $H_{\breve{\Phi}}$ is $\Theta H^2_{E'}$ for some inner function Θ, then the dimension of E' can be computed by the degree of non-cyclicity of $\{\Phi_+\}$ (see Theorem 2.13).

When Δ is an inner function, we may ask when it is possible to complement Δ to a two-sided inner function by aid of an inner function Ω; in other words, when is $[\Delta, \Omega]$ a two-sided inner function, where $[\Delta(\cdot), \Omega(\cdot)]$ is understood as an 1×2 operator matrix defined on the unit circle $\mathbb{T}$? (It turns out that this question can be answered by using the Complementing Lemma; see [VN] or [Ni1]). The following question refers to more general cases.

Question 1.4. If Δ is an $n \times r$ inner matrix function, which condition on Δ allows us to complement Δ to an $n \times (r+q)$ inner matrix function using an $n \times q$ inner matrix function ?

An answer to Question 1.4 is also subject to the degree of non-cyclicity of $\{\Delta\}$ (see Theorem 2.21).

By the Beurling-Lax-Halmos Theorem, we saw that the kernel of the adjoint of a Hankel operator with a strong L^2-symbol is of the form $\Delta H^2_{E'}$ for some inner function Δ. In view of its converse, we may ask:

Question 1.5. Is every shift-invariant subspace $\Delta H^2_{E'}$ represented by the kernel of $H^*_{\breve{\Phi}}$ with some strong L^2-symbol Φ with values in $\mathcal{B}(D, E)$?

Question 1.5 asks whether a strong L^2-solution Φ always exists for the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$ for a given inner function Δ. In Theorem 4.1 we give an affirmative answer to Question 1.5. The matrix-valued version of this result is as follows (see Corollary 4.2): for a given $n \times r$ inner matrix function Δ, there always exists a solution $\Phi \in L^\infty_{M_{n\times m}}$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$, for some $m \le r+1$. In view of this, it is reasonable to ask whether such a solution $\Phi \in L^2_{M_{n\times m}}$ exists for each $m = 1, 2, \cdots$. But the answer to this question is negative (see Remark 4.4).

It is then natural to ask how to determine a possible dimension of D for which there exists a strong L^2-solution Φ (with values in $\mathcal{B}(D, E)$) of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$. In fact, we would like to ask what is the infimum of $\dim D$ that guarantees the existence of a strong L^2-solution Φ. To find a way to determine such an infimum, we introduce the notion of "Beurling degree" for an inner function. We do this by employing the canonical decomposition of a strong L^2-function induced by the given inner function: if Δ is an inner function with values in $\mathcal{B}(E', E)$, then the *Beurling degree*, denoted by $\deg_B(\Delta)$, of Δ is defined by the infimum of the dimension of the nonzero space D for which there exists a pair (A, B) such that $\Phi \equiv \Delta A^* + B$ is a canonical decomposition of a strong L^2-function Φ with values in $\mathcal{B}(D, E)$ (Definition 4.5).

We now recall that the Model Theorem ([Ni1], [SFBK]) states that if a bounded linear operator T acting on a Hilbert space $\mathcal{H}$ (in symbols, $T \in \mathcal{B}(\mathcal{H})$) is a contraction (i.e., $||T|| \le 1$) satisfying

$$\lim_{n\to\infty} T^n x = 0 \quad \text{for each } x \in \mathcal{H}, \tag{4}$$

then T is unitarily equivalent to a truncated backward shift $S^*_E|_{\mathcal{H}(\Delta)}$ for some inner function Δ with values in $\mathcal{B}(E', E)$, where $E = \mathrm{cl}\,\mathrm{ran}(I - T^*T)$. In this case, $S^*_E|_{\mathcal{H}(\Delta)}$ is called the the *model operator* of T and Δ is called the *characteristic function* of T. We often write $T \in C_{0\boldsymbol{\cdot}}$ for a contraction $T \in \mathcal{B}(\mathcal{H})$ satisfying the condition (4).

We can now prove that if Δ is the characteristic function of the model operator T with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$ (in particular, when Δ is an inner matrix function), then the spectral multiplicity of the model operator is equal to the Beurling degree of Δ. Equivalently, given an inner function Δ with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$, let $T := S_E^*|_{\mathcal{H}(\Delta)}$. Then

$$\mu_T = \deg_B(\Delta) \tag{5}$$

(see Theorem 4.6). The equality (5) is the second objective of this paper. It is somewhat surprising that the spectral multiplicity of the model operator can be computed by a function-theoretic property of the corresponding characteristic function. As a result, we give an answer to Question 1.1 (see Corollary 4.9).

The third objective of this paper is to consider the case of $\mu_T = 1$, i.e., when the operator T has a cyclic vector. In general, if $T \in \mathcal{B}(\mathcal{H})$ is such that $\mu_T = 1$, then T is said to be *multiplicity-free*. To avoid confusion, we regard T to be multiplicity-free if the operator T acts on the zero space. Thus we are interested in the following question on the characteristic function Δ of T.

Question 1.6. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$. For which inner function Δ does it follow that T is multiplicity-free?

To get an answer to Question 1.6, we consider the notion of "characteristic scalar" inner function, which is a generalization of the case of two-sided inner matrix function (and we often call it *square inner* matrix function) (cf. [Hel], [SFBK], [CHL3]). This will be done in Subsection 5.1. Let $\widetilde{\Delta}(z) := \Delta(\overline{z})^*$. Then we can get an answer to Question 1.6, as follows:

> If $T := S_E^*|_{\mathcal{H}(\Delta)}$, where Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$ and $\widetilde{\Delta}$ is an outer function, then T is multiplicity-free (see Theorem 5.15).

The organization of this paper is as follows. The main theorems of this paper are Theorem 3.1 (a canonical decomposition of strong L^2-functions), Theorem 4.6 (the Beurling degree and the spectral multiplicity), Theorem 5.15 (multiplicity-free model operators). To prove those theorems, we need to consider several questions emerging from the Beurling-Lax-Halmos Theorem. We also consider several auxiliary lemmas, and new notions of complementary factors of inner functions, the degree of non-cyclicity, bounded type strong L^2-functions, and the Beurling degree of an inner function.

In Subsection 2.1 we give the notations and the basic definitions. In Subsections 2.2-2.4 we introduce notions of complementary factors of inner functions and the degree of non-cyclicity, and then give answers to Question 1.3 and Question 1.4. In Subsection 2.5 we introduce the notion of "bounded type" strong L^2-functions, which correspond to the functions whose entries are of bounded type in the matrix-valued case.

In Section 3 we establish a canonical decomposition of a strong L^2-functions Φ, which reduces to the Douglas-Shapiro-Shields factorization of Φ if $\breve{\Phi}$ is of bounded type. In Section 4 we give an answer to Question 1.5 and then establish a connection between the spectral multiplicity of the model operator and the Beurling degree of the corresponding characteristic function.

In Section 5 we consider the spectral multiplicity of model operators by using the notion of meromorphic pseudo-continuation of bounded type in the complement of the closed unit disk and then give an answer to Question 1.6. In Section 6 we address some unsolved problems.

In Appendix A we study operator-valued strong L^2-functions and then prove some properties which we have not been able to find in the literature, but we need to use in this paper. In Appendix B, we consider a reduction to the case of C_0-contractions for the spectral multiplicity of model operators. In Appendix C, by using the preceding results, we analyze the left and right coprime-ness, the model operator and an interpolation problem for operator-valued functions.

2 Preliminaries and auxiliary lemmas

In this section we provide notations, notions and auxiliary lemmas which will be used in this paper. In the course, we will coin the new notions of complementary factor of an inner function, degree of non-cyclicity, strong L^2-functions of bounded type, and meromorphic pseudo-continuation of bounded type for operator-valued functions.

2.1 Basic notions

We write $\mathbb{D}$ for the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ for the unit circle in $\mathbb{C}$. To avoid a confusion, we will write z for points on $\mathbb{T}$ and ζ for points in $\mathbb{C} \setminus \mathbb{T}$. For $\phi \in L^2$, write

$$\breve{\phi}(z) := \phi(\overline{z}) \quad \text{and} \quad \widetilde{\phi}(z) := \overline{\phi(\overline{z})}.$$

For $\phi \in L^2$, write

$$\phi_+ := P_+\phi \quad \text{and} \quad \breve{\phi}_- := P_-\phi,$$

where P_+ and P_- are the orthogonal projections from L^2 onto H^2 and $L^2 \ominus H^2$, respectively. Thus, we may write $\phi = \breve{\phi}_- + \phi_+$.

Throughout the paper, we assume that

$$X \text{ and } Y \text{ are complex Banach spaces;}$$
$$D \text{ and } E \text{ are separable complex Hilbert spaces.}$$

We write $\mathcal{B}(X,Y)$ for the set of all bounded linear operators from X to Y and abbreviate $\mathcal{B}(X,X)$ to $\mathcal{B}(X)$. For a complex Banach space X, we write X^* for its dual. We write $M_{n\times m}$ for the set of $n \times m$ complex matrices, and abbreviate $M_{n\times n}$ to M_n. We also write g.c.d.$(\cdot)$ and l.c.m.$(\cdot)$ denote the greatest common inner divisor and the least common inner multiple, respectively, while left-g.c.d.$(\cdot)$ and left-l.c.m.$(\cdot)$ denote the greatest common left inner divisor and the least common left inner multiple, respectively.

If $A : D \to E$ is a linear operator whose domain is a subspace of D, then A is also a linear operator from the closure of the domain of A into E. So we will only consider those A such that the domain of A is dense in D. Such an operator A is said to be *densely defined.* If $A : D \to E$ is densely defined, we write $\operatorname{dom} A$, $\ker A$, and $\operatorname{ran} A$ for the domain, the kernel, and the range of A, respectively. If $A : D \to E$ is densely defined, write

$$\operatorname{dom} A^* = \big\{e \in E : d \mapsto \langle Ad, e\rangle \text{ is a bounded linear functional on } \operatorname{dom} A\big\}.$$

If $e \in \operatorname{dom} A^*$, then there exists a unique $f \in E$ such that $\langle Ad, e\rangle = \langle d, f\rangle$ for all $d \in \operatorname{dom} A$. Denote this unique vector f by $f \equiv A^*e$. Thus $\langle Ad, e\rangle = \langle d, A^*e\rangle$ for all $d \in \operatorname{dom} A$ and $e \in \operatorname{dom} A^*$. We call A^* the adjoint of A. It is well known from unbounded operator theory (cf. [Gol], [Con]) that if A is densely defined, then $\ker A^* = (\operatorname{ran} A)^\perp$, so that $\ker A^*$ is closed even though $\ker A$ may not be closed.

We recall ([Abr], [Co2], [GHR], [Ni1]) that a meromorphic function $\phi : \mathbb{D} \to \mathbb{C}$ is said to be of *bounded type* (or in the Nevanlinna class $\mathcal{N}$) if there are functions $\psi_1, \psi_2 \in H^\infty$ such that

$$\phi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

It is well known that ϕ is of bounded type if and only if $\phi = \frac{\psi_1}{\psi_2}$ for some $\psi_i \in H^p$ $(p > 0,\ i = 1, 2)$. If $\psi_2 = \psi^i\psi^e$ is the inner-outer factorization of ψ_2, then $\phi = \overline{\psi^i}\frac{\psi_1}{\psi^e}$. Thus if $\phi \in L^2$ is of bounded type, then ϕ can be written as

$$\phi = \overline{\theta} a,$$

where θ is inner, $a \in H^2$ and θ and a are coprime.

Write $\mathbb{D}^e := \{z : 1 < |z| \le \infty\}$. For a function $g : \mathbb{D}^e \to \mathbb{C}$, define a function $g_{\mathbb{D}} : \mathbb{D} \to \mathbb{C}$ by

$$g_{\mathbb{D}}(\zeta) := \overline{g(1/\overline{\zeta})} \quad (\zeta \in \mathbb{D}).$$

For a function $g : \mathbb{D}^e \to \mathbb{C}$, we say that g belongs to $H^p(\mathbb{D}^e)$ if $g_{\mathbb{D}} \in H^p$ $(1 \le p \le \infty)$. A function $g : \mathbb{D}^e \to \mathbb{C}$ is said to be of *bounded type* if $g_{\mathbb{D}}$ is of bounded type. If $f \in H^2$, then the function $\hat{f}$ defined in $\mathbb{D}^e$ is called a *pseudo-continuation* of f if $\hat{f}$ is a function of bounded type and $\hat{f}(z) = f(z)$ for almost all $z \in \mathbb{T}$ (cf. [BoB], [Ni1], [Sha]). Then we can easily show that $\breve{f}$ is of bounded type if and only if f has a pseudo-continuation $\hat{f}$. In this case, $\hat{f}_{\mathbb{D}}(z) = \overline{f(z)}$ for almost all $z \in \mathbb{T}$. In particular,

$$\phi \equiv \breve{\phi}_- + \phi_+ \in L^2 \text{ is of bounded type} \iff \phi_- \text{ has a pseudo-continuation.} \tag{6}$$

For a complex Banach space X and $1 \le p \le \infty$, let L^p_X be the space of X-valued L^p-functions and H^p_X be the Hardy space of X-valued H^p-functions. For a detailed explanation for L^p_X and H^p_X, see Appendix A.

To examine Question 1.1, we need to consider operator-valued functions defined on the unit circle constructed by arranging the vectors in F as their column vectors. Using this viewpoint, we will consider operator-valued functions whose "column" vectors are L^2-functions. Note that (bounded linear) operators between separable Hilbert spaces may be represented as infinite matrices, so that column vectors of operators are well justified. This viewpoint leads us to define (operator-valued) strong L^2-functions.

For $1 \le p < \infty$, we define the class $L^p_s(\mathcal{B}(D, E)) \equiv L^p_s(\mathbb{T}, \mathcal{B}(D, E))$ as the set of all (WOT) measurable $\mathcal{B}(D, E)$-valued functions Φ on $\mathbb{T}$ such that $\Phi(\cdot)x \in L^p_E$. A function $\Phi \in L^p_s(\mathcal{B}(D, E))$ is called a *strong L^p-function.*

If $\Phi \in L^1_s(\mathcal{B}(D, E))$ and $x \in D$, then $\Phi(\cdot)x \in L^1_E$. Thus the n-th Fourier coefficient $\widehat{\Phi(\cdot)x}(n)$ of $\Phi(\cdot)x$ is given by

$$\widehat{\Phi(\cdot)x}(n) = \int_{\mathbb{T}} \overline{z}^n \Phi(z)x \, dm(z).$$

We now define the n-th Fourier coefficient of $\Phi \in L^1_s(\mathcal{B}(D,E))$, denoted by $\widehat{\Phi}(n)$, by

$$\widehat{\Phi}(n)x := \widehat{\Phi(\cdot)x}(n) \quad (n \in \mathbb{Z},\ x \in D).$$

We define

$$H^2_s(\mathcal{B}(D,E)) \equiv H^2_s(\mathbb{T}, \mathcal{B}(D,E)) := \big\{\Phi \in L^2_s(\mathcal{B}(D,E)) :\ \widehat{\Phi}(n) = 0 \text{ for } n < 0\big\},$$

or equivalently, $H^2_s(\mathcal{B}(D,E))$ is the set of all (WOT) measurable functions Φ on $\mathbb{T}$ such that $\Phi(\cdot)x \in H^2_E$ for each $x \in D$. The terminology of a "strong H^2-function" is reserved for the operator-valued functions on the unit disk $\mathbb{D}$, following to N.K. Nikolskii [Ni1]: A function $\Phi : \mathbb{D} \to \mathcal{B}(D,E)$ is called a *strong H^2-function* if $\Phi(\cdot)x \in H^2(\mathbb{D}, E)$ for each $x \in D$. Let $L^\infty(\mathcal{B}(D,E))$ be the space of all bounded (WOT) measurable $\mathcal{B}(D,E)$-valued functions on $\mathbb{T}$ and let

$$H^\infty(\mathcal{B}(D,E)) := \big\{\Phi \in L^\infty(\mathcal{B}(D,E)) :\ \widehat{\Phi}(n) = 0 \text{ for } n < 0\big\}.$$

In Appendix A, we provide some properties of strong L^2-functions, $H^2_s(\mathcal{B}(D,E))$-functions, strong H^2-functions, and connections between them in addition with $H^2_{\mathcal{B}(D,E))}$-functions, which we have not been able to find in the literature.

A function $\Delta \in H^\infty(\mathcal{B}(D,E))$ is called an *inner* function with values in $\mathcal{B}(D,E)$ if $\Delta(z)$ is an isometric operator from D into E for almost all $z \in \mathbb{T}$, i.e., $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$. Δ is called a *two-sided inner* function if $\Delta\Delta^* = I_E$ a.e. on $\mathbb{T}$ and $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$. If Δ is an inner function with values in $\mathcal{B}(D,E)$, we may assume that D is a subspace of E, and if further Δ is two-sided inner then we may assume that $D = E$.

We write $\mathcal{P}_D$ for the set of all polynomials with values in D, i.e., $p(z) = \sum_{k=0}^n \widehat{p}(k)z^k$, where $\widehat{p}(k) \in D$. If $F \in H^2_s(B(D,E))$, then the function Fp belongs to H^2_E for all $p \in \mathcal{P}_D$. A function $F \in H^2_s(B(D,E))$ is called *outer* if $\operatorname{cl} F\mathcal{P}_D = H^2_E$. We then have an analogue of the scalar factorization theorem:

Inner-Outer Factorization for H^2_s-functions. If $F \in H^2_s(B(D,E))$, then F can be expressed in the form

$$F = F^i F^e,$$

where F^e is an outer function with values in $\mathcal{B}(D,E')$ and F^i is an inner function with values in $\mathcal{B}(E',E)$ for some subspace E' of E.

The proof of the above inner-outer factorization for H^2_s-functions is same as that for strong H^2-function (cf. [Ni1, Corollary I.9]).

For a function $\Phi : \mathbb{T} \to \mathcal{B}(D,E)$, write

$$\breve{\Phi}(z) := \Phi(\overline{z}), \quad \widetilde{\Phi} := \breve{\Phi}^*.$$

We call $\breve{\Phi}$ the *flip* of Φ. For $\Phi \in L^2_s(\mathcal{B}(D,E))$, we denote by $\breve{\Phi}_- \equiv \mathbb{P}_-\Phi$ and $\Phi_+ \equiv \mathbb{P}_+\Phi$ the functions

$$\begin{aligned} ((\mathbb{P}_-\Phi)(\cdot))x &:= P_-(\Phi(\cdot)x) \quad \text{a.e. on } \mathbb{T} \quad (x \in D); \\ ((\mathbb{P}_+\Phi)(\cdot))x &:= P_+(\Phi(\cdot)x) \quad \text{a.e. on } \mathbb{T} \quad (x \in D), \end{aligned}$$

where P_+ and P_- are the orthogonal projections from L^2_E onto H^2_E and $L^2_E \ominus H^2_E$, respectively. Then we may write $\Phi \equiv \breve{\Phi}_- + \Phi_+$. Note that if $\Phi \in L^2_s(\mathcal{B}(D,E))$, then $\Phi_+,\ \Phi_- \in H^2_s(\mathcal{B}(D,E))$.

In the sequel, we will often encounter the adjoints of inner matrix functions. If Δ is a two-sided inner matrix function, it is easy to show that Δ^* is of bounded type, i.e., all entries of Δ^* are of bounded type (see p. 4). We may predict that if Δ is an inner matrix function then Δ^* is of bounded type. However the following example shows that this is not the case.

Example 2.1. Let $h(z) := e^{\frac{1}{z-3}}$. Then $h \in H^\infty$ and $\overline{h}$ is not of bounded type. Let

$$f(z) := \frac{h(z)}{\sqrt{2}||h||_\infty}.$$

Clearly, $\overline{f}$ is not of bounded type. Let $h_1(z) := \sqrt{1-|f(z)|^2}$. Then $h_1 \in L^\infty$ and $|h_1| \geq \frac{1}{\sqrt{2}}$. Thus there exists an outer function g such that $|h_1| = |g|$ a.e. on $\mathbb{T}$ (see [Do1, Corollary 6.25]). Put

$$\Delta := \begin{bmatrix} f \\ g \end{bmatrix} \quad (f, g \in H^\infty).$$

Then $\Delta^*\Delta = |f|^2 + |g|^2 = |f|^2 + |h_1|^2 = 1$ a.e. on $\mathbb{T}$, which implies that Δ is an inner function. Note that Δ^* is not necessarily of bounded type.

For a function $\Phi \in H^2_s(\mathcal{B}(D,E))$, we say that an inner function Δ with values in $\mathcal{B}(D',E)$ is a *left inner divisor* of Φ if $\Phi = \Delta A$ for $A \in H^2_s(\mathcal{B}(D,D'))$. For $\Phi \in H^2_s(\mathcal{B}(D_1,E))$ and $\Psi \in H^2_s(\mathcal{B}(D_2,E))$, we say that Φ and Ψ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary operator. Also, we say that Φ and Ψ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. The determination of left or right coprime-ness seems to be a somewhat delicate problem. For matrix-valued functions, left and right coprime-ness was developed in [CHKL], [CHL1], [CHL2], [CHL3] and [FF].

Lemma 2.2. If Θ is a two-sided inner function, then any left inner divisor of Θ is two-sided inner.

Proof. Straightforward. □

For an inner function $\Delta \in H^\infty(\mathcal{B}(E',E))$, $\mathcal{H}(\Delta)$ denotes the orthogonal complement of the subspace $\Delta H^2_{E'}$ in H^2_E, i.e.,

$$\mathcal{H}(\Delta) := H^2_E \ominus \Delta H^2_{E'}.$$

The space $\mathcal{H}(\Delta)$ is often called a *model space* or a *de Branges-Rovnyak space* (cf. [dBR], [Sa2], [SFBK]).

2.2 The Beurling-Lax-Halmos Theorem

We first review a few essential facts for (vectorial) Toeplitz operators and (vectorial) Hankel operators, and for that we will use [BS], [Do1], [Do2], [MR], [Ni1], [Ni2], and [Pel] for general references. For $\Phi \in L^2_s(\mathcal{B}(D,E))$, the Hankel operator $H_\Phi : H^2_D \to H^2_E$ is a densely defined operator defined by

$$H_\Phi p := JP_-(\Phi p) \quad (p \in \mathcal{P}_D),$$

where J denotes the unitary operator from L^2_E to L^2_E given by $(Jg)(z) := \overline{z}g(\overline{z})$ for $g \in L^2_E$. Also a Toeplitz operator $T_\Phi : H^2_D \to H^2_E$ is a densely defined operator defined by

$$T_\Phi p := P_+(\Phi p) \quad (p \in \mathcal{P}_D).$$

The following lemma gives a characterization of bounded Hankel operators on H^2_D.

Lemma 2.3. [Pel, Theorem 2.2] Let $\Phi \in L^2_s(B(D,E))$. Then H_Φ is extended to a bounded operator on H^2_D if and only if there exists a function $\Psi \in L^\infty(\mathcal{B}(D,E))$ such that $\widehat{\Psi}(n) = \widehat{\Phi}(n)$ for $n < 0$ and

$$||H_\Phi|| = \text{dist}_{L^\infty}(\Psi, H^\infty(\mathcal{B}(D,E)).$$

The following basic properties can be easily derived: If D, E, and D' are separable complex Hilbert spaces and $\Phi \in L^\infty(\mathcal{B}(D,E))$, then

$$T^*_\Phi = T_{\Phi^*},\ H^*_\Phi = H_{\widetilde{\Phi}}; \tag{7}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi} \quad \text{if } \Psi \in H^\infty(\mathcal{B}(D',D)); \tag{8}$$

$$H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad \text{if } \Psi \in H^\infty(\mathcal{B}(E,D')). \tag{9}$$

A *shift* operator S_E on H^2_E is defined by

$$(S_E f)(z) := zf(z) \quad \text{for each } f \in H^2_E.$$

Thus we may write $S_E = T_{zI_E}$.

The following theorem is a fundamental result in modern operator theory.

The Beurling-Lax-Halmos Theorem. [Beu], [Lax], [Ha1], [FF], [Pel] A subspace M of H^2_E is invariant for the shift operator S_E on H^2_E if and only if

$$M = \Delta H^2_{E'},$$

where E' is a subspace of E and Δ is an inner function with values in $\mathcal{B}(E',E)$. Furthermore, Δ is unique up to a unitary constant right factor, i.e., if $M = \Theta H^2_{E''}$, where Θ is an inner function with values in $\mathcal{B}(E'',E)$, then $\Delta = \Theta V$, where V is a unitary operator from E' onto E''.

As customarily done, we say that two inner functions $A, B \in H^\infty(\mathcal{B}(D,E))$ are *equal* if they are equal up to a unitary constant right factor. If $\Phi \in L^\infty(\mathcal{B}(D,E))$, then by (8) and (9),

$$H_{\Phi^*} S_E = S_E^* H_{\Phi^*},$$

which implies that the kernel of the Hankel operator H_{Φ^*} is an invariant subspace of the shift operator S_E on H^2_E. Thus, by the Beurling-Lax-Halmos Theorem,

$$\ker H_{\Phi^*} = \Delta H^2_{E'}$$

for some inner function Δ with values in $\mathcal{B}(E', E)$. We note that E' may be the zero space and Δ need not be two-sided inner.

However, we have:

Lemma 2.4. If $\Phi \in L^\infty(\mathcal{B}(D,E))$ and Δ is a two-sided inner function with values in $\mathcal{B}(E)$, then the following are equivalent:

(a) $\ker H_{\Phi^*} = \Delta H^2_E$;

(b) $\Phi = \Delta A^*$, where $A \in H^\infty(\mathcal{B}(E,D))$ is such that Δ and A are right coprime;

(c) $\bigvee_{n=1}^\infty S_E^{*n} \mathbb{P}_+ \Phi D = \mathcal{H}(\Delta)$.

Proof. Let $\Phi \in L^\infty(\mathcal{B}(D,E))$ and Δ be a two-sided inner function with values in $\mathcal{B}(E)$.

(a) $\Rightarrow$ (b): Suppose $\ker H_{\Phi^*} = \Delta H^2_E$. If we put $A := \Phi^* \Delta$ then, by Lemma A.12 (see Appendix A), $A \in H^\infty(\mathcal{B}(E,D))$ and $\Phi = \Delta A^*$. We now claim that Δ and A are right coprime. To see this, suppose Ω is a common left inner divisor, with values in $\mathcal{B}(E', E)$, of $\widetilde{\Delta}$ and $\widetilde{A}$. Then we may write $\widetilde{\Delta} = \Omega \widetilde{\Delta}_1$ and $\widetilde{A} = \Omega \widetilde{A}_1$, where $\widetilde{\Delta}_1 \in H^\infty(\mathcal{B}(E,E'))$ and $\widetilde{A}_1 \in H^\infty(\mathcal{B}(D,E'))$. Since Δ is two-sided inner, it follows from Lemma 2.2 and Lemma A.12 (see Appendix A) that Ω and Δ_1 are two-sided inner. Since $\Phi = \Delta_1 A_1^*$, we have

$$\Delta_1 H^2_{E'} \subseteq \ker H_{\Phi^*} = \Delta H^2_E = \Delta_1 \widetilde{\Omega} H^2_E,$$

which implies $H^2_{E'} = \widetilde{\Omega} H^2_E$. Thus by the Beurling-Lax-Halmos Theorem, $\widetilde{\Omega}$ is a unitary constant and so is Ω. Therefore, Δ and A are right coprime.

(b) $\Rightarrow$ (a): Suppose (b) holds. Clearly, $\Delta H^2_E \subseteq \ker H_{\Phi^*}$. By the Beurling-Lax-Halmos Theorem, $\ker H_{\Phi^*} = \Theta H^2_{E'}$ for some inner function Θ, so that $\Delta H^2_E \subseteq \Theta H^2_{E'}$. Thus Θ is a left inner divisor of Δ (cf. [FF], [Pel]) so that, by Lemma 2.2, we may write $\Delta = \Theta \Delta_0$ for some two-sided inner function Δ_0 with values in $\mathcal{B}(E, E')$. Put $G := \Phi^* \Theta \in H^\infty(\mathcal{B}(E', D))$. Then $G = A \Delta_0^*$, and hence, $\widetilde{A} = \widetilde{\Delta}_0 \widetilde{G}$. But since Δ and A are right coprime, $\widetilde{\Delta}_0$ is a unitary operator, and so is Δ_0. Therefore $\ker H_{\Phi^*} = \Delta H^2_E$, which proves (a).

(b) $\Leftrightarrow$ (c): See the proof of [FB, Theorem 4.7.1]. $\square$

We recall that the factorization in Lemma 2.4(b) is called the (*canonical*) *Douglas-Shapiro-Shields factorization* of $\Phi \in L^\infty(\mathcal{B}(D,E))$ (see [DSS], [FB], [Fu2]). Consequently, Lemma 2.4 may be rephrased as: If $\Phi \in L^\infty(\mathcal{B}(D,E))$, then the following are equivalent:

(a) Φ admits a Douglas-Shapiro-Shields factorization;

(b) $\ker H_{\Phi^*} = \Delta H^2_E$ for some two-sided inner function $\Delta \in H^\infty(\mathcal{B}(E))$.

The following lemma will be frequently used in the sequel.

Complementing Lemma. [Ni1, p. 49, p. 53] Let $\Psi \in H^\infty(\mathcal{B}(E', E))$ with $E' \subseteq E$ and $\dim E' < \infty$, and let θ be a scalar inner function. Then the following statements are equivalent:

(a) There exists a function G in $H^\infty(\mathcal{B}(E, E'))$ such that $G\Psi = \theta I_{E'}$;

(b) There exist functions Φ and Ω in $H^\infty(\mathcal{B}(E))$ with $\Phi|_{E'} = \Psi$, $\Phi|_{(E\ominus E')}$ being an inner function such that $\Omega\Phi = \Phi\Omega = \theta I_E$.

In addition, if dim $E < \infty$, then (a) and (b) are equivalent to the following statement:

(c) $\text{ess inf}_{z\in\mathbb{T}} \min\big\{||\Psi(z)x|| : ||x|| = 1\big\} > 0$.

We recall that if Φ is a strong H^2-function with values in $\mathcal{B}(D, E)$, with $\dim E < \infty$, the *local rank* of Φ is defined by (cf. [Ni1])

$$\text{Rank}\,\Phi := \max_{\zeta\in\mathbb{D}} \text{rank}\,\Phi(\zeta),$$

where $\text{rank}\,\Phi(\zeta) := \dim \Phi(\zeta)(D)$.

As we have remarked in the Introduction, if Φ is a strong L^2-function with values in $\mathcal{B}(D, E)$, then $H^*_{\breve{\Phi}}$ need not be a Hankel operator. Of course, if $\Phi \in L^\infty(\mathcal{B}(D, E))$, then by (7), $H^*_{\breve{\Phi}} = H_{\widetilde{\breve{\Phi}}} = H_{\Phi^*}$. By contrast, for a strong L^2-function Φ with values in $\mathcal{B}(D, E)$, $H^*_{\breve{\Phi}} \neq H_{\Phi^*}$ in general even though Φ^* is also a strong L^2-function. We note that if Φ^* is a strong L^2-function with values in $\mathcal{B}(E, D)$, then $\ker H_{\Phi^*}$ is possibly trivial because H_{Φ^*} is defined in the dense subset of polynomials in H^2_E. Thus it is much better to deal with $H^*_{\breve{\Phi}}$ in place of H_{Φ^*}. Even though $H^*_{\breve{\Phi}}$ need not be a Hankel operator, we can show that the kernel of $H^*_{\breve{\Phi}}$ is still of the form $\Delta H^2_{D'}$ for some inner function Δ. To see this, we observe:

Lemma 2.5. Let Φ be a strong L^2-function with values in $\mathcal{B}(D, E)$. Then

$$\ker H^*_{\breve{\Phi}} = \Big\{ f \in H^2_E : \big\langle \Phi(z)x,\ z^n f(z)\big\rangle_{L^2_E} = 0 \quad \text{for all } x \in D \text{ and } n = 1, 2, 3, \cdots \Big\}.$$

Proof. Observe that

$$\begin{aligned} f \in \ker H^*_{\breve{\Phi}} &\Longleftrightarrow \big\langle H_{\breve{\Phi}}p,\ f\big\rangle_{L^2_E} = 0 \quad \text{for all } p \in \mathcal{P}_D \\ &\Longleftrightarrow \big\langle \breve{\Phi}(z)p(z),\ (Jf)(z)\big\rangle_{L^2_E} = 0 \quad \text{for all } p \in \mathcal{P}_D \\ &\Longleftrightarrow \int_{\mathbb{T}} \big\langle \Phi(\overline{z})xz^k,\ \overline{z}f(\overline{z})\big\rangle_E dm(z) = 0 \quad \text{for all } x \in D \text{ and } k = 0, 1, 2, \cdots \\ &\Longleftrightarrow \int_{\mathbb{T}} \big\langle \Phi(z)x,\ z^n f(z)\big\rangle_E dm(z) = 0 \quad \text{for all } x \in D \text{ and } n = 1, 2, 3, \cdots, \end{aligned}$$

which gives the result. $\square$

We then have:

Lemma 2.6. If Φ is a strong L^2-function with values in $\mathcal{B}(D,E)$, then

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}, \tag{10}$$

where E' is a subspace of E and Δ is an inner function with values in $\mathcal{B}(E',E)$.

Proof. By Lemma 2.5, if $f \in \ker H^*_{\breve{\Phi}}$, then $zf \in \ker H^*_{\breve{\Phi}}$. Since $\ker H^*_{\breve{\Phi}}$ is always closed, it follows that $\ker H^*_{\breve{\Phi}}$ is an invariant subspace for S_E. Thus, by the Beurling-Lax-Halmos Theorem, there exists an inner function Δ with values in $\mathcal{B}(E',E)$ such that $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$ for a subspace E' of E. □

2.3 Complementary factors of inner functions

For $\Phi \in L^\infty(\mathcal{B}(D,E))$, we symbolically define the kernel of Φ by

$$\ker \Phi := \{f \in H^2_D : \Phi(z)f(z) = 0 \text{ for almost all } z \in \mathbb{T}\}.$$

Note that the kernel of Φ consists of functions in H^2_D, but not in L^2_D, such that $\Phi f = 0$ a.e. on $\mathbb{T}$. Since $\ker \Phi$ is an invariant subspace for S_D, it follows from the Beurling-Lax-Halmos Theorem that $\ker \Phi = \Omega H^2_{D'}$, for some inner function $\Omega \in H^\infty(D',D)$.

We now recall a notion from classical Banach space theory, about regarding a vector as an operator acting on the scalars. This notion is important as motivation for the study of strong L^2-functions. Let E be a separable complex Hilbert space. For a function $f : \mathbb{T} \to E$, define $[f] : \mathbb{T} \to \mathcal{B}(\mathbb{C}, E)$ by

$$[f](z)\alpha := \alpha f(z) \quad (\alpha \in \mathbb{C}) \tag{11}$$

(see Appendix A). Let Δ be an inner function with values in $\mathcal{B}(D,E)$. If $g \in \ker \Delta^*$, then $g \in H^2_E$ so that, by Lemma A.7, $[g] \in H^2_s(\mathcal{B}(\mathbb{C},E))$. Write

$$[g] = [g]^i[g]^e \quad \text{(inner-outer factorization)},$$

where $[g]^e$ is an outer function with values in $\mathcal{B}(\mathbb{C},E')$ and $[g]^i$ is an inner function with values in $\mathcal{B}(E',E)$ for some subspace E' of E. If $g \neq 0$, then $[g]^e$ is a nonzero outer function, so that $E' = \mathbb{C}$. Thus, $[g]^i \in H^\infty(\mathcal{B}(\mathbb{C},E))$. If instead $g = 0$, then $E' = \{0\}$. Therefore, in this case, $[g]^i \in H^\infty(\mathcal{B}(\{0\},E))$.

We then have:

Lemma 2.7. Let Δ be an inner function with values in $\mathcal{B}(D,E)$. Then we may write

$$\ker \Delta^* = \Omega H^2_{D'} \tag{12}$$

for some inner function Ω with values in $B(D',E)$. Put

$$\Delta_c := \text{left-g.c.d.}\{\, [g]^i : g \in \ker \Delta^*\}. \tag{13}$$

Then we have

(a) $\Omega = \Delta_c$;

(b) $[\Delta, \Delta_c]$ is an inner function with values in $\mathcal{B}(D \oplus D', E)$;

(c) $\ker H_{\Delta^*} = [\Delta, \Delta_c] H^2_{D \oplus D'} \equiv \Delta H^2_D \bigoplus \Delta_c H^2_{D'}$,

where $[\Delta, \Delta_c]$ is obtained by complementing Δ_c to Δ, in other words, $[\Delta, \Delta_c]$ is regarded as a 1×2 operator matrix.

Definition 2.8. The inner function Δ_c in (13) is said to be the *complementary factor* of the inner function Δ.

Proof of Lemma 2.7. If $\ker \Delta^* = \{0\}$, then (a) and (b) are trivial. Suppose that $\ker \Delta^* \neq \{0\}$. Note that

$$\Delta_c := \text{left-g.c.d.}\left\{ [g]^i : g \in \ker \Delta^* \right\} \in H^\infty(\mathcal{B}(D'', E)), \tag{14}$$

where D'' is a nonzero subspace of E. If $g \in \ker \Delta^*$, then it follows from (12) that

$$\begin{aligned} \Delta_c H^2_{D''} &= \bigvee \left\{ [g]^i H^2 : g \in \ker \Delta^* \right\} \\ &= \bigvee \left\{ [g] \mathcal{P}_{\mathbb{C}} : g \in \ker \Delta^* \right\} \\ &\subseteq \ker \Delta^* = \Omega H^2_{D'}. \end{aligned}$$

For the reverse inclusion, let $0 \neq g \in \ker \Delta^*$. Then it follows that

$$g(z) = [g](z)1 = ([g]^i [g]^e)(z)1 = [g]^i(z)\big([g]^e(z)1\big) \in [g]^i H^2.$$

Thus we have

$$\Omega H^2_{D'} = \ker \Delta^* \subseteq \bigvee \left\{ [g]^i H^2 : g \in \ker \Delta^* \right\} = \Delta_c H^2_{D''}.$$

Therefore, by the Beurling-Lax-Halmos Theorem, $\Omega = \Delta_c$ and $D' = D''$, which gives (a). Note that $\Delta^* \Delta_c = 0$. We thus have

$$\begin{bmatrix} \Delta^* \\ \Delta_c^* \end{bmatrix} [\Delta, \Delta_c] = \begin{bmatrix} I_D & 0 \\ 0 & I_{D'} \end{bmatrix},$$

which implies that $[\Delta, \Delta_c]$ is an inner function with values in $\mathcal{B}(D \oplus D', E)$, which gives (b). For (c), we first note that ΔH^2_D and $\ker \Delta^*$ are orthogonal and

$$\Delta H^2_D \bigoplus \ker \Delta^* \subseteq \ker H_{\Delta^*}.$$

For the reverse inclusion, suppose that $f \in H^2_E$ and $f \notin \Delta H^2_D \bigoplus \ker \Delta^* \equiv M$. Write

$$f_1 := P_M f \quad \text{and} \quad f_2 := f - f_1 \neq 0.$$

Since $f_2 \in H^2_E \ominus M = \mathcal{H}(\Delta) \cap (H^2_E \ominus \ker \Delta^*)$, it follows from Corollary A.15 (see Appendix A) that $\Delta^* f_2 \in L^2_D \ominus H^2_D$ and $\Delta^* f_2 \neq 0$. We thus have $H_{\Delta^*} f = J(\Delta^* f_2)$, and hence, $||H_{\Delta^*} f|| = ||\Delta^* f_2|| \neq 0$, which implies that $f \notin \ker H_{\Delta^*}$. We thus have that

$$\ker H_{\Delta^*} = \Delta H^2_D \bigoplus \ker \Delta^*.$$

Thus it follows from (a) that

$$\ker H_{\Delta^*} = \Delta H^2_D \bigoplus \Delta_c H^2_{D'} = [\Delta, \Delta_c] H^2_{D \oplus D'},$$

which gives (c). This completes the proof. □

2.4 The degree of non-cyclicity: Answers to Question 1.3 and Question 1.4

For a subset F of H^2_E, let E^*_F denote the smallest S^*_E-invariant subspace containing F, i.e.,

$$E^*_F = \bigvee \{S^{*n}_E F : \ n \geq 0\}.$$

Then by the Beurling-Lax-Halmos Theorem, $E^*_F = \mathcal{H}(\Delta)$ for an inner function Δ with values in $\mathcal{B}(D, E)$. In general, if $\dim E = 1$, then every S^*_E-invariant subspace M admits a cyclic vector, i.e., $M = E^*_f$ for some $f \in H^2$. However, if $\dim E \geq 2$, then this is not such a case. For example, if $M = \mathcal{H}(\Delta)$ with $\Delta = \left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$, then M does not admit a cyclic vector, i.e., $M \neq E^*_f$ for any vector $f \in H^2_{\mathbb{C}^2}$.

If $\Phi \in H^2_s(\mathcal{B}(D, E))$ and $\{d_k\}_{k \geq 1}$ is an orthonormal basis for D, write

$$\phi_k := \Phi d_k \in H^2_E \cong H^2_s(\mathcal{B}(\mathbb{C}, E)).$$

We then define

$$\{\Phi\} := \{\phi_k\}_{k \geq 1} \subseteq H^2_E.$$

Hence, $\{\Phi\}$ may be regarded as the set of "column" vectors ϕ_k (in H^2_E) of Φ, in which case we may think of Φ as an infinite matrix-valued function.

Lemma 2.9. For $\Phi \in H^2_s(\mathcal{B}(D, E))$, we have

$$E^*_{\{\Phi\}} = \text{cl ran}\, H_{\overline{z}\breve{\Phi}}. \tag{15}$$

Remark 2.10. By definition, $\{\Phi\}$ depends on the orthonormal basis of D. However, Lemma 2.9 shows that $E^*_{\{\Phi\}}$ is independent of a particular choice of the orthonormal basis of D because the right-hand side of (15) is independent of the orthonormal basis of D.

Proof of Lemma 2.9. We first claim that if $f \in H^2_E$, then

$$E^*_f = \text{cl ran} H_{[\overline{z}\breve{f}]}. \tag{16}$$

Indeed, for each $k = 1, 2, \cdots$,

$$S_E^{*k} f = \sum_{j=0}^{\infty} \widehat{f}(k+j) z^j = JP_- \left(z^{k-1} \breve{f} \right) = H_{[\overline{z}\breve{f}]} z^k,$$

which gives (16). Let $\{d_k\}_{k \geq 1}$ be an orthonormal basis for D, and let $\phi_k := \Phi d_k$. Since by (16), $E^*_{\phi_k} = \text{cl ran}\, H_{[\overline{z}\breve{\phi}_k]}$ for each $k = 1, 2, 3, \cdots$, it follows that

$$E^*_{\{\Phi\}} = \bigvee \text{ran}\, H_{[\overline{z}\breve{\phi}_k]} = \text{cl ran}\, H_{\overline{z}\breve{\Phi}},$$

which gives the result. $\square$

The Douglas-Shapiro-Shields factorization in [DSS] actually has the form $\Phi = \overline{z} \cdot \Delta B^*$. With the aid of Lemma 2.9, we can get another version of Lemma 2.4 (c) for the original Douglas-Shapiro-Shields factorization.

Corollary 2.11. Let $\Phi \in L^\infty(\mathcal{B}(D, E))$ and Δ be a two-sided inner function with values in $\mathcal{B}(E)$. If

$$\Phi = \overline{z} \cdot \Delta B^*,$$

where $B \in H^\infty(\mathcal{B}(E, D))$ is such that Δ and B are right coprime, then

$$\bigvee_{n=0}^{\infty} S_E^{*n} \mathbb{P}_+ \Phi D = E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta).$$

Proof. Write $\Phi := \breve{\Phi}_- + \Phi_+$. Then $\overline{z}\breve{\Phi} = \overline{z}\Phi_- + \overline{z}\breve{\Phi}_+$, so that

$$H_{\overline{z}\breve{\Phi}} = H_{\overline{z}\Phi_-} + H_{\overline{z}\breve{\Phi}_+} = H_{\overline{z}\breve{\Phi}_+} \quad \text{and} \quad \overline{z}\breve{\Phi} = \breve{\Delta}\widetilde{B} = \widetilde{B\Delta^*}.$$

It thus follows from Lemma 2.9 that

$$\begin{aligned} \bigvee_{n=0}^{\infty} S_E^{*n} \mathbb{P}_+ \Phi D = E^*_{\{\Phi_+\}} &= \text{cl ran}\, H_{\overline{z}\breve{\Phi}_+} = \text{cl ran}\, H_{\overline{z}\breve{\Phi}} \\ &= \left(\ker H^*_{\overline{z}\breve{\Phi}}\right)^\perp = \left(\ker H_{B\Delta^*}\right)^\perp = \mathcal{H}(\Delta), \end{aligned}$$

where the last equality comes from Lemma 2.4. $\square$

We now introduce:

Definition 2.12. Let $F \subseteq H^2_E$. The *degree of non-cyclicity*, denoted by $\text{nc}(F)$, of F is defined by the number

$$\text{nc}(F) := \sup_{\zeta \in \mathbb{D}} \dim \big\{ g(\zeta) : g \in H^2_E \ominus E^*_F \big\}.$$

We will often refer to $\text{nc}(F)$ as the nc-*number* of F.

Since E_F^* is an invariant subspace for S_E^*, it follows from the Beurling-Lax-Halmos Theorem that $E_F^* = \mathcal{H}(\Delta)$ for some inner function Δ with values in $\mathcal{B}(D, E)$. Thus

$$\operatorname{nc}(F) = \sup_{\zeta \in \mathbb{D}} \dim \left\{ g(\zeta) : \, g \in \Delta H_D^2 \right\} = \dim D.$$

In particular, $\operatorname{nc}(F) \le \dim E$. We note that $\operatorname{nc}(F)$ may take ∞. So it is customary to make the following conventions: (i) if n is a natural number then $n + \infty = \infty$; (ii) $\infty + \infty = \infty$. If $\dim E = r < \infty$, then $\operatorname{nc}(F) \le r$ for every subset $F \subseteq H_E^2$. If $F \subseteq H_E^2$ and $\dim E = r < \infty$, then the *degree of cyclicity*, denoted by $\operatorname{dc}(F)$, of $F \subseteq H_E^2$ is defined by the number (cf. [VN])

$$\operatorname{dc}(F) := r - \operatorname{nc}(F). \tag{17}$$

In particular, if $E_F^* = \mathcal{H}(\Delta)$, then Δ is two-sided inner if and only if $\operatorname{nc}(F) = r$.

The following theorem gives an answer to Question 1.3.

Theorem 2.13. (An answer to Question 1.3) Let Φ be a strong L^2-function with values in $\mathcal{B}(D, E)$. In view of the Beurling-Lax-Halmos Theorem and Lemma 2.6, we may write

$$E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta) \quad \text{and} \quad \ker H^*_{\breve{\Phi}} = \Theta H^2_{E'},$$

for some inner functions Δ and Θ with values in $\mathcal{B}(E'', E)$ and $\mathcal{B}(E', E)$, respectively. Then

$$\Delta = \Theta \Delta_1 \tag{18}$$

for some two-sided inner function Δ_1 with values in $\mathcal{B}(E'', E')$. Hence, in particular,

$$\ker H^*_{\breve{\Phi}} = \Theta H^2_{E'} \iff \operatorname{nc}\{\Phi_+\} = \dim E'. \tag{19}$$

Proof. Suppose that $\ker H^*_{\breve{\Phi}} = \Theta H^2_{E'}$ for some inner function Θ with values in $\mathcal{B}(E', E)$ and $E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta)$ for some inner function Δ with values in $\mathcal{B}(E'', E)$. Then it follows from Lemma 2.9 that

$$\mathcal{H}(\Delta) = E^*_{\{\Phi_+\}} = \operatorname{cl} \operatorname{ran} H_{\overline{z}\breve{\Phi}} = \left(\ker H^*_{\overline{z}\breve{\Phi}}\right)^{\perp}.$$

It thus follows from Lemma 2.5 that

$$\begin{aligned}
\Delta H^2_{E''} &= \ker H^*_{\overline{z}\breve{\Phi}} \\
&= \left\{ f \in H^2_E : \left\langle \Phi(z)x, \ z^n f(z) \right\rangle_{L^2_E} = 0 \quad \text{for all } x \in D \textit{ and } \ n = 0, 1, 2, 3, \cdots \right\} \\
&\subseteq \left\{ f \in H^2_E : \left\langle \Phi(z)x, \ z^n f(z) \right\rangle_{L^2_E} = 0 \quad \text{for all } x \in D \textit{ and } \ n = 1, 2, 3, \cdots \right\} \\
&= \ker H^*_{\breve{\Phi}} = \Theta H^2_{E'},
\end{aligned}$$

which implies that Θ is a left inner divisor of Δ. Thus we can write

$$\Delta = \Theta \Delta_1 \tag{20}$$

for some inner function $\Delta_1 \in H^\infty(\mathcal{B}(E'', E'))$. By the same argument as above, we also have $z\Theta H^2_{E'} \subseteq \Delta H^2_{E''}$, so that we may write $z\Theta = \Delta\Delta_2$ for some inner function $\Delta_2 \in H^\infty(\mathcal{B}(E', E''))$. Therefore by (20), we have $zI_{E'} = \Delta_1 \Delta_2$, and hence by Lemma 2.2, Δ_1 is two-sided inner. This proves (18) and in turn (19). This completes the proof. $\square$

Corollary 2.14. If $\Phi = \Theta A^* = \overline{z} \cdot \Delta B^*$ with Θ two-sided inner, A and B in operator-valued H^∞, Δ inner, (Θ, A) and (Δ, B) right coprime, then $\Delta = \Theta\Delta_1$ with Δ_1 also two-sided inner.

Proof. By Lemma 2.4, $\ker H_{\Phi^*} = \Theta H^2_E$ and by Corollary 2.11, $E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta)$. Thus the result follows at once from Theorem 2.13. □

If we write Corollary 2.14 in full, then we have: if $\Phi \in L^\infty(\mathcal{B}(D,E))$ and Θ is a two-sided inner functions with values in $\mathcal{B}(E)$ satisfying

$$\bigvee_{n=1}^{\infty} S_E^{*n}\mathbb{P}_+\Phi D = \mathcal{H}(\Theta),$$

and the inner Δ is chosen so that

$$\bigvee_{n=0}^{\infty} S_E^{*n}\mathbb{P}_+\Phi D = \mathcal{H}(\Delta).$$

Then $\Delta = \Theta\Delta_1$ with Δ_1 a two-sided inner.

On the other hand, from the proof of Theorem 2.13, we see that Δ_1 is an inner divisor of $zI_{E'}$. In this case, if E' is finite dimensional and hence Δ_1 is two-sided inner matrix function then by Lemma 2.5 of [CHL2], we can see that Δ_1 is a Blaschke-Potapov factor $zP + (I_{E'} - P)$, where P is the orthogonal projection on $\mathbb{C}^n$. However, if E' is infinite dimensional, we have been unable to decide whether Δ_1 is a Blaschke-Potapov factor.

From Theorem 2.13, we get several corollaries.

Corollary 2.15. Let Φ be a strong L^2-function with value in $\mathcal{B}(D,E)$. Then the following statements are equivalent:

(a) $E^*_{\{\Phi_+\}} = H^2_E$;

(b) $\text{nc}\{\Phi_+\} = 0$;

(c) $\ker H^*_{\breve{\Phi}} = \{0\}$.

Proof. Immediate from Theorem 2.13. □

Corollary 2.16. Let Δ be an inner function with values in $\mathcal{B}(D,E)$. If Δ_c is the complementary factor of Δ, with values in $\mathcal{B}(D',E)$, then

$$\text{nc}\{\Delta\} = \dim D + \dim D'.$$

Proof. Immediate from Lemma 2.7(c) and Theorem 2.13. □

Corollary 2.17. If Φ is an $n \times m$ matrix L^2-function, i.e., $\Phi \in L^2_{M_{n\times m}}$, then the following are equivalent:

(a) Φ is of bounded type;

(b) $\ker H_\Phi^* = \Delta H^2_{\mathbb{C}^n}$ for some two-sided inner matrix function Δ;

(c) $\mathrm{nc}\,\{\Phi_-\} = n$.

Proof. The equivalence (a) $\Leftrightarrow$ (c) follows from [Ni1, Corollary 2, p. 47] and (6), and the equivalence (b) $\Leftrightarrow$ (c) follows at once from Theorem 2.13. □

The equivalence (a) $\Leftrightarrow$ (b) of Corollary 2.17 was known from [GHR] for the cases of $\Phi \in L^\infty_{M_n}$. On the other hand, it was known ([Abr, Lemma 4]) that if $\phi \in L^\infty$, then

$$\phi \text{ is of bounded type} \Longleftrightarrow \ker H_\phi \neq \{0\}. \tag{21}$$

The following corollary shows that (21) still holds for L^2-functions.

Corollary 2.18. If $\phi \in L^2$, then ϕ is of bounded type if and only if $\ker H_\phi^* \neq \{0\}$.

Proof. Immediate from Corollary 2.17. □

Corollary 2.19. If Δ is an $n \times r$ inner matrix function then the following are equivalent:

(a) Δ^* is of bounded type;

(b) $\breve{\Delta}$ is of bounded type;

(c) $[\Delta, \Delta_c]$ is two-sided inner,

where Δ_c is the complementary factor of Δ.

Proof. The equivalence (a) $\Leftrightarrow$ (b) is trivial. The equivalence (b) $\Leftrightarrow$ (c) follows from Lemma 2.7 and Corollary 2.17. □

Remark 2.20. R.G. Douglas and J.W. Helton [DH] have considered a problem from engineering circuit theory called Darlington synthesis which mathematically translates to: given a contractive analytic operator-valued functions on the unit disk, can one embed S into a two-sided 2×2 inner matrix function $\Theta = \left[\begin{smallmatrix} S & A \\ B & C \end{smallmatrix}\right]$? The special case where $S = \Delta$ is inner and the second block-row is vacuous amounts to our problem of finding Ω so that $[\Delta, \Omega]$ is two-sided inner. Thus, Corollary 2.19 can be obtained from [DH, Theorem].

The following theorem gives an answer to Question 1.4.

Theorem 2.21. (An answer to Question 1.4) If Δ is an $n \times r$ inner matrix function, then $[\Delta, \Omega]$ is inner for some $n \times q$ ($q \geq 1$) inner matrix function Ω if and only if

$$q \leq \mathrm{nc}\{\Delta\} - r.$$

In particular, Δ is complemented to a two-sided inner function if and only if $\mathrm{nc}\{\Delta\} = n$.

Proof. Suppose that $[\Delta, \Omega]$ is an inner matrix function for some $n \times q$ ($q \geq 1$) inner matrix function Ω. Then

$$I_{r+q} = [\Delta, \Omega]^* [\Delta, \Omega] = \begin{bmatrix} I_r & \Delta^*\Omega \\ \Omega^*\Delta & I_q \end{bmatrix},$$

which implies that $\Omega H^2_{\mathbb{C}^q} \subseteq \ker \Delta^*$. Since by Lemma 2.7, $\ker \Delta^* = \Delta_c H^2_{\mathbb{C}^p}$, it follows that $\Omega H^2_{\mathbb{C}^q} \subseteq \Delta_c H^2_{\mathbb{C}^p}$, so that Δ_c is a left inner divisor of Ω. Thus we can write

$$\Omega = \Delta_c \Omega_1 \quad \text{for some } p \times q \text{ inner matrix function } \Omega_1.$$

Thus we have $q \leq p$. But since by Corollary 2.16, $\text{nc}\{\Delta\} = r + p$, it follows that $q \leq \text{nc}\{\Delta\} - r$. For the converse, suppose that $q \leq \text{nc}\{\Delta\} - r$. Then it follows from Corollary 2.16 that the complementary factor Δ_c of Δ is in $H^\infty_{M_{n\times p}}$ for some $p \geq q$. Thus if we take $\Omega := \Delta_c|_{\mathbb{C}^q}$, then $[\Delta, \Omega]$ is inner. $\square$

We give an illuminating example of how to find the nc number.

Example 2.22. Let f and g be given in Example 2.1, and let

$$\Phi := \begin{bmatrix} f & f & 0 \\ g & g & 0 \\ 0 & 0 & a \end{bmatrix} \qquad (a \in H^\infty)$$

To find the degree of non-cyclicity of Φ, write $\Psi := \left[\begin{smallmatrix} f & f \\ g & g \end{smallmatrix}\right]$. Then it follows that

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \in \ker H^*_{\breve{\Phi}} \Longleftrightarrow \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \ker H_{\Psi^*} \text{ and } h_3 \in \ker H_{\overline{a}}.$$

Case 1: If $\overline{a}$ is not of bounded type, then $\ker H^*_{\breve{\Phi}} = [f\ g\ 0]^t H^2$. By Theorem 2.13, $\text{nc}\{\Phi\} = 1$.
Case 2: If $\overline{a}$ is of bounded type of the form $a = \theta\overline{b}$ (coprime), then

$$\ker H^*_{\breve{\Phi}} = \begin{bmatrix} f & 0 \\ g & 0 \\ 0 & \theta \end{bmatrix} H^2_{\mathbb{C}^2}.$$

By Theorem 2.13, $\text{nc}\{\Phi\} = 2$.

2.5 Strong L^2-functions of bounded type

We introduce the notion of "bounded type" for strong L^2-functions. Recall that a matrix-valued function of bounded type was defined by a matrix whose entries are of bounded type (see p. 4). But this definition is not appropriate for operator-valued functions, in particular strong L^2-functions, even though the terminology of "entry" can be properly interpreted. Thus we need a new idea about how to define a "bounded type" strong L^2-functions, which is equivalent to the condition that each entry is of bounded type when the function is matrix-valued. Our motivation stems from the equivalence (a)⇔(b) in Corollary 2.17.

Definition 2.23. A strong L^2-function Φ with values in $\mathcal{B}(D,E)$ is said to be of *bounded type* if $\ker H^*_\Phi = \Theta H^2_E$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$.

On the other hand, in [FB], it was shown that if Φ belongs to $L^\infty(\mathcal{B}(D,E))$, then Φ admits a Douglas-Shapiro-Shields factorization (see p. 14) if and only if $E^*_{\{\Phi_+\}} = \mathcal{H}(\Theta)$ for a two-sided inner function Θ. Thus, by Theorem 2.13, we can see that if $\Phi \in L^\infty(\mathcal{B}(D,E))$, then

$$\breve{\Phi} \text{ is of bounded type} \Longleftrightarrow \Phi \text{ admits a Douglas-Shapiro-Shields factorization.} \qquad (22)$$

We can prove more:

Lemma 2.24. Let Φ be a strong L^2-function with values in $\mathcal{B}(D,E)$. Then the following are equivalent:

(a) $\breve{\Phi}$ is of bounded type;

(b) $E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta)$ for some two-sided inner function Δ with values in $\mathcal{B}(E)$;

(c) $E^*_{\{\Phi_+\}} \subseteq \mathcal{H}(\Theta)$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$;

(d) $\{\Phi_+\} \subseteq \mathcal{H}(\Theta)$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$;

(e) For $\{\varphi_{k_1}, \varphi_{k_2}, \cdots\} \subseteq \{\Phi\}$, write $\Psi \equiv [\varphi_{k_1}, \varphi_{k_2}, \cdots]$. Then $\breve{\Psi}$ is of bounded type.

Proof. (a) $\Rightarrow$ (b): Suppose that $\breve{\Phi}$ is of bounded type. Then $\ker H^*_{\breve{\Phi}} = \Theta H^2_E$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$. It thus follows from Theorem 2.13 that $E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta)$ for some two-sided inner function Δ with values in $\mathcal{B}(E)$.

(b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d): Clear.

(d) $\Rightarrow$ (e): Suppose that $\{\varphi_{k_1}, \varphi_{k_2}, \cdots\} \subseteq \{\Phi\}$ and $\{\Phi_+\} \subseteq \mathcal{H}(\Theta)$ for some two-sided inner function $\Theta \in H^\infty(\mathcal{B}(E))$. Write $\Psi \equiv [\varphi_{k_1}, \varphi_{k_2}, \cdots]$. Then $\{\Psi_+\} \subseteq \mathcal{H}(\Theta)$, so that $E^*_{\{\Psi_+\}} \subseteq \mathcal{H}(\Theta)$. Suppose that $E^*_{\{\Psi_+\}} = \mathcal{H}(\Delta)$ for some inner function Δ with values in $\mathcal{B}(D', E)$. Thus $\Theta H^2_E \subseteq \Delta H^2_{D'}$, so that by Lemma 2.2, Δ is two-sided inner. Thus, by Theorem 2.13, $\ker H^*_{\breve{\Psi}} = \Omega H^2_E$ for some two-sided inner function Ω with values in $\mathcal{B}(E)$, so that $\breve{\Psi}$ is of bounded type.

(e) $\Rightarrow$ (a): Clear. $\square$

Corollary 2.25. Let Δ be an inner function with values in $\mathcal{B}(D,E)$. Then

$$\breve{\Delta} \text{ is of bounded type} \Longleftrightarrow [\Delta, \Delta_c] \text{ is two-sided inner,}$$

where Δ_c is the complementary factor of Δ. Hence, in particular, if Δ is a two-sided inner function with values in $\mathcal{B}(E)$, then $\breve{\Delta}$ is of bounded type.

Proof. The first assertion follows from Lemma 2.7. The second assertion follows from the first assertion together with the observation that if Δ is two-sided inner then $[\Delta, \Delta_c] = \Delta$. $\square$

Corollary 2.26. Let Δ be an inner function with values in $\mathcal{B}(D,E)$. Then $[\Delta,\Omega]$ is two-sided inner for some inner function Ω with values in $\mathcal{B}(D',E)$ if and only if $\breve{\Delta}$ is of bounded type.

Proof. Suppose that $[\Delta,\Omega]$ is two-sided inner for some inner function Ω with values in $\mathcal{B}(D',E)$. Then $\Delta^*\Omega=0$, so that $\Omega H^2_{D'}\subseteq \ker\Delta^*=\Delta_c H^2_{D''}$. Thus Δ_c is a left inner divisor of Ω, and hence $[\Delta,\Delta_c]$ is a left inner divisor of $[\Delta,\Omega]$. Therefore by Lemma 2.2, $[\Delta,\Delta_c]$ is two-sided inner, so that by Corollary 2.25, $\breve{\Delta}$ is of bounded type. The converse follows at once from Corollary 2.25 with $\Omega=\Delta_c$. $\square$

Also, as we noticed in Remark 2.20, the matrix-valued cases of Corollary 2.25 and Corollary 2.26 can be also obtained from [DH, Theorem].

We now ask: If $\Delta\equiv[\delta_1,\delta_2,\cdots,\delta_m]\in H^\infty_{M_{n\times m}}$ is an inner matrix function, does there exist j $(1\le j\le m)$ such that $\mathrm{dc}\{\delta_j\}=\mathrm{dc}\{\Delta\}$? (Recall from the definition of the dc-number given in (17) that $\mathrm{dc}=r-\mathrm{nc}$.) The answer, however, is negative. To see this, let f and g be given in Example 2.1 and let

$$\Delta:=\begin{bmatrix} f & 0\\ g & 0\\ 0 & f\\ 0 & g\end{bmatrix}\equiv\begin{bmatrix}\delta_1,\delta_2\end{bmatrix}.$$

Since

$$\begin{bmatrix} f & 0 & 0\\ g & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\ \text{is inner,}$$

in view of Theorem 2.21, we have $\mathrm{dc}(\delta_1)\le 1$. But since $\mathrm{dc}(\delta_1)\ne 0$ (because δ_1^* is not of bounded type), it follows that $\mathrm{dc}\{\delta_1\}=1$. Similarly, $\mathrm{dc}\{\delta_2\}=1$. However, we have $\mathrm{dc}\{\Delta\}=2$, because we can show that $\Delta_c=0$.

2.6 Meromorphic pseudo-continuations of bounded type

In general, if a strong L^2-function Φ is of bounded type then we cannot guarantee that each entry $\phi_{ij}\equiv\langle\Phi d_j,\ e_i\rangle$ is of bounded type, where $\{d_j\}$ and $\{e_i\}$ are orthonormal bases of D and E, respectively. But if we strengthen the assumption then we may have the assertion. To see this, for a function $\Psi:\mathbb{D}^e\equiv\{z:1<|z|\le\infty\}\to\mathcal{B}(D,E)$, we define $\Psi_{\mathbb{D}}:\mathbb{D}\to\mathcal{B}(E,D)$ by

$$\Psi_{\mathbb{D}}(\zeta):=\Psi^*(1/\overline{\zeta})\quad\text{for }\zeta\in\mathbb{D}.$$

If $\Psi_{\mathbb{D}}$ is a strong H^2-function, inner, and two-sided inner with values in $\mathcal{B}(E,D)$ (see Appendix A), then we shall say that Ψ is a strong H^2-function, inner, and two-sided inner in $\mathbb{D}^e$ with values in $\mathcal{B}(D,E)$, respectively.

A $\mathcal{B}(D,E)$-valued function Ψ is said to be *meromorphic of bounded type* in $\mathbb{D}^e$ if it can be represented by

$$\Psi=\frac{G}{\theta},$$

where G is a strong H^2-function in $\mathbb{D}^e$, with values in $\mathcal{B}(D,E)$ and θ is a scalar inner function in $\mathbb{D}^e$. (cf. [Fu2]). A function $\Phi \in L^2_s(\mathcal{B}(D,E))$ is said to have a *meromorphic pseudo-continuation* $\hat{\Phi}$ *of bounded type* in $\mathbb{D}^e$ if $\hat{\Phi}$ is meromorphic of bounded type in $\mathbb{D}^e$ and Φ is the nontangential SOT limit of $\hat{\Phi}$, that is, for all $x \in D$,

$$\Phi(z)x = \hat{\Phi}(z)x := \lim_{rz\to z} \hat{\Phi}(rz)x \quad \text{for almost all } z \in \mathbb{T}.$$

Note that for almost all $z \in \mathbb{T}$,

$$\Phi(z)x = \lim_{rz\to z} \hat{\Phi}(rz)x = \lim_{rz\to z} \hat{\Phi}^*_{\mathbb{D}}(r^{-1}z)x = \hat{\Phi}^*_{\mathbb{D}}(z)x \quad (x \in D).$$

We then have:

Lemma 2.27. Let Φ be a strong L^2-function with values in $\mathcal{B}(D,E)$. If Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then $\breve{\Phi}$ is of bounded type.

Proof. Suppose that Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Thus the meromorphic pseudo-continuation $\hat{\Phi}$ of Φ can be written as

$$\hat{\Phi}(\zeta) := \frac{G(\zeta)}{\delta(\zeta)} \quad (\zeta \in \mathbb{D}^e),$$

where G is a strong H^2-function in $\mathbb{D}^e$, with values in $\mathcal{B}(D,E)$ and δ is a scalar inner function in $\mathbb{D}^e$. Then for all $x \in D$,

$$\Phi(z)x = \hat{\Phi}^*_{\mathbb{D}}(z)x = \delta_{\mathbb{D}}(z)G^*_{\mathbb{D}}(z)x \quad \text{ for almost all } z \in \mathbb{T}.$$

Thus for all $x \in D$, $p \in \mathcal{P}_E$, and $n = 1, 2, 3, \cdots$,

$$\begin{aligned}\int_{\mathbb{T}} \big\langle \Phi(z)x,\, z^n \delta_{\mathbb{D}}(z)p(z)\big\rangle_E\, dm(z) &= \int_{\mathbb{T}} \big\langle G^*_{\mathbb{D}}(z)x,\ z^n p(z)\big\rangle_E\, dm(z)\\ &= \big\langle x,\ z^n G_{\mathbb{D}}(z)p(z)\big\rangle_{L^2_D} = 0,\end{aligned}$$

where the last equality follows from the fact that $z^n G_{\mathbb{D}}(z)p(z) \in zH^2_D$. Thus by Lemma 2.5, we can see that

$$\delta_{\mathbb{D}} H^2_E = \operatorname{cl} \delta_{\mathbb{D}} \mathcal{P}_E \subseteq \ker H^*_{\breve{\Phi}}. \tag{23}$$

In view of Lemma 2.6, $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$ for some inner function Δ with values in $\mathcal{B}(E', E)$. Thus Δ is a left inner divisor of $\delta_{\mathbb{D}} I_E$ (cf. [FF], [Pel]). Thus, it follows from Lemma 2.2 that that Δ is two-sided inner, so that $\breve{\Phi}$ is of bounded type. $\square$

The following lemma was proved in [Fu1] under the more restrictive setting of $H^\infty(\mathcal{B}(D,E))$.

Lemma 2.28. Let $\Phi \in L^\infty(\mathcal{B}(D,E))$. Then the following are equivalent:

(a) Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$;

(b) $\theta H^2_E \subseteq \ker H_{\Phi^*}$ for some scalar inner function θ;

(c) $\Phi = \theta A^*$ for a scalar inner function θ and some $A \in H^\infty(\mathcal{B}(E, D))$.

Proof. First of all, recall that $L^\infty(\mathcal{B}(D,E)) \subseteq L^2_s(\mathcal{B}(D,E))$.

(a) $\Rightarrow$ (b): This follows from (23) in the proof of Lemma 2.27.

(b) $\Rightarrow$ (c): Suppose that $\theta H^2_E \subseteq \ker H_{\Phi^*}$ for some scalar inner function θ. Put $A := \theta\Phi^*$. Then A belongs to $H^\infty(\mathcal{B}(E,D))$ and $\Phi = \theta A^*$.

(c) $\Rightarrow$ (a): Suppose that $\Phi = \theta A^*$ for a scalar inner function θ and some $A \in H^\infty(\mathcal{B}(E,D))$. Thus it follows from Lemma A.6 that A is a strong H^2-function. Let

$$\hat{\Phi}(\zeta) := \frac{A^*(1/\overline{\zeta})}{\overline{\theta}(1/\overline{\zeta})} \quad (\zeta \in \mathbb{D}^e).$$

Then $\hat{\Phi}$ is meromorphic of bounded type in $\mathbb{D}^e$ and for all $x \in D$,

$$\hat{\Phi}(z)x = \frac{A^*(z)x}{\overline{\theta}(z)} = \theta(z)A^*(z)x = \Phi(z)x \quad \text{for almost all } z \in \mathbb{T},$$

which implies that Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. $\square$

An examination of the proof of Lemma 2.28 shows that Lemma 2.28 still holds for every function $\Phi \in L^2_{\mathcal{B}(D,E)}$.

Corollary 2.29. If $\Phi \in L^2_{\mathcal{B}(D,E)}$, then Lemma 2.28 holds with $A \in H^2_{\mathcal{B}(E,D)}$ in place of $A \in H^\infty(\mathcal{B}(E,D))$.

The following proposition gives an answer to the opening remark of this subsection.

Proposition 2.30. Let D and E be separable complex Hilbert spaces and let $\{d_j\}$ and $\{e_i\}$ be orthonormal bases of D and E, respectively. If $\Phi \in L^2_{\mathcal{B}(D,E)}$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then $\breve{\phi}_{ij}(z) \equiv \langle \breve{\Phi}(z)d_j,\ e_i\rangle_E$ is of bounded type for each i, j.

Proof. Let $\Phi \in L^2_{\mathcal{B}(D,E)}$. Suppose that Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then by Corollary 2.29, $\Phi = \theta A^*$ for a scalar inner function θ and some $A \in H^2_{\mathcal{B}(E,D)}$. Write

$$\phi_{ij}(z) := \langle \Phi(z)d_j,\ e_i\rangle_E \quad \text{and} \quad a_{ij}(z) := \langle \widetilde{A}(z)d_j,\ e_i\rangle_E.$$

Then for each i, j,

$$\begin{aligned}\int_{\mathbb{T}} |\phi_{ij}(z)|^2 dm(z) &= \int_{\mathbb{T}} |\langle \Phi(z)d_j,\ e_i\rangle_E|^2 dm(z) \\ &\le \int_{\mathbb{T}} ||\Phi(z)||^2_{\mathcal{B}(D,E)} dm(z) < \infty,\end{aligned}$$

which implies $\phi_{ij} \in L^2$. Similarly, $a_{ij} \in L^2$ and for $n = 1, 2, 3, \cdots$,

$$\widehat{a_{ij}}(-n) = \int_{\mathbb{T}} z^n \langle \widetilde{A}(z)d_j,\ e_i\rangle_E dm(z) = \langle d_j,\ z^{-n}\breve{A}(z)e_i\rangle_{L^2_D} = 0,$$

which implies $a_{ij} \in H^2$. Note that

$$\breve{\phi}_{ij}(z) = \breve{\theta}(z)\langle \widetilde{A}(z)d_j,\ e_i\rangle_E = \breve{\theta}(z)a_{ij}(z)\,,$$

which implies that $\breve{\phi}_{ij}$ is of bounded type for each i, j. $\square$

Example 2.31. The converse of Lemma 2.27 is not true in general. To see this, let $\{\alpha_n\}$ be a sequence of distinct points in $\mathbb{D}$ such that $\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$ and put $\Delta := \text{diag}(b_{\alpha_n})$, where $b_{\alpha_n}(z) := \frac{z-\alpha_n}{1-\overline{\alpha}_n z}$. Then Δ is two-sided inner, and hence by Lemma 2.25, $\breve{\Delta}$ is of bounded type. On the other hand, by Lemma 2.7, $\ker H_{\Delta^*} = \Delta H^2_{\ell^2}$. Thus if Δ had a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then by Lemma 2.28, we would have $\theta H^2_{\ell^2} \subseteq \Delta H^2_{\ell^2}$ for a scalar inner function θ, so that we should have $\theta(\alpha_n) = 0$ for each $n = 1, 2, \cdots$, and hence $\theta = 0$, a contradiction. Therefore, Δ cannot have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$.

For matrix-valued cases, a function having a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$ is actually a function whose flip is of bounded type.

Corollary 2.32. For $\Phi \equiv [\phi_{ij}] \in L^2_{M_{n\times m}}$, the following are equivalent:

(a) Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$;

(b) $\breve{\Phi}$ is of bounded type;

(c) $\breve{\phi}_{ij}$ is of bounded type for each i, j.

Proof. (a) $\Rightarrow$ (b): This follows from Lemma 2.27.

(b) $\Rightarrow$ (a): Suppose that $\breve{\Phi}$ is of bounded type. Then $\ker H^*_{\breve{\Phi}} = \Theta H^2_{\mathbb{C}^n}$ for some two-sided inner function $\Theta \in H^\infty_{M_n}$. Thus by the Complementing Lemma (cf. p. 15), there exist a scalar inner function θ and a function G in $H^\infty_{M_n}$ such that $G\Theta = \Theta G = \theta I_n$, and hence, $\theta H^2_{\mathbb{C}^n} = \Theta G H^2_{\mathbb{C}^n} \subseteq \Theta H^2_{\mathbb{C}^n} = \ker H^*_{\breve{\Phi}}$. It thus follows from Corollary 2.29 that $\breve{\Phi}$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$.

(a) $\Leftrightarrow$ (c): This follows from Corollary 2.29 and Proposition 2.30. $\square$

However, by contrast to the matrix-valued case, it may happen that an L^∞-function Φ is not of bounded type in the sense of Definition 2.23 even though each entry ϕ_{ij} of Φ is of bounded type.

Example 2.33. Let $\{\alpha_j\}$ be a sequence of distinct points in $(0,1)$ satisfying $\sum_{j=1}^{\infty}(1-\alpha_j) < \infty$. For each $j \in \mathbb{Z}_+$, choose a sequence $\{\alpha_{ij}\}$ of distinct points on the circle $C_j := \{z \in \mathbb{C} : |z| = \alpha_j\}$. Let

$$B_{ij} := \frac{\overline{b}_{\alpha_{ij}}}{(i+j)!} \quad (i, j \in \mathbb{Z}_+),$$

where $b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}$, and let

$$\Phi := [B_{ij}] = \begin{bmatrix} \frac{\overline{b}_{\alpha_{11}}}{2!} & \frac{\overline{b}_{\alpha_{12}}}{3!} & \frac{\overline{b}_{\alpha_{13}}}{4!} & \cdots \\ \frac{\overline{b}_{\alpha_{21}}}{3!} & \frac{\overline{b}_{\alpha_{22}}}{4!} & \frac{\overline{b}_{\alpha_{23}}}{5!} & \cdots \\ \frac{\overline{b}_{\alpha_{31}}}{4!} & \frac{\overline{b}_{\alpha_{32}}}{5!} & \frac{\overline{b}_{\alpha_{33}}}{6!} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Observe that

$$\sum_{i,j} |B_{ij}(z)|^2 = \sum_i \frac{i}{((1+i)!)^2} \le \sum_i \frac{1}{(1+i)^2} < \infty,$$

which implies that $\Phi \in L^\infty(\mathcal{B}(\ell^2))$. For a function $f \in H^2_{\ell^2}$, we write $f = (f_1, f_2, f_3, \cdots)^t$ ($f_n \in H^2$). Thus if $f = (f_1, f_2, f_3, \cdots)^t \in \ker H_\Phi$, then $\sum_j \frac{\overline{b}_{\alpha_{ij}}}{(i+j)!} f_j \in H^2$ for each $i \in \mathbb{Z}_+$, which forces that $f_j(\alpha_{ij}) = 0$ for each i, j. Thus $f_j = 0$ for each j (by the Identity Theorem). Therefore we can conclude that $\ker H^*_{\widetilde{\Phi}} = \{0\}$, so that $\widetilde{\Phi}$ is not of bounded type. But we note that every entry of $\widetilde{\Phi}$ is of bounded type.

We conclude this section with an application to C_0-contractions.

The class $C_{0\,\bullet}$ denotes the set of all contractions $T \in \mathcal{B}(\mathcal{H})$ satisfying the condition (4). The class C_{00} denotes the set of all contractions $T \in \mathcal{B}(\mathcal{H})$ such that $\lim_{n\to\infty} T^n x = 0$ and $\lim_{n\to\infty} T^{*n} x = 0$ for each $x \in H$. It was known ([Ni1, p.43]) that if T is a $C_{0\,\bullet}$-contraction with characteristic function Δ (i.e., $T \cong S_E^*|_{\mathcal{H}(\Delta)}$), then

$$T \in C_{00} \iff \Delta \text{ is } \textit{two-sided} \text{ inner.} \tag{24}$$

A contraction $T \in \mathcal{B}(\mathcal{H})$ is called a *completely non-unitary* (c.n.u.) if there exists no nontrivial reducing subspace on which T is unitary. The class C_0 is the set of all c.n.u. contractions T such that there exists a nonzero function $\varphi \in H^\infty$ annihilating T, i.e., $\varphi(T) = 0$, where $\varphi(T)$ is given by the calculus of Sz.-Nagy and Foiaş. We can easily check that $C_0 \subseteq C_{00}$. Moreover, it is well known ([Ni1, p.73]) that if $T := P_{\mathcal{H}(\Delta)} S_E|_{\mathcal{H}(\Delta)} \in C_{00}$ and $\varphi \in H^\infty$, then

$$\varphi(T) = 0 \iff \exists\, G \in H^\infty(\mathcal{B}(E)) \text{ such that } G\Delta = \Delta G = \varphi I_E. \tag{25}$$

The theory of spectral multiplicity for operators of class C_0 has been well developed (see [Ni1, Appendix 1], [SFBK]). If $T \in C_0$, then there exists an inner function m_T such that $m_T(T) = 0$ and

$$\varphi \in H^\infty,\ \varphi(T) = 0 \implies \varphi/m_T \in H^\infty.$$

The function m_T is called the *minimal annihilator* of the operator T.

In view of (24), we may ask what is a condition on the characteristic function Δ of T for a $C_{0\,\bullet}$-contraction T to belong to the class C_0. The following proposition gives an answer.

Proposition 2.34. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ with values in $\mathcal{B}(D, E)$. Then the following are equivalent:

(a) $T \in C_0$;

(b) Δ is two-sided inner and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$.

Hence, in particular, if Δ is an inner matrix function then $T \in C_0$ if and only if $T \in C_{00}$.

Proof. (a) $\Rightarrow$ (b): Suppose $T \in C_0$, and hence $\varphi(T) = 0$ for some nonzero function $\varphi \in H^\infty$. Then $T \in C_{00}$, so that by the above remark, Δ is two-sided inner. Thus by the Model Theorem (cf. [Ni1, p.75]), we have

$$T \cong P_{\mathcal{H}(\widetilde{\Delta})} S_E|_{\mathcal{H}(\widetilde{\Delta})}.$$

It thus follows from (25) that there exists $\Omega \in H^\infty(\mathcal{B}(E))$ such that $\widetilde{\Delta}\Omega = \Omega\widetilde{\Delta} = \varphi I_E$. Thus $H_{\Delta^*}\big(\widetilde{\varphi} H^2_E\big) = H_{\Delta^*}\big(\Delta\widetilde{\Omega} H^2_E\big) = 0$. We thus have

$$\widetilde{\varphi}^i H^2_E \subseteq \mathrm{cl}\, \widetilde{\varphi} H^2_E \subseteq \ker H_{\Delta^*}.$$

It thus follows from Lemma 2.28 that Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. This gives the implication (a)$\Rightarrow$(b).

(b) $\Rightarrow$ (a): Suppose that Δ is two-sided inner and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then by Lemma 2.7 and Lemma 2.28, there exists a scalar function δ such that $\delta H^2_E \subseteq \ker H_{\Delta^*} = \Delta H^2_E$. Thus we may write $\delta I_E = \Delta\Omega = \Omega\Delta$ for some $\Omega \in H^\infty(\mathcal{B}(E))$. Thus we have

$$\delta\big(P_{\mathcal{H}(\Delta)} S_E|_{\mathcal{H}(\Delta)}\big) = P_{\mathcal{H}(\Delta)}(\delta I_E)|_{\mathcal{H}(\Delta)} = 0,$$

so that

$$\widetilde{\delta}(T) = \big(\delta(T^*)\big)^* = \Big(\delta\big(P_{\mathcal{H}(\Delta)} S_E|_{\mathcal{H}(\Delta)}\big)\Big)^* = 0,$$

which gives $T \in C_0$. This prove the implication (b)$\Rightarrow$(a).

The second assertion follows from the first together with Corollary 2.25 and Corollary 2.32. $\square$

3 A canonical decomposition of strong L^2-functions

In this section, we first give an answer to Question 1.2.

3.1 An answer to Question 1.2

To better understand the canonical decomposition, we first consider an example of a matrix-valued L^2-function that does not admit a Douglas-Shapiro-Shields factorization. Suppose that θ_1 and θ_2 are coprime inner functions. Consider

$$\Phi := \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & a \end{bmatrix} \equiv [\phi_1, \phi_2, \phi_3] \in H^\infty_{M_3},$$

where $a \in H^\infty$ is such that $\overline{a}$ is not of bounded type. Then a direct calculation shows that

$$\ker H_{\Phi^*} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} H^2_{\mathbb{C}^2} \equiv \Delta H^2_{\mathbb{C}^2}.$$

Since Δ is not two-sided inner, it follows from Lemma 2.4 that Φ does not admit a Douglas-Shapiro-Shields factorization. For a decomposition of Φ, suppose that

$$\Phi = \Omega A^*, \tag{26}$$

where $\Omega, A \in H^2_{M_{3\times k}} (k = 1, 2)$, Ω is an inner function, and Ω and A are right coprime. We then have

$$\Phi^* \Omega = A \in H^2_{M_{3\times k}}. \tag{27}$$

But since $\overline{a}$ is not of bounded type, it follows from (27) that the 3rd row vector of Ω is zero. Thus by (26), we must have $a = 0$, a contradiction. Therefore we could not get any decomposition of the form $\Phi = \Omega A^*$ with a $3 \times k$ inner matrix function Ω for each $k = 1, 2, 3$. To get another idea, we note that $\ker \Delta^* = [0\ 0\ 1]^t H^2 \equiv \Delta_c H^2$. Then by a direct manipulation, we can get

$$\Phi = \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^* + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & a \end{bmatrix} \equiv \Delta A^* + \Delta_c C \tag{28}$$

where Δ and A are right coprime because $\widetilde{\Delta} H^2_{\mathbb{C}^3} \bigvee \widetilde{A} H^2_{\mathbb{C}^3} = H^2_{\mathbb{C}^2}$.

To encounter another situation, consider

$$\Phi := \begin{bmatrix} f & f & 0 \\ g & g & 0 \\ 0 & 0 & \theta\overline{a} \end{bmatrix} \equiv [\phi_1, \phi_2, \phi_3] \in H^\infty_{M_3},$$

where f and g are given in Example 2.1, θ is inner, and $a \in H^\infty$ is such that θ and a are coprime. It then follows from Lemma 2.7 that

$$\ker H_{[\overline{f}\ \overline{g}]} = \begin{bmatrix} f \\ g \end{bmatrix} H^2.$$

We thus have that

$$\ker H_{\Phi^*} = \ker H_{[\overline{f}\ \overline{g}]} \bigoplus \ker H_{\overline{\theta} a} = \begin{bmatrix} f & 0 \\ g & 0 \\ 0 & \theta \end{bmatrix} H^2_{\mathbb{C}^2} \equiv \Delta H^2_{\mathbb{C}^2}.$$

Thus by Lemma 2.4, Φ does not admit a Douglas-Shapiro-Shields factorization. Observe that

$$\Phi = \begin{bmatrix} f & f & 0 \\ g & g & 0 \\ 0 & 0 & \theta\overline{a} \end{bmatrix} = \begin{bmatrix} f & 0 \\ g & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & a \end{bmatrix}^* = \Delta A^*. \tag{29}$$

Since $\widetilde{\theta}$ and $\widetilde{a}$ are coprime, it follows that Δ and A are right coprime. Note that Δ is not two-sided inner and $\ker \Delta^* = \{0\}$.

The above examples (28) and (29) seem to signal that the decomposition of a matrix-valued H^2-functions Φ satisfying $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^n}$ may be affected by the kernel of Δ^* and in turn, the complementary factor Δ_c of Δ. Indeed, if we regard Δ^* as an operator acting from L^2_E, and hence $\ker \Delta^* \subseteq L^2_E$, then B in the canonical decomposition (30) satisfies the inclusion $\{B\} \subseteq \ker \Delta^*$. The following theorem gives a canonical decomposition of strong L^2-functions which realizes the idea inside those examples.

We are ready for an answer to Question 1.2:

Theorem 3.1. (A canonical decomposition of strong L^2-functions) If Φ is a strong L^2-function with values in $\mathcal{B}(D,E)$, then Φ can be expressed in the form

$$\Phi = \Delta A^* + B, \tag{30}$$

where

(i) Δ is an inner function with values in $\mathcal{B}(E',E)$, $\widetilde{A} \in H^2_s(\mathcal{B}(D,E'))$, and $B \in L^2_s(\mathcal{B}(D,E))$;

(ii) Δ and A are right coprime;

(iii) $\Delta^* B = 0$;

(iv) $\mathrm{nc}\{\Phi_+\} \le \dim E'$.

(v) In particular, if $\dim E' < \infty$ (for instance, if $\dim E < \infty$), then the expression (30) is unique (up to a unitary constant right factor).

Proof. If $\ker H^*_{\breve{\Phi}} = \{0\}$, take $E' := \{0\}$ and $B := \Phi$. Then $\widetilde{\Delta}$ and $\widetilde{A}$ are zero operator with codomain $\{0\}$. Thus $\Phi = \Delta A^* + B$, where Δ and A are right coprime. It also follows from Theorem 2.13 that $\mathrm{nc}\{\Phi_+\} = 0$, which gives the inequality (iv).

If instead $\ker H^*_{\breve{\Phi}} \neq \{0\}$, then in view of Lemma 2.6, we may suppose $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$ for some nonzero inner function Δ with values in $\mathcal{B}(E',E)$. Put $A := \Phi^* \Delta$. Then it follows from Lemma A.13 (see Appendix A) that A^* is a strong L^2-function with values in $\mathcal{B}(D,E')$. Thus $\widetilde{A} = \breve{A}^*$ is a strong L^2-function with values in $\mathcal{B}(D,E')$. Since $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$, it follows that for all $p \in \mathcal{P}_D$ and $h \in H^2_{E'}$

$$\begin{aligned}
0 &= \langle H_{\breve{\Phi}} p,\ \Delta h\rangle_{L^2_E} \\
&= \int_{\mathbb{T}} \big\langle \breve{\Phi}(z)p(z),\ \overline{z}\Delta(\overline{z})h(\overline{z})\big\rangle_E dm(z) \\
&= \int_{\mathbb{T}} \big\langle \widetilde{\Delta}(z)\breve{\Phi}(z)p(z),\ \overline{z}h(\overline{z})\big\rangle_{E'} dm(z) \\
&= \big\langle H_{\widetilde{A}} p,\ h\big\rangle_{L^2_{E'}},
\end{aligned}$$

which implies $H_{\widetilde{A}} = 0$. Thus by Lemma 2.3, $\widetilde{A}$ belongs to $H^2_s(\mathcal{B}(D,E))$. Put $B := \Phi - \Delta A^*$. Then by Lemma A.13 (see Appendix A), B is a strong L^2-function with values in $\mathcal{B}(D,E)$. Observe that

$$\Phi = \Delta A^* + B \quad \text{and} \quad \Delta^* B = 0.$$

This completes the proof of (30) and assertions (i) and (iii).

To prove assertion (ii) we must show that Δ and A are right coprime.

To see this, we suppose that Ω is a common left inner divisor, with values in $\mathcal{B}(E'', E')$, of $\widetilde{\Delta}$ and $\widetilde{A}$. Then we may write

$$\widetilde{\Delta} = \Omega \widetilde{\Delta}_1 \quad \text{and} \quad \widetilde{A} = \Omega \widetilde{A}_1,$$

where $\widetilde{\Delta}_1 \in H^\infty(\mathcal{B}(E, E''))$ and $\widetilde{A}_1 \in H^2_s(\mathcal{B}(D, E''))$. Thus we have

$$\Delta = \Delta_1 \widetilde{\Omega} \quad \text{and} \quad A = A_1 \widetilde{\Omega}. \tag{31}$$

Since Ω is inner, it follows that $\Delta_1 = \Delta \widetilde{\Omega}^*$. By Lemma A.12 (see Appendix A), $\Delta_1 \in H^\infty(\mathcal{B}(E'', E))$ and

$$\Delta_1^* \Delta_1 = \widetilde{\Omega} \Delta^* \Delta \widetilde{\Omega}^* = \widetilde{\Omega^* \Omega} = I,$$

which implies that Δ_1 is inner. We now claim that

$$\Delta_1 H^2_{E''} = \ker H^*_{\check{\Phi}} = \Delta H^2_{E'}. \tag{32}$$

Since Ω is an inner function with values in $\mathcal{B}(E'', E')$, we know that $\widetilde{\Omega} \in H^\infty(\mathcal{B}(E', E''))$ by Lemma A.12 (see Appendix A). Thus it follows from Corollary A.14 (see Appendix A) and (31) that

$$\Delta H^2_{E'} = \Delta_1 \widetilde{\Omega} H^2_{E'} \subseteq \Delta_1 H^2_{E''}.$$

For the reverse inclusion, by (31), we may write $\Phi = \Delta_1 A_1^* + B$. Since $0 = \Delta^* B = \widetilde{\Omega}^* \Delta_1^* B$, it follows that $\Delta_1^* B = 0$. Therefore for all $f \in H^2_{E''}$, $x \in D$ and $n = 1, 2, \cdots$, we have

$$\begin{aligned}
\int_{\mathbb{T}} \big\langle \Phi(z)x,\ z^n \Delta_1(z) f(z) \big\rangle_E dm(z) &= \int_{\mathbb{T}} \big\langle \big(\Delta_1(z) A_1^*(z) + B(z)\big)x,\ z^n \Delta_1(z) f(z) \big\rangle_E dm(z) \\
&= \int_{\mathbb{T}} \big\langle A_1^*(z)x,\ z^n f(z) \big\rangle_{E''} dm(z) \\
&= \big\langle A_1^*(z)x,\ z^n f(z) \big\rangle_{L^2_{E''}} \\
&= 0,
\end{aligned}$$

where the last equality follows from the fact that $A_1^*(z)x = \widetilde{A}_1(\overline{z})x \in L^2_{E''} \ominus z H^2_{E''}$. Thus by Lemma 2.5, we have

$$\Delta_1 H^2_{E''} \subseteq \ker H^*_{\check{\Phi}} = \Delta H^2_{E'},$$

which proves (32). Thus it follows from the Beurling-Lax-Halmos Theorem and (31) that $\widetilde{\Omega}$ is a unitary operator, and so is Ω. Therefore A and Δ are right coprime. This completes the proof of assertion (ii).

Assertion (iv) on the nc-number comes from Theorem 2.13. We have now completed the proof (i)–(iv).

It remains to verify the uniqueness assertion (v). To see this, suppose $\dim E' < \infty$. For the uniqueness of the expression (30), we suppose that $\Phi = \Delta_1 A_1^* + B_1 = \Delta_2 A_2^* + B_2$ are two canonical decompositions of Φ. We want to show that $\Delta_1 = \Delta_2$, which gives

$$A_1^* = \Delta_1^*(\Delta_1 A_1^* + B_1) = \Delta_2^*(\Delta_2 A_2^* + B_2) = A_2^*$$

and in turn, $B_1 = B_2$, which implies that the representation (30) is unique. To prove $\Delta_1 = \Delta_2$, it suffices to show that if $\Phi = \Delta A^* + B$ is a canonical decomposition of Φ, then

$$\ker H_{\breve{\Phi}}^* = \Delta H_{E'}^2. \tag{33}$$

If $E' = \{0\}$, then $\mathrm{nc}\{\Phi_+\} = 0$. Thus it follows from Corollary 2.15 that

$$\ker H_{\breve{\Phi}}^* = \{0\} = \Delta H_{E'}^2,$$

which proves (33). If instead $E' \neq \{0\}$, then we suppose $r := \dim E' < \infty$. Thus, we may assume that $E' = \mathbb{C}^r$, so that Δ is an inner function with values in $\mathcal{B}(\mathbb{C}^r, E)$. Suppose that $\Phi = \Delta A^* + B$ is a canonical decomposition of Φ in $L_s^2(\mathcal{B}(D,E))$. We first claim that

$$\Delta H_{\mathbb{C}^r}^2 \subseteq \ker H_{\breve{\Phi}}^*. \tag{34}$$

Observe that for each $g \in H_{\mathbb{C}^r}^2$, $x \in D$ and $k = 1, 2, 3, \cdots$,

$$\begin{aligned}\int_{\mathbb{T}} \big\langle \Phi(z)x,\ z^k \Delta(z) g(z)\big\rangle_E dm(z) &= \int_{\mathbb{T}} \big\langle A^*(z)x,\ z^k g(z)\big\rangle_{\mathbb{C}^r} dm(z)\\ &= \big\langle \widetilde{A}(\overline{z})x,\ z^k g(z)\big\rangle_{L^2_{\mathbb{C}^r}}\\ &= 0.\end{aligned}$$

It thus follows from Lemma 2.5 that $\Delta H_{\mathbb{C}^r}^2 \subseteq \ker H_{\breve{\Phi}}^*$, which proves (34). In view of Lemma 2.6, we may assume that $\ker H_{\breve{\Phi}}^* = \Theta H_{E''}^2$ for some inner function Θ with values in $\mathcal{B}(E'', E)$. Then by Theorem 2.13,

$$p \equiv \dim E'' = \mathrm{nc}\,\{\Phi_+\} \leq r. \tag{35}$$

Thus we may assume $E'' \equiv \mathbb{C}^p$. Since

$$\Delta H_{\mathbb{C}^r}^2 \subseteq \ker H_{\breve{\Phi}}^* = \Theta H_{\mathbb{C}^p}^2, \tag{36}$$

it follows that Θ is left inner divisor of Δ, i.e., there exists a $p \times r$ inner matrix function Δ_1 such that $\Delta = \Theta \Delta_1$. Since Δ_1 is inner, it follows that $r \leq p$. But since by (35), $p \leq r$, we must have $r = p$, which implies that Δ_1 is two-sided inner. Thus we have

$$\Theta^* \Phi = \Delta_1 A^* + \Delta_1 \Delta^* B = \Delta_1 A^*. \tag{37}$$

Since $\ker H_{\breve{\Phi}}^* = \Theta H_{\mathbb{C}^r}^2$, it follows from Lemma 2.5 and (37) that for all $f \in H_{\mathbb{C}^r}^2$, $x \in D$ and $n = 1, 2, \cdots$,

$$\int_{\mathbb{T}} \big\langle \Delta_1(z) A^*(z)x,\ z^n f(z)\big\rangle_{\mathbb{C}^r} dm(z) = \int_{\mathbb{T}} \big\langle \Phi(z)x,\ z^n \Theta(z) f(z)\big\rangle_E dm(z) = 0. \tag{38}$$

Write $\Psi := \Delta_1 A^*$. Then by Lemma A.13 (see Appendix A), $\Psi \in L^2_s(\mathcal{B}(D, \mathbb{C}^r))$. Thus by Lemma 2.3, Lemma 2.5 and (38), we have $\breve{\Psi} \in H^2_s(\mathcal{B}(D, \mathbb{C}^r))$. Since $\widetilde{A} = \widetilde{\Delta}_1 \breve{\Psi}$, it follows that $\widetilde{\Delta}_1$ is a common left inner divisor of $\widetilde{\Delta}$ and $\widetilde{A}$. But since Δ and A are right coprime, it follows that $\widetilde{\Delta}_1$ is a unitary matrix, and so is Δ_1, which proves (33). This completes the proof of assertion (v).

This completes the proof. □

The proof of Theorem 3.1 shows that the inner function Δ in a canonical decomposition (30) of a strong L^2-function Φ can be obtained from equation

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$$

which is guaranteed by the Beurling-Lax-Halmos Theorem (see Corollary 2.6). In this case, the expression (30) will be called the *BLH-canonical decomposition* of Φ in the viewpoint that Δ comes from the Beurling-Lax-Halmos Theorem. However, if $\dim E' = \infty$ (even though $\dim D < \infty$), then it is possible to get another inner function Θ of a canonical decomposition (30) for the same function: in this case, $\ker H^*_{\breve{\Phi}} \neq \Theta H^2_{E''}$. Indeed, the following remark shows that the canonical decomposition (30) is not unique in general.

Remark 3.2. If $\dim E' = \infty$ (even though $\dim D < \infty$), the canonical decomposition (30) may not be unique even if $\breve{\Phi}$ is of bounded type. To see this, let Φ be an inner function with values in $\mathcal{B}(\mathbb{C}^2, \ell^2)$ defined by

$$\Phi := \begin{bmatrix} \theta_1 & 0 \\ 0 & 0 \\ 0 & \theta_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix},$$

where θ_1 and θ_2 are scalar inner functions. Then

$$\ker H^*_{\breve{\Phi}} = \ker H_{\Phi^*} = \operatorname{diag}(\theta_1, 1, \theta_2, 1, 1, 1, \cdots) H^2_{\ell^2} \equiv \Theta H^2_{\ell^2},$$

which implies that $\breve{\Phi}$ is of bounded type since Θ is two-sided inner (see Definition 2.23). Let

$$A := \Phi^* \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \quad \text{and} \quad B := 0.$$

Then $\widetilde{A}$ belongs to belongs to $H^2_s(\mathcal{B}(\mathbb{C}^2, \ell^2))$ and $\widetilde{\Theta} H^2_{\ell^2} \bigvee \widetilde{A} H^2_{\mathbb{C}^2} = H^2_{\ell^2}$, which implies that Θ and A are right coprime. Clearly, $\Theta^* B = 0$ and $\operatorname{nc}\{\Phi_+\} \leq \dim \ell^2 = \infty$. Therefore,

$$\Phi = \Theta A^*$$

is the BLH-canonical decomposition of Φ. On the other hand, to get another canonical decomposition of Φ, let

$$\Delta := \begin{bmatrix} \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \theta_2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then Δ is an inner function. If we define

$$A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \quad \text{and} \quad B := 0\,,$$

then $\widetilde{A}_1$ belongs to $H^2_s(B(\mathbb{C}^2, \ell^2))$ such that Δ and A_1 are right coprime, $\Delta^* B = 0$ and $\mathrm{nc}\{\Phi_+\} \le \dim \ell^2 = \infty$. Therefore $\Phi = \Delta A_1^*$ is also a canonical decomposition of Φ. In this case, $\ker H^*_{\breve{\Phi}} \neq \Delta H^2_{\ell^2}$. Therefore, the canonical decomposition of Φ is not unique.

Remark 3.3. Let Δ be an inner matrix function with values in $\mathcal{B}(E', E)$. Then Theorem 3.1 says that if $\dim E' < \infty$, the expression (30) satisfying the conditions (i) - (iv) in Theorem 3.1 gives $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$. We note that the condition (iv) on nc-number cannot be dropped from the assumptions of Theorem 3.1. To see this, let

$$\Delta := \frac{1}{\sqrt{2}} \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad A := \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \text{and} \quad B := 0.$$

If

$$\Phi := \Delta A^* + B = \begin{bmatrix} z & 0 \\ 1 & 0 \end{bmatrix},$$

then Φ satisfies the conditions (i), (ii), and (iii), but $\ker H^*_{\breve{\Phi}} = zH^2 \oplus H^2 \neq \Delta H^2$. Note that by Theorem 2.13, $\mathrm{nc}\{\Phi_+\} = 2$, which does not satisfy the condition on nc-number, say $\mathrm{nc}\{\Phi_+\} \le 1$.

3.2 Special cases

In this subsection we give several corollaries to Theorem 3.1.

Corollary 3.4. If $\breve{\Delta}$ is of bounded type then B in (30) is given by

$$B = \Delta_c \Delta_c^* \Phi,$$

where Δ_c is the complementary factor of Δ, with values in $\mathcal{B}(D', E)$. Moreover, if $\dim E' < \infty$, then $\dim D'$ can be computed by the formula

$$\dim D' = \mathrm{nc}\{\Delta\} - \mathrm{nc}\{\Phi_+\}.$$

Proof. Suppose that $\breve{\Delta}$ is of bounded type. Then by Corollary 2.25, $[\Delta, \Delta_c]$ is two-sided inner, where Δ_c is the complementary factor of Δ, with values in $\mathcal{B}(D', E)$. We thus have

$$I = [\Delta, \Delta_c][\Delta, \Delta_c]^* = \Delta\Delta^* + \Delta_c\Delta_c^*,$$

so that

$$B = \Phi - \Delta A^* = (I - \Delta\Delta^*)\Phi = \Delta_c\Delta_c^*\Phi.$$

This proves the first assertion. The second assertion follows at once from the facts that $\mathrm{nc}\{\Phi_+\} = \dim E' < \infty$ (by Theorem 2.13) and $\mathrm{nc}\{\Delta\} = \dim E' + \dim D'$ (by Corollary 2.16). □

The following corollary is an extension of Lemma 2.4 (the Douglas-Shapiro-Shields factorization) to strong L^2-functions.

Corollary 3.5. If Φ is a strong L^2-function with values in $\mathcal{B}(D, E)$, then the following are equivalent:

(a) The flip $\breve{\Phi}$ of Φ is of bounded type;

(b) $\Phi = \Delta A^*$ (Δ is two-sided inner) is a canonical decomposition of Φ.

Proof. The implication (a)⇒(b) follows from the proof of Theorem 3.1. For the implication (b)⇒(a), suppose $\Phi = \Delta A^*$ (Δ is two-sided inner) is a canonical decomposition of Φ. By Lemma 2.6, there exists an inner function Θ with values in $\mathcal{B}(D', E)$ such that $\ker H^*_{\breve{\Phi}} = \Theta H^2_{D'}$. Then it follows from Lemma 2.5 that $\Delta H^2_E \subseteq \ker H^*_{\breve{\Phi}} = \Theta H^2_{D'}$. Since Δ is two-sided inner, we have that by Lemma 2.2, Θ is two-sided inner, and hence the flip $\breve{\Phi}$ of Φ is of bounded type. This completes the proof. □

If Δ is an inner matrix function such that $\Delta\Delta^*\Phi$ is analytic (even though $\breve{\Delta}$ is not of bounded type) then the perturbation part B of the canonical decomposition may be also determined in terms of the complementary factor of Δ.

Corollary 3.6. Let Φ be an $n \times m$ matrix-valued H^2-function. Then the following are equivalent:

(a) $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$ for an $n \times r$ inner matrix function Δ such that $\Delta\Delta^*\Phi$ is analytic;

(b) $\Phi = \Delta A^* + \Delta_c\Delta_c^*\Phi$ is a canonical decomposition of Φ, where Δ_c is the complementary factor of Δ.

Proof. (a)⇒(b): Suppose that $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$ for an $n \times r$ inner matrix function Δ such that $\Delta\Delta^*\Phi$ is analytic. Then by the proof of Theorem 3.1, we can write

$$\Phi = \Delta A^* + B,$$

where $B = (I - \Delta\Delta^*)\Phi$. Write $\Phi \equiv [\phi_1, \phi_1, \cdots, \phi_m]$. Since $\Delta\Delta^*\Phi \in H^2_{M_{n\times m}}$ and $\Delta^*(I - \Delta\Delta^*) = 0$, it follows from Corollary A.14 (see Appendix A) and Lemma 2.7 that for each $j = 1, 2, \cdots m$,

$$(I - \Delta\Delta^*)\phi_j \in \ker \Delta^* = \Delta_c H^2_{\mathbb{C}^p},$$

which implies that $B=(I-\Delta\Delta^*)\Phi=\Delta_c D$ for some $D\in H^2_{M_{p\times m}}$. Thus

$$\Delta_c^* B=\Delta_c^*(I-\Delta\Delta^*)\Phi=D,$$

so that

$$B=\Delta_c D=\Delta_c\Delta_c^*(I-\Delta\Delta^*)\Phi=\Delta_c\Delta_c^*\Phi.$$

(b)⇒(a): Suppose that $\Phi=\Delta A^*+\Delta_c\Delta_c^*\Phi$ is a canonical decomposition of Φ. Since Φ is a matrix-valued function, it follows from Theorem 3.1 that

$$\Delta_c\Delta_c^*\Phi=B=(I-\Delta\Delta^*)\Phi,$$

so that

$$\Phi=\Delta_c\Delta_c^*\Phi+\Delta\Delta^*\Phi.$$

But since $\langle\Delta_c\Delta_c^*\phi_j,\ \Delta\Delta^*\phi_j\rangle=0$ for all $j=1,2,\cdots,m$, it follows that $\Delta\Delta^*\Phi\in H^2_{M_{n\times m}}$. This completes the proof. □

Corollary 3.7. Let Φ be an $n\times m$ matrix-valued H^2-function satisfying $\ker H^*_{\breve{\Phi}}=\Delta H^2_{\mathbb{C}^r}$ for an $n\times r$ inner matrix function Δ such that $\Delta\Delta^*$ is analytic. Then Φ can be written as

$$\Phi=\Delta A^*+\Delta_c C\quad(\text{with } C:=P_+\Delta_c^*\Phi\in H^2_{M_{p\times m}}),\tag{39}$$

where Δ_c is the complementary factor of Δ.

Proof. We claim that if $\Delta\Delta^*$ is analytic, then

$$(I-\Delta\Delta^*)\,H^2_{\mathbb{C}^n}=\Delta_c H^2_{\mathbb{C}^p}.\tag{40}$$

To see this, let $f\in\Delta_c H^2_{\mathbb{C}^p}$. Then $f=\Delta_c g$ for some $g\in H^2_{\mathbb{C}^p}$. Observe that

$$(I-\Delta\Delta^*)f=(I-\Delta\Delta^*)\Delta_c g=\Delta_c g=f,$$

which implies that $f\in(I-\Delta\Delta^*)H^2_{\mathbb{C}^n}$. Thus we have $\Delta_c H^2_{\mathbb{C}^p}\subseteq(I-\Delta\Delta^*)H^2_{\mathbb{C}^n}$. The converse inclusion follows from the proof of Corollary 3.6. This proves (40). Thus $I-\Delta\Delta^*$ is the orthogonal projection that maps from $H^2_{\mathbb{C}^n}$ onto $\Delta_c H^2_{\mathbb{C}^p}$. Therefore by the Projection Lemma in [Ni1, P. 43], we have

$$(I-\Delta\Delta^*)\,|_{H^2_{\mathbb{C}^n}}=\Delta_c P_+\Delta_c^*,$$

so that

$$\Phi=\Delta A^*+B=\Delta A^*+\Delta_c P_+\Delta_c^*\Phi,$$

as desired □

4 The Beurling degree

In this section we first give an answer to Question 1.5. Then we introduce a new notion of the “Beurling degree” and establish a connection between the Beurling degree and the spectral multiplicity of the model operator. Consequently, we give an answer to Question 1.1.

4.1 An answer to Question 1.5

We first consider Question 1.5. Question 1.5 can be rephrased as: *If Δ is an inner function with values in $\mathcal{B}(E', E)$, does there exist a strong L^2-function Φ with values in $\mathcal{B}(D, E)$ satisfying the equation*

$$\ker H^*_{\check{\Phi}} = \Delta H^2_{E'} \ ? \tag{41}$$

To closely understand an answer to Question 1.5, we examine a question whether there exists an inner function Ω satisfying $\ker H_{\Omega^*} = \Delta H^2_{E'}$ if Δ is an inner function with values in $\mathcal{B}(E', E)$. In fact, the answer to this question is negative. Indeed, if $\ker H_{\Omega^*} = \Delta H^2_{E'}$ for some inner function $\Omega \in H^\infty(\mathcal{B}(D, E))$, then by Lemma 2.7, we have $[\Omega, \Omega_c] = \Delta$, and hence $\Delta_c = 0$. Conversely, if $\Delta_c = 0$ then by again Lemma 2.7, we should have $\ker H_{\Delta^*} = \Delta H^2_{E'}$. Consequently, $\ker H_{\Omega^*} = \Delta H^2_{E'}$ for some inner function Ω if and only if $\Delta_c = 0$. Thus if

$$\Delta := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then there exists no inner function Ω such that $\ker H_{\Omega^*} = \Delta H^2$. On the other hand, we note that the solution Φ is not unique although there exists an inner function Φ satisfying the equation (41). For example, if $\Delta := \operatorname{diag}(z, 1, 1)$, then the following Φ are such solutions:

$$\Phi = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta.$$

The following theorem gives an affirmative answer to Question 1.5: indeed, we can always find a strong L^2-function Φ with values in $\mathcal{B}(D, E)$ satisfying the equation $\ker H^*_{\check{\Phi}} = \Delta H^2_{E'}$.

Theorem 4.1. (An answer to Question 1.5) Let Δ be an inner function with values in $\mathcal{B}(E', E)$. Then there exists a function Φ in $H^2_s(\mathcal{B}(D, E))$, with either $D = E'$ or $D = \mathbb{C} \oplus E'$, satisfying

$$\ker H^*_{\check{\Phi}} = \Delta H^2_{E'}.$$

Proof. If $\ker \Delta^* = \{0\}$, take $\Phi = \Delta$. Then it follows from Lemma 2.7 that

$$\ker H^*_{\check{\Phi}} = \ker H_{\Delta^*} = \Delta H^2_{E'}.$$

If instead $\ker \Delta^* \neq \{0\}$, let Δ_c be the complementary factor of Δ with values in $\mathcal{B}(E'', E)$ for some nonzero Hilbert space E''. Choose a cyclic vector $g \in H^2_{E''}$ of $S^*_{E''}$ and define

$$\Phi := \big[[z\Delta_c g], \Delta \big],$$

where $[z\Delta_c g](z) : \mathbb{C} \to E$ is given by $[z\Delta_c g](z)\alpha := \alpha z \Delta_c(z) g(z)$. Then it follows from Lemma A.7 and Corollary A.14 (see Appendix A) that Φ belongs to $H^2_s(\mathcal{B}(D, E))$, where

$D = \mathbb{C} \oplus E'$. For each $x \equiv \alpha \oplus x_0 \in D$, $f \in H^2_{E'}$, and $n = 1, 2, 3, \cdots$, we have

$$\begin{aligned}\int_{\mathbb{T}} \big\langle \Phi(z)x,\ z^n \Delta(z) f(z)\big\rangle_E dm(z) &= \int_{\mathbb{T}} \big\langle \alpha z \Delta_c(z) g(z) + \Delta(z) x_0,\ z^n \Delta(z) f(z)\big\rangle_E dm(z)\\ &= \int_{\mathbb{T}} \big\langle x_0,\ z^n f(z)\big\rangle_{E'} dm(z) \quad (\text{since } \Delta^*\Delta_c = 0)\\ &= 0.\end{aligned}$$

It thus follows from Lemma 2.5 that

$$\Delta H^2_{E'} \subseteq \ker H^*_{\breve{\Phi}}. \tag{42}$$

For the reverse inclusion, suppose $h \in \ker H^*_{\breve{\Phi}}$. Then by Lemma 2.5, we have that for each $x_0 \in E'$ and $n = 1, 2, 3, \cdots$,

$$\int_{\mathbb{T}} \big\langle \Delta(z) x_0,\ z^n h(z)\big\rangle_E dm(z) = 0,$$

which implies, by Lemma 2.5, that $h \in \ker H_{\Delta^*}$. It thus follows from Lemma 2.7 that

$$\ker H^*_{\breve{\Phi}} \subseteq \ker H_{\Delta^*} = \Delta H^2_{E'} \bigoplus \Delta_c H^2_{E''}. \tag{43}$$

Assume to the contrary that $\ker H^*_{\breve{\Phi}} \neq \Delta H^2_{E'}$. Then by (42) and (43), there exists a nonzero function $f \in H^2_{E''}$ such that $\Delta_c f \in \ker H^*_{\breve{\Phi}}$. It thus follows from Lemma 2.5 that for each $x \equiv \alpha \oplus x_0 \in D$ and $n = 1, 2, 3, \cdots$,

$$\begin{aligned}0 &= \int_{\mathbb{T}} \big\langle \Phi(z)x,\ z^n \Delta_c(z) f(z)\big\rangle_E dm(z)\\ &= \int_{\mathbb{T}} \big\langle \alpha z \Delta_c(z) g(z) + \Delta(z) x_0,\ z^n \Delta_c(z) f(z)\big\rangle_E dm(z)\\ &= \int_{\mathbb{T}} \big\langle z[g](z)\alpha,\ z^n f(z)\big\rangle_{E''} dm(z) \quad (\text{since } \Delta^*\Delta_c = 0),\end{aligned}$$

which implies that $f \in \ker H^*_{\overline{z}[\breve{g}]}$. Since g is a cyclic vector of $S^*_{E''}$, it thus follows from Lemma 2.9 that

$$f \in \big(\text{cl ran}\, H_{\overline{z}[\breve{g}]}\big)^{\perp} = \big(E^*_g\big)^{\perp} = \{0\},$$

which is a contradiction. This completes the proof. $\square$

If Δ is an $n \times r$ inner matrix function, then we can find a solution $\Phi \in H^\infty_{M_{n\times m}}$ (with $m \le r+1$) of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$.

Corollary 4.2. For a given $n \times r$ inner matrix function Δ, there exists at least a solution $\Phi \in H^\infty_{M_{n\times m}}$ (with $m \le r+1$) of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$.

Proof. If $\ker \Delta^* = \{0\}$, then this is obvious. Let $\ker \Delta^* \neq \{0\}$ and $\Delta_c \in H^\infty_{M_{n\times p}}$ be the complementary factor of Δ. Then by Lemma 2.7, $1 \le p \le n-r$. For $j = 1, 2, \cdots, p$, put

$$g_j := e^{\frac{1}{z-\alpha_j}},$$

where α_j are distinct points in the interval $[2,3]$. Then it is known that (cf. [Ni1, P. 55])

$$g := \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_p \end{bmatrix} \in H^\infty_{\mathbb{C}^p}$$

is a cyclic vector of $S^*_{\mathbb{C}^p}$. Put $\Phi := \big[[z\Delta_c g], \Delta\big]$. Then by Lemma A.7 (see Appendix A), we have $\Phi \in H^\infty_{M_{n\times(r+1)}}$. The same argument as the proof of Theorem 4.1 gives the result. $\square$

Corollary 4.3. If Δ is an inner function with values in $\mathcal{B}(E', E)$, then there exists a function $\Phi \in L^2_s(\mathcal{B}(D,E))$ (with $D = E'$ or $D = \mathbb{C} \oplus E'$) such that $\Phi \equiv \Delta A^* + B$ is the BLH-canonical decomposition of Φ.

Proof. By Theorem 4.1, there exists a function $\Phi \in H^2_s(\mathcal{B}(D,E))$ such that $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$, with $D = E'$ or $D = \mathbb{C} \oplus E'$. If we put $A := \Phi^*\Delta$ and $B := \Phi - \Delta A^*$, then by the proof of the first assertion of Theorem 3.1, $\Phi = \Delta A^* + B$ is the BLH-canonical decomposition of Φ. $\square$

Remark 4.4. In view of Corollary 4.2, it is reasonable to ask whether such a solution $\Phi \in L^2_{M_{n\times m}}$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$ (Δ an $n \times r$ inner matrix function) exists for each $m = 1, 2, \cdots$ even though it exists for some m. For example, let

$$\Delta := \frac{1}{\sqrt{2}} \begin{bmatrix} z \\ 1 \end{bmatrix}. \tag{44}$$

Then, by Corollary 4.2, there exists a solution $\Phi \in L^2_{M_{2\times m}}$ ($m = 1$ or 2) of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2$. For $m = 2$, let

$$\Phi := \begin{bmatrix} z & za \\ 1 & -a \end{bmatrix} \in H^\infty_{M_2}, \tag{45}$$

where $a \in H^\infty$ is such that $\overline{a}$ is not of bounded type. Then a direct calculation shows that $\ker H^*_{\breve{\Phi}} = \ker H_{\Phi^*} = \Delta H^2$. We may then ask how about the case $m = 1$. In this case, the answer is affirmative. To see this, let

$$\Psi := \begin{bmatrix} z + za \\ 1 - a \end{bmatrix} \in H^\infty_{M_{2\times 1}},$$

where $a \in H^\infty$ is such that $\overline{a}$ is not of bounded type. Then a direct calculation shows that $\ker H_{\Psi^*} = \Delta H^2$. Therefore, if Δ is given by (44), then we may assert that there exists a solution $\Phi \in L^2_{M_{n\times m}}$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2$ for each $m = 1, 2$. However, this assertion is not true in general, i.e., a solution exists for some m, but may not exist for another $m_0 < m$. To see this, let

$$\Delta := \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in H^\infty_{M_{4\times 3}}.$$

Then Δ is inner. We will show that there exists no solution $\Phi \in L^2_{M_{4\times 1}}$ (i.e., the case $m = 1$) of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^3}$. Assume to the contrary that $\Phi \in L^2_{M_{4\times 1}}$ is a solution of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^3}$. By Theorem 3.1, Φ can be written as

$$\Phi = \Delta A^* + B,$$

where $A \in H^2_{M_{1\times 3}}$ is such that Δ and A are right coprime. But since $\widetilde{\Delta} H^2_{\mathbb{C}^4} = zH^2 \oplus zH^2 \oplus H^2$, it follows that

$$\widetilde{\Delta} H^2_{\mathbb{C}^4} \bigvee \widetilde{A} H^2 \neq H^2_{\mathbb{C}^3},$$

which implies that Δ and A are not right coprime, a contradiction. Therefore we cannot find any solution Φ, in $L^2_{M_{4\times 1}}$ (the case $m = 1$), of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^3}$. By contrast, if $m = 2$, then we can find a solution $\Phi \in L^2_{M_{4\times 2}}$. Indeed, let

$$\Phi := \begin{bmatrix} z & 0 \\ 0 & z \\ 0 & 0 \\ a & 0 \end{bmatrix},$$

where $a \in H^\infty$ is such that $\overline{a}$ is not of bounded type. Then $\ker H_{\Phi^*} = zH^2 \oplus zH^2 \oplus H^2 \oplus \{0\} = \Delta H^2_{\mathbb{C}^3}$. Thus we obtain a solution for $m = 2$ although there exists no solution for $m = 1$.

4.2 The Beurling degree and the spectral multiplicity: An answer to Question 1.1

Let Δ be an inner function with values in $\mathcal{B}(E', E)$. In view of Remark 4.4, we may ask how to determine a possible dimension of D for which there exists a solution $\Phi \in L^2_s(\mathcal{B}(D, E))$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$. In fact, if we have a solution $\Phi \in L^2_s(\mathcal{B}(D, E))$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$, then a solution $\Psi \in L^2_s(D', E))$ also exists if D' is a separable complex Hilbert space containing D: indeed, if $\mathbf{0}$ denotes the zero operator in $\mathcal{B}(D' \ominus D, E)$ and $\Psi := [\Phi, \mathbf{0}]$, then it follows from Lemma 2.5 that $\ker H^*_{\breve{\Phi}} = \ker H^*_{\breve{\Psi}}$. Thus we would like to ask what is the infimum of $\dim D$ such that there exists a solution $\Phi \in L^2_s(\mathcal{B}(D, E))$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$. To answer this question, we introduce a notion of the "Beurling degree" for an inner function, by employing a canonical decomposition of strong L^2-functions induced by the given inner function.

Definition 4.5. Let Δ be an inner function with values in $\mathcal{B}(E', E)$. Then the *Beurling degree* of Δ, denoted by $\deg_B(\Delta)$, is defined by

$$\begin{aligned}\deg_B(\Delta) := \inf\Big\{\dim D \in \mathbb{Z}_+ \cup \{\infty\} : \exists \text{ a pair } (A, B) \text{ such that}\\ \Phi = \Delta A^* + B \text{ is a canonical decomposition of } \Phi \in L^2_s(B(D, E))\Big\}\end{aligned}$$

Note. By Corollary 4.3, $\deg_B(\Delta)$ is well-defined: indeed, $1 \le \deg_B(\Delta) \le 1 + \dim E'$. In particular, if $E' = \{0\}$, then $\deg_B(\Delta) = 1$. Also if Δ is a unitary operator then clearly, $\deg_B(\Delta) = 1$.

We are ready for:

Theorem 4.6. (The Beurling degree and the spectral multiplicity) Given an inner function Δ with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$, let $T := S_E^*|_{\mathcal{H}(\Delta)}$. Then

$$\mu_T = \deg_B(\Delta). \tag{46}$$

Proof. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$. We first claim that

$$\begin{aligned}\deg_B(\Delta) = \inf\big\{\dim D :\ \ker H^*_{\breve{\Phi}} = \Delta H^2_{E'} \text{ for some } \Phi \in L^2_s(\mathcal{B}(D, E))\\ \text{with } D \neq \{0\}\big\}.\end{aligned} \tag{47}$$

To see this, let Δ be an inner function with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$. Suppose that $\Phi = \Delta A^* + B$ is a canonical decomposition of Φ in $L^2_s(\mathcal{B}(D, E))$. Then by the uniqueness of Δ in Theorem 3.1, we have

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}, \tag{48}$$

which implies

$$\begin{aligned}\deg_B(\Delta) \ge \inf\big\{\dim D :\ \ker H^*_{\breve{\Phi}} = \Delta H^2_{E'} \text{ for some } \Phi \in L^2_s(\mathcal{B}(D, E))\\ \text{with } D \neq \{0\}\big\}.\end{aligned} \tag{49}$$

For the reverse inequality of (49), suppose $\Phi \in L^2_s(\mathcal{B}(D, E))$ satisfies $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$. Then by the same argument as in the proof of the first assertion of Theorem 3.1,

$$\Phi = \Delta A^* + B \quad (A := \Phi^*\Delta \text{ and } B := \Phi - \Delta A^*)$$

is a canonical decomposition of Φ, and hence we have the reverse inequality of (49). This proves the claim (47). We will next show that

$$\deg_B(\Delta) \le \mu_T. \tag{50}$$

If $\mu_T = \infty$, then (50) is trivial. Suppose $p \equiv \mu_T < \infty$. Then there exists a subset $G = \{g_1, g_2, \cdots g_p\} \subseteq H^2_E$ such that $E^*_G = \mathcal{H}(\Delta)$. Put

$$\Psi := z[G].$$

Then by Lemma A.7 (see Appendix A), $\Psi \in H^2_s(\mathcal{B}(\mathbb{C}^p, E))$. It thus follows from Lemma 2.9 that

$$\mathcal{H}(\Delta) = E^*_G = \text{cl ran}\, H_{\overline{z}[\breve{G}]} = \text{cl ran}\, H_{\breve{\Psi}},$$

which implies $\ker H^*_{\breve{\Psi}} = \Delta H^2_{E'}$. Thus by (47), $\deg_B(\Delta) \le p = \mu_T$, which proves (50). For the reverse inequality of (50), suppose that $r \equiv \dim E' < \infty$, Write $m_0 \equiv \deg_B(\Delta)$. Then it follows from Theorem 4.1 and (47) that $m_0 \le r + 1 < \infty$ and there exists a function $\Phi \in L^2_s(\mathcal{B}(\mathbb{C}^{m_0}, E))$ such that

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}. \tag{51}$$

Now let

$$G := \Phi_+ - \widehat{\Phi}(0).$$

Thus we may write $G = zF$ for some $F \in H^2_s(\mathcal{B}(\mathbb{C}^{m_0}, E))$. Then by Lemma 2.3 and Lemma 2.9, we have that

$$E^*_{\{F\}} = \text{cl ran}\, H_{\breve{G}} = \left(\ker H^*_{\breve{\Phi}}\right)^{\perp} = \mathcal{H}(\Delta),$$

which implies $\mu_T \le m_0 = \deg_B(\Delta)$. This completes the proof. □

Corollary 4.7. Let $T := S^*_E|_{\mathcal{H}(\Delta)}$. If $\text{rank}\,(I - T^*T) < \infty$, then

$$\mu_T = \deg_B(\Delta).$$

Proof. This follows at once from Theorem 4.6 together with the observation that if Δ is an inner function with values in $\mathcal{B}(E', E)$, then $\dim\, E' \le \dim\, E = \text{rank}\,(I - T^*T) < \infty$, where the second equality comes from the Model Theorem (cf. p.7, paragraph containing (4)). □

Remark 4.8. We conclude with some observations on Theorem 4.6.

(a) From a careful analysis of the proof of Theorem 4.6, we can see that (50) holds in general without the assumption "$\dim\, E' < \infty$": more concretely, given an inner function Δ with values in $\mathcal{B}(E', E)$, if $T := S^*_E|_{\mathcal{H}(\Delta)}$, then

$$\deg_B(\Delta) \le \mu_T.$$

(b) From Remark 4.4 and (47), we see that if

$$\Delta := \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\deg_B(\Delta) = 2$. Let $T := S^*_{\mathbb{C}^4}|_{\mathcal{H}(\Delta)}$. Observe that

$$\mathcal{H}(\Delta) = \mathcal{H}(z) \oplus \mathcal{H}(z) \oplus \{0\} \oplus H^2.$$

Since $\mathcal{H}(z) \oplus \mathcal{H}(z)$ has no cyclic vector, we must have $\mu_T \neq 1$. In fact, if we put

$$f = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

where $\overline{a}$ is not of bounded type, then $E^*_{\{f,g\}} = \mathcal{H}(\Delta)$, which implies $\mu_T = 2$. This illustrates Theorem 4.6.

We now give an answer to Question 1.1.

Corollary 4.9. (An answer to Question 1.1) Suppose Δ is an inner function with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$. If $\Phi = \Delta A^* + B$ is a canonical decomposition of Φ in $L^2_s(\mathcal{B}(D, E))$, we define a function F by

$$F(z) := \overline{z}\big(\Phi_+(z) - \hat{\Phi}(0)\big).$$

We then have

$$E^*_{\{F\}} = \mathcal{H}(\Delta). \tag{52}$$

Proof. If $\Phi = \Delta A^* + B$ is a canonical decomposition of Φ in $L^2_s(\mathcal{B}(D, E))$, then by Theorem 3.1 we have

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}.$$

It thus follows from the proof of Theorem 4.6 that $E^*_{\{F\}} = \mathcal{H}(\Delta)$. $\square$

Remark 4.10. If Δ is two-sided inner with values in $\mathcal{B}(E)$, and therefore $B = 0$ in Corollary 4.9, then (52) can be obtained by the equivalence (b) $\Leftrightarrow$ (c) in Lemma 2.4 (cf. [FB, Theorem 4.7.1]): indeed, if $F(z) = \overline{z}\big(\Phi_+(z) - \hat{\Phi}(0)\big) = S^*_E \Phi_+$, then

$$E^*_{\{F\}} = \bigvee_{n=0}^{\infty} S^{*n}_E S^*_E \Phi_+ D = \bigvee_{n=1}^{\infty} S^{*n}_E \mathbb{P}_+ \Phi D = \mathcal{H}(\Delta).$$

5 Multiplicity free model operators

In this section, we consider Question 1.6: *Let $T := S^*_E|_{\mathcal{H}(\Delta)}$. For which inner function Δ with values in $\mathcal{B}(E', E)$, does it follow that*

$$T \textit{ is multiplicity-free, i.e., } \mu_T = 1?$$

If $\dim E' < \infty$, then in the viewpoint of Theorem 4.6, Question 1.6 is equivalent to the following: if T is the truncated backward shift $S^*_E|_{\mathcal{H}(\Delta)}$, which inner function Δ guarantees that $\deg_B(\Delta) = 1$? To answer Question 1.6, in Subsection 5.1, we consider the notion of the characteristic scalar inner function of operator-valued inner functions having a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e \equiv \{z : 1 < |z| \leq \infty\}$. In Subsection 5.2, we give an answer to Question 1.6.

5.1 Characteristic scalar inner functions

In this subsection we consider the characteristic scalar inner functions of operator-valued inner functions, by using the results of Subsection 2.6. The characteristic scalar inner function of a two-sided inner matrix function has been studied in [Hel], [SFBK] and [CHL3].

Let $\Delta \in H^\infty(\mathcal{B}(D,E))$ have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then by Lemma 2.28, there exists a scalar inner function δ such that $\delta H^2_E \subseteq \ker H_{\Delta^*}$. Put $G := \delta\Delta^* \in H^\infty(\mathcal{B}(E,D))$. If further Δ is inner then $G\Delta = \delta I_D$, so that

$$\text{g.c.d.}\left\{\delta :\ G\Delta = \delta I_D \text{ for some } G \in H^\infty(\mathcal{B}(E,D))\right\}$$

always exists. Thus the following definition makes sense.

Definition 5.1. Let Δ be an inner function with values in $\mathcal{B}(D,E)$. If Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, define

$$m_\Delta := \text{g.c.d.}\left\{\delta :\ G\Delta = \delta I_D \text{ for some } G \in H^\infty(\mathcal{B}(E,D))\right\},$$

where δ is a scalar inner function. The inner function m_Δ is called the *characteristic scalar inner function* of Δ.

The notion of m_Δ arises in the Sz.-Nagy and Foiaş theory of contraction operators T of class C_0 (cf. p.29): the *minimal annihilator* m_T of the C_0-contraction operator T amounts to our m_{Θ_T}, where Θ_T is the characteristic function of T (cf. [Ber], [SFBK], [CHL3]).

We would like to remark that

$$\text{g.c.d.}\left\{\delta :\ G\Delta = \delta I_D \text{ for some } G \in H^\infty(\mathcal{B}(E,D))\right\} \tag{53}$$

may exist for some inner function Δ having no meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. To see this, let

$$\Delta := \begin{bmatrix} f \\ g \end{bmatrix} \quad (f, g \in H^\infty), \tag{54}$$

where f and g are given in Example 2.1. Then Δ is an inner function. Since $\breve{f}$ is not of bounded type it follows from Corollary 2.32 that Δ has no meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. On the other hand, since Δ is inner, by the Complementing Lemma, there exists a function $G \in H^\infty_{M_{1\times 2}}$ such that $G\Delta$ is a scalar inner function, so that (53) exists.

If Δ is an $n \times n$ square inner matrix function then we may write $\Delta \equiv [\theta_{ij}\bar{b}_{ij}]$, where θ_{ij} is inner and θ_{ij} and $b_{ij} \in H^\infty$ are coprime for each $i, j = 1, 2, \cdots, n$. In Lemma 4.12 of [CHL3], it was shown that

$$m_\Delta = \text{l.c.m.}\left\{\theta_{ij} : i, j = 1, 2, \cdots, n\right\}.$$

In this section, we examine the cases of general inner functions that have meromorphic pseudo-continuations of bounded type in $\mathbb{D}^e$.

On the other hand, if $\Phi \in H^\infty(\mathcal{B}(D,E))$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then by Lemma 2.28, $\delta H^2_E \subseteq \ker H_{\Phi^*}$ for some scalar inner function δ. Thus we may also define

$$\omega_\Phi := \text{g.c.d.}\left\{\delta : \ \delta H^2_E \subseteq \ker H_{\Phi^*} \text{ for some scalar inner function } \delta\right\}.$$

If Δ is an inner function with values in $\mathcal{B}(D,E)$ and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then ω_Δ is called the *pseudo-characteristic scalar inner function* of Δ. Note that m_Δ is an inner divisor of ω_Δ. If further Δ is two-sided inner, then

$$\delta H^2_E \subseteq \ker H_{\Delta^*} \iff G \equiv \delta\Delta^* \in H^\infty(\mathcal{B}(E)) \iff G\Delta = \Delta G = \delta I_E, \tag{55}$$

which implies $m_\Delta = \omega_\Delta$.

The following lemma shows a way to determine ω_Φ more easily.

Lemma 5.2. Let D and E be separable complex Hilbert spaces and let $\{d_j\}$ and $\{e_i\}$ be orthonormal bases of D and E, respectively. Suppose $\Phi \in H^\infty(\mathcal{B}(D,E))$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. In view of Proposition 2.30, we may write

$$\phi_{ij} \equiv \langle \Phi d_j,\ e_i\rangle_E = \theta_{ij}\overline{a}_{ij},$$

where θ_{ij} is inner and θ_{ij} and $a_{ij} \in H^\infty$ are coprime. Then we have

$$\omega_\Phi = \text{l.c.m.}\left\{\theta_{ij} : i,j = 1,2,\cdots,\right\}.$$

Proof. Let $\Phi \in H^\infty(\mathcal{B}(D,E))$ have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. By Lemma 2.28, we may write $\Phi = \theta A^*$ for some $A \in H^\infty(\mathcal{B}(E,D))$ and a scalar inner function θ. Also by an analysis of the proof of Proposition 2.30, we can see that $\theta_0 \equiv \text{l.c.m.}\left\{\theta_{ij} : i,j = 1,2,\cdots,\right\}$ is an inner divisor of θ. Thus by Lemma 2.28, θ_0 is an inner divisor of ω_Φ. Since $\Phi \in H^\infty(\mathcal{B}(D,E))$, it follows that for all $f \in H^2_E$ and $j,n \geq 1$,

$$\langle \Phi(z)d_j,\ z^n\theta_0(z)f(z)\rangle_E \in L^2. \tag{56}$$

On the other hand, for all $f \in H^2_E$,

$$f(z) = \sum_{i\geq 1}\langle f(z),\ e_i\rangle e_i \equiv \sum_{i\geq 1} f_i(z)e_i \quad \text{for almost all } z \in \mathbb{T} \quad (f_i \in H^2). \tag{57}$$

Since $\theta_0 = \text{l.c.m.}\left\{\theta_{ij} : i,j = 1,2,\cdots,\right\}$, it follows from (56) and (57) that for all $j,n \geq 1$,

$$\begin{aligned}\int_{\mathbb{T}}\langle \Phi(z)d_j,\ z^n\theta_0(z)f(z)\rangle_E dm(z) &= \int_{\mathbb{T}} \overline{z}^n \sum_{i\geq 1}\overline{f_i}(z)\overline{\theta_0}(z)\theta_{ij}(z)\overline{a}_{ij}(z)dm(z)\\ &= 0,\end{aligned}$$

where the last equality follows from the fact that $\overline{z}^n \sum_{i\geq 1} \overline{f_i}(z)\overline{\theta_0}(z)\theta_{ij}(z)\overline{a}_{ij}(z) \in L^2 \ominus H^2$. Since $\{d_i\}$ is an orthonormal basis for D, it follows from Fatou's Lemma that for all $x \in D$ and $n = 1, 2, 3, \cdots$,

$$\int_{\mathbb{T}} \langle \Phi(z)x,\ z^n\theta_0(z)f(z)\rangle_E dm(z) = 0.$$

Thus by Lemma 2.5, $\theta_0 H^2_E \subseteq \ker H_{\Phi^*}$, so that ω_Φ is an inner divisor of θ_0, and therefore $\theta_0 = \omega_\Phi$. This complete the proof. □

Corollary 5.3. Let Δ be a two-sided inner matrix function. Thus, in view of Corollary 2.32, we may write $\Delta \equiv \left[\theta_{ij}\overline{b}_{ij}\right]$, where θ_{ij} is an inner function and θ_{ij} and $b_{ij} \in H^\infty$ are coprime for each $i, j = 1, 2, \cdots$. Then

$$\omega_\Delta = m_\Delta = \text{l.c.m.}\left\{\theta_{ij} : i, j = 1, 2, \cdots,\right\}.$$

Proof. Immediate from Lemma 5.2. □

Remark 5.4. If Δ is not two-sided inner then Corollary 5.3 may fail. To see this, let

$$\Delta := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ z \end{bmatrix}.$$

Then by Corollary 2.32, Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. It thus follows from Lemma 5.2 that $\omega_\Delta = z$. On the other hand, let $G := \begin{bmatrix}\sqrt{2} & 0\end{bmatrix}$. Then $G\Delta = 1$, so that $m_\Delta = 1 \neq z = \omega_\Delta$. Note that, by Corollary 5.3,

$$[\Delta, \Delta_c] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} \quad \text{and} \quad m_{[\Delta,\Delta_c]} = \omega_{[\Delta,\Delta_c]} = z.$$

The following lemma shows that Remark 5.4 is not an accident.

Lemma 5.5. Let Δ be an inner function and have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then

$$m_{[\Delta,\Delta_c]} = \omega_{[\Delta,\Delta_c]} = \omega_\Delta$$

and Δ_c has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$: in this case, ω_{Δ_c} is an inner divisor of ω_Δ.

Proof. Suppose that Δ is an inner function with values in $\mathcal{B}(D, E)$ and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then it follows from Corollary 2.25 and Lemma 2.27 that $[\Delta, \Delta_c]$ is two-sided inner. On the other hand, it follows from Lemma 2.7 that

$$\ker H_{\Delta^*} = [\Delta, \Delta_c]H^2_{D\oplus D'} = \ker H_{[\Delta,\Delta_c]^*}.$$

Thus by Lemma 2.28, $[\Delta, \Delta_c]$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$ and $m_{[\Delta,\Delta_c]} = \omega_{[\Delta,\Delta_c]} = \omega_\Delta$. This proves the first assertion. Since $[\Delta, \Delta_c]$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, it follows from Lemma 2.28 that Δ_c has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. On the other hand, by Lemma 2.7(b), $\Delta_c^*\Delta = 0$. Thus, by Lemma 2.7(a), $\Delta H^2_D \subseteq \ker \Delta_c^* = \Delta_{cc}H^2_{D''}$, which implies that Δ_{cc} is a left inner divisor of Δ. Thus, $[\Delta_{cc}, \Delta_c]$ is a left inner divisor of $[\Delta, \Delta_c]$, so that $\omega_{\Delta_c} = \omega_{[\Delta_{cc},\Delta_c]}$ is an inner divisor of $\omega_\Delta = \omega_{[\Delta,\Delta_c]}$. This proves the second assertion. □

5.2 An answer to Question 1.6

In this subsection we give an answer to Question 1.6. This is accomplished by several lemmas.

Lemma 5.6. Let $\Phi \in H^\infty(\mathcal{B}(D,E))$ have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then for each cyclic vector g of S_D^*,

$$\ker H^*_{[z\Phi g]^\smile} = \ker \Phi^*,$$

where $[z\Phi g]^\smile$ denotes the flip of $[z\Phi g]$.

Proof. Let $\Phi \in H^\infty(\mathcal{B}(D,E))$ have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then by Lemma 2.28, there exists a scalar inner function δ such that $\delta H_E^2 \subseteq \ker H_{\Phi^*}$. We thus have

$$\delta\Phi^* h \in H_D^2 \quad \text{for any } h \in H_E^2. \tag{58}$$

Let g be a cyclic vector of S_D^* and $h \in \ker H^*_{[z\Phi g]^\smile}$. Then it follows from Lemma 2.5 that for all $n = 1, 2, 3 \cdots$,

$$\begin{aligned}
0 &= \int_{\mathbb{T}} \big\langle z\Phi(z)g(z),\ z^n\delta(z)h(z)\big\rangle_E dm(z)\\
&= \int_{\mathbb{T}} \big\langle S_D^{*(n-1)}g(z),\ \delta(z)\Phi^*(z)h(z)\big\rangle_D dm(z)\\
&= \big\langle S_D^{*(n-1)}g(z),\ \delta(z)\Phi^*(z)h(z)\big\rangle_{L_D^2},
\end{aligned}$$

which implies, by (58), that $\delta\Phi^* h = 0$, and hence $h \in \ker \Phi^*$. We thus have

$$\ker H^*_{[zg_\Phi]^\smile} \subseteq \ker \Phi^* .$$

The reverse inclusion follows at once from Lemma 2.5. This completes the proof. $\square$

Lemma 5.7. Let $\Phi \in H^\infty(\mathcal{B}(D,E))$ have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then for each cyclic vector g of S_D^*,

$$E^*_{\{\Phi g\}} = \mathcal{H}((\Phi^i)_c), \tag{59}$$

where Φ^i denotes the inner part in the inner-outer factorization of Φ. Hence, in particular, $S_E^*|_{\mathcal{H}((\Phi^i)_c)}$ is multiplicity-free.

Proof. Let $\Phi \equiv \Phi^i\Phi^e$ be the inner-outer factorization of Φ. Since Φ^e has dense range, $(\Phi^e)^*$ is one-one, so that $\ker \Phi^* = \ker (\Phi^i)^*$. It thus follows from Lemma 2.7, Lemma 2.9 and Lemma 5.6 that

$$E^*_{\{\Phi g\}} = \big(\ker H^*_{[z\Phi g]^\smile}\big)^\perp = \big(\ker \Phi^*\big)^\perp = \mathcal{H}((\Phi^i)_c),$$

which proves (59). This completes the proof. $\square$

As a straightforward consequence of Lemma 5.7, we have:

Corollary 5.8. If Φ is a two-sided inner function with values in $\mathcal{B}(E)$ and g is a cyclic vector for S_E^*, then Φg is also a cyclic vector for S_E^*.

The following corollary is a matrix-valued version of Lemma 5.7.

Corollary 5.9. Let Δ be an $n \times r$ inner matrix function such that $\breve{\Delta}$ is of bounded type. If g is a cyclic vector of $S_{\mathbb{C}^r}^*$, then $E_{\{\Delta g\}}^* = \mathcal{H}(\Delta_c)$.

Proof. It follows from Corollary 2.32 and Lemma 5.7. □

The following lemma shows that the flip of the adjoint of an inner function may be an outer function.

Lemma 5.10. Let Δ be an inner function with values in $\mathcal{B}(D, E)$, with its complementary factor Δ_c with values in $\mathcal{B}(D', E)$. If $\dim D' < \infty$, then $\widetilde{\Delta_c}$ is an outer function.

Proof. If $D' = \{0\}$, then this is trivial. Suppose that $D' = \mathbb{C}^p$ for some $p \geq 1$. Write

$$\widetilde{\Delta_c} \equiv (\widetilde{\Delta_c})^i(\widetilde{\Delta_c})^e \quad \text{(inner-outer factorization)}, \tag{60}$$

where $(\widetilde{\Delta_c})^i \in H^\infty_{M_{p\times q}}$ and $(\widetilde{\Delta_c})^e \in H^\infty(\mathcal{B}(E, \mathbb{C}^q))$ for some $q \leq p$. It thus follows that

$$q = \text{Rank}\,(\widetilde{\Delta_c})^i \geq \text{Rank}\,\widetilde{\Delta_c} = \max_{\zeta\in\mathbb{D}} \text{rank}\,\widetilde{\Delta_c}(\zeta)\widetilde{\Delta_c}(\zeta)^* = p,$$

which implies $p = q$. Since $(\widetilde{\Delta_c})^i \in H^\infty_{M_p}$ is two-sided inner, by the Complementing Lemma, there exists a function $G \in H^\infty_{M_p}$ and a scalar inner function θ such that $G(\widetilde{\Delta_c})^i = \theta I_p$. Thus by (60), we have $G\widetilde{\Delta_c} = \theta I_p(\widetilde{\Delta_c})^e$, and hence we have

$$\breve{\theta} I_E \Delta_c \widetilde{G} = \widetilde{\overline{\theta} I_p G \widetilde{\Delta_c}} = \widetilde{(\widetilde{\Delta_c})^e} \in H^\infty(\mathcal{B}(\mathbb{C}^p, E)).$$

Thus we have

$$\breve{\theta} I_E \Delta_c \widetilde{G} H^2_{\mathbb{C}^p} \subseteq H^2_E. \tag{61}$$

It thus follows from Lemma 2.7 and (61) that

$$\Delta_c \breve{\theta} I_p \widetilde{G} H^2_{\mathbb{C}^p} = \breve{\theta} I_E \Delta_c \widetilde{G} H^2_{\mathbb{C}^p} \subseteq \ker \Delta^* = \Delta_c H^2_{\mathbb{C}^p},$$

which implies $\breve{\theta} I_p \widetilde{G} H^2_{\mathbb{C}^p} \subseteq H^2_{\mathbb{C}^p}$. We thus have $\breve{\theta} I_p \widetilde{G} \in H^\infty_{M_p}$, so that $\overline{\theta} I_p G \in H^\infty_{M_p}$. Therefore we may write $G = \theta I_p G_1$ for some $G_1 \in H^\infty_{M_p}$. It thus follows that

$$\theta I_p = G(\widetilde{\Delta_c})^i = \theta I_p G_1 (\widetilde{\Delta_c})^i,$$

which gives that $G_1(\widetilde{\Delta_c})^i = I_p$. Therefore we have

$$H^2_{\mathbb{C}^p} = \widetilde{(\widetilde{\Delta_c})^i}\widetilde{G_1} H^2_{\mathbb{C}^p} \subseteq \widetilde{(\widetilde{\Delta_c})^i} H^2_{\mathbb{C}^p}, \tag{62}$$

which implies that $\widetilde{(\widetilde{\Delta_c})^i}$ is a unitary matrix, and so is $(\widetilde{\Delta_c})^i$. Thus, $\widetilde{\Delta_c}$ is an outer function. This completes the proof. □

Corollary 5.11. If Δ is an inner matrix function, then Δ_c^t is an outer function.

Proof. Immediate from Lemma 5.10. $\square$

Remark 5.12. Let $T := S^*_{\mathbb{C}^n}|_{\mathcal{H}(\Delta)}$ for some non-square inner matrix function Δ. Then Corollary 5.9 shows that if $\Delta = \Omega_c$ for an inner matrix function Ω such that $\breve{\Omega}$ is of bounded type, then T is multiplicity-free. However, the converse is not true in general, i.e., the condition "multiplicity-free" does not guarantee that $\Delta = \Omega_c$. To see this, let $\Delta := [0\ z]^t$. Then Δ is inner and $\breve{\Delta}$ is of bounded type. Since $\Delta^t = [0\ z]$ is not an outer function, it follows from Corollary 5.11 that $\Delta \neq \Omega_c$ for any inner matrix function. Let $f := (a\ 1)^t$ ($\overline{a}$ is not of bounded type). Then $E_f^* = \mathcal{H}(\Delta)$, so that T is multiplicity-free.

Lemma 5.13. Let Δ be an inner function and have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. If $\widetilde{\Delta}$ is an outer function and $\ker \Delta^* = \{0\}$, then Δ is a unitary operator.

Proof. Let Δ be an inner function with values in $\mathcal{B}(D, E)$ and have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then by Lemma 2.27, $\breve{\Delta}$ is of bounded type. Suppose that $\widetilde{\Delta}$ is an outer function and $\ker \Delta^* = \{0\}$. Then by Lemma 2.7, Corollary 2.25 and Lemma 2.27, Δ is two-sided inner, and so is $\widetilde{\Delta}$. Thus Δ is a unitary operator, as desired. $\square$

The following lemma is a key idea for an answer to Question 1.6.

Lemma 5.14. Let Δ be an inner function and have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. If $\widetilde{\Delta}$ is an outer function, then

$$\Delta_{cc} = \Delta.$$

Proof. Let Δ be an inner function with values in $\mathcal{B}(D, E)$ and have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Also, suppose $\widetilde{\Delta}$ is an outer function. If $\ker \Delta^* = \{0\}$, then the result follows at one from Lemma 5.13. Assume that $\ker \Delta^* \neq \{0\}$. By Lemma 2.27, $\breve{\Delta}$ is of bounded type, so that by Corollary 2.25, $[\Delta, \Delta_c]$ is a two-sided inner function with values in $\mathcal{B}(D \oplus D', E)$ for some nonzero Hilbert space D'. We now claim that

$$\Delta = \Delta_{cc}\Omega \quad \text{for a two-sided inner function } \Omega \text{ with values in } \mathcal{B}(D). \tag{63}$$

Since Δ_{cc} is a left inner divisor of Δ (cf. the Proof of Lemma 5.5), we may write

$$\Delta = \Delta_{cc}\Omega \tag{64}$$

for an inner function Ω with values in $\mathcal{B}(D, D'')$. Assume to the contrary that Ω is not two-sided inner. Since Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, it follows from Lemma 2.28 that

$$\theta H_E^2 \subseteq \ker H_{\Delta^*} = \ker H_{\Omega^* \Delta_{cc}^*}$$

for some scalar inner function θ. Thus $\Omega^*\Delta_{cc}^*\theta H_E^2 \subseteq H_D^2$. In particular, we have

$$\Omega^*\theta H_{D''}^2 = \Omega^*\Delta_{cc}^*\theta\Delta_{cc}H_{D''}^2 \subseteq H_D^2,$$

and hence $\theta H_{D''}^2 \subseteq \ker H_{\Omega^*}$, which implies, by Lemma 2.28, that Ω has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Thus by Lemma 2.27, $\breve{\Omega}$ is of bounded type. It thus follows from Lemma 2.7 that

$$[\Omega, \Omega_c] \text{ is two-sided inner,}$$

where Ω_c is the complementary factor of Ω, with values in $\mathcal{B}(D_1, D'')$ for some nonzero Hilbert space D_1. On the other hand, it follows from (64) that for all $f \in H_{D_1}^2$,

$$[\Delta, \Delta_c]^*\Delta_{cc}\Omega_c f = \begin{bmatrix} \Omega^*\Omega_c f \\ \Delta_c^*\Delta_{cc}\Omega_c f \end{bmatrix} = 0,$$

which implies that $D_1 = \{0\}$, a contradiction. This proves (63). Thus we may write

$$\widetilde{\Delta} = \widetilde{\Omega}\widetilde{\Delta_{cc}} \tag{65}$$

for a two-sided inner function $\widetilde{\Omega}$ with values in $\mathcal{B}(D)$. Since $\widetilde{\Delta}$ is an outer function and $\widetilde{\Omega}$ is two-sided inner, it follows from (65) that $\widetilde{\Omega}$ is a unitary operator, and so is Ω. This completes the proof. $\square$

Lemma 5.14 may fail if the condition "Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$" is dropped. To see this, let

$$\Delta := \begin{bmatrix} f \\ g \\ 0 \end{bmatrix},$$

where f and g are given in Example 2.1. Then $\widetilde{\Delta}$ is an outer function. A straightforward calculation shows that

$$\Delta_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \Delta_{cc} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \Delta.$$

Note that $\breve{\Delta}$ is not of bounded type. Thus, by Corollary 2.32, Δ has no meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$.

We are ready to give an answer to Question 1.6.

Theorem 5.15. (An answer to Question 1.6) Let $T := S_E^*|_{\mathcal{H}(\Delta)}$. If Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$ and $\widetilde{\Delta}$ is an outer function, then T is multiplicity-free.

Proof. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$. Suppose that Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$ and $\widetilde{\Delta}$ is an outer function. If $\ker \Delta^* = \{0\}$, then by Lemma 5.13, Δ is a unitary operator, so that T is multiplicity-free. If instead $\ker \Delta^* \neq \{0\}$, then by Lemma 5.5, Δ_c has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Since $\widetilde{\Delta}$ is an outer function it follows from Lemma 5.14 that $\Delta = \Delta_{cc}$. Applying Lemma 5.7 with $\Phi \equiv \Delta_c$, we can see that T has a cyclic vector, i.e., T is multiplicity-free. □

The following corollary is an immediate consequence of Theorem 5.15.

Corollary 5.16. Let $T := S_{\mathbb{C}^n}^*|_{\mathcal{H}(\Delta)}$ for an inner matrix function Δ whose flip $\breve{\Delta}$ is of bounded type. If Δ^t is an outer function, then T is multiplicity-free.

Proof. This follows from Theorem 5.15 and Corollary 2.32. □

If Δ is an inner matrix function then the converse of Lemma 5.14 is also true.

Corollary 5.17. Let Δ be an inner matrix function whose flip $\breve{\Delta}$ is of bounded type. Then the following are equivalent:

(a) Δ^t is an outer function;

(b) $\widetilde{\Delta}$ is an outer function;

(c) $\Delta_{cc} = \Delta$;

(d) $\Delta = \Omega_c$ for some inner matrix function Ω.

Hence, in particular, $\Delta_{ccc} = \Delta_c$.

Proof. The implication (a)⇒(b) is clear and the implication (b)⇒(c) follows from Corollary 2.32 and Lemma 5.14. Also the implication (c)⇒(d) is clear and the implication (d)⇒(a) follows from Corollary 5.11. The second assertion follows from the first assertion together with Corollary 2.25 and Corollary 5.11. □

6 Some unsolved problems

In this paper we have explored the Beurling-Lax-Halmos Theorem and have tried to answer several outstanding questions. In this process, we have gotten results on a canonical decomposition of strong L^2-functions, a connection between the Beurling degree and the spectral multiplicity, and the multiplicity-free model operators. However there are still open questions in which we are interested. In this section, we pose some unsolved problems.

6.1. The Beurling degree of an inner matrix functions. The theory of spectral multiplicity for operators of the class C_0 has been well developed (see [Ni1, Appendix 1], [SFBK]). For an inner matrix function $\Delta \in H^\infty_{M_N}$ and $k = 0, 1, \cdots, N$, let

$$\delta_k := \text{g.c.d.}\,\{\text{all inner parts of the minors of order } N-k \text{ of } \Delta\}. \tag{66}$$

Then it is well-known that if $T \in C_0$ with characteristic function $\Delta \in H^\infty_{M_N}$, then

$$\mu_T = \min\big\{k: \ \delta_k = \delta_{k+1}\big\}. \tag{67}$$

In fact, the proof for "$\geq$" in (67) is not difficult. But the proof for "$\leq$" is so complicated. However, Theorem 4.6 gives a simple proof for "$\leq$" in (67) with the aid of the Moore-Nordgren Theorem. To see this, we recall that for an inner function Δ_k $(k = 1, 2)$ with values in M_N, Δ_1 and Δ_2 are called quasi-equivalent if there exist functions $X, Y \in H^\infty_{M_N}$ such that $X\Delta_1 = \Delta_2 Y$ and such that the inner parts $(\det X)^i$ and $(\det Y)^i$ of the corresponding determinants are coprime to $(\det \Delta_k)^i$ $(k = 1, 2)$.

The following theorem shows that the spectral multiplicity of C_0-operators with square-inner characteristic functions can be computed by studying diagonal characteristic functions (cf. [Nor], [MN], [Ni1]):

Nordgren-Moore Theorem.

(a) Let Δ_k $(k = 1, 2)$ be an inner function with values in M_N and let $T_k := P_{\mathcal{H}(\Delta)} S_{\mathbb{C}^N}|_{\mathcal{H}(\Delta_k)}$ $(k = 1, 2)$. If Δ_1 and Δ_2 are quasi-equivalent then $\mu_{T_1} = \mu_{T_2}$.

(b) Let Δ be an inner function with values in M_N. Then Δ is quasi-equivalent to a unique diagonal inner function

$$\operatorname{diag}(\delta_0/\delta_1, \delta_1/\delta_2, \cdots, \delta_{N-1}/\delta_N).$$

By the Nordgren-Moore Theorem (a), the Model theorem and Theorem 4.6, we can see that if Δ_1 and Δ_2 are quasi-equivalent square inner matrix functions then

$$\deg_B(\widetilde{\Delta}_1) = \deg_B(\widetilde{\Delta}_2). \tag{68}$$

We now have:

Proposition 6.1. If Δ is an $N \times N$ square-inner matrix function then

$$\deg_B(\Delta) \leq \min\big\{k: \ \delta_k = \delta_{k+1}\big\}. \tag{69}$$

Proof. Let $m := \min\big\{k : \delta_k = \delta_{k+1}\big\}$. Then by the Nordgren-Moore Theorem, Δ is quasi-equivalent to $\Theta \equiv \operatorname{diag}(\delta_0/\delta_1, \cdots, \delta_{m-1}/\delta_m, 1, \cdots, 1)$. We now take

$$\Phi := \begin{bmatrix} \delta_0/\delta_1 & 0 & \cdots & 0 \\ 0 & \delta_1/\delta_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \delta_{m-1}/\delta_m \\ 0 & \cdots & 0 & 1 \\ & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in H^\infty_{M_{N\times m}}.$$

Then a direct calculation shows that

$$\ker H_{\Phi^*} = \left(\sum_{k=1}^{m} \bigoplus (\delta_{k-1}/\delta_k) H^2\right) \bigoplus H^2_{\mathbb{C}^{N-m}} = \Theta H^2_{\mathbb{C}^n}.$$

It thus follows from (47) and (68) that $\deg_B(\Delta) = \deg_B(\Theta) \le m$. □

Corollary 6.2. If Θ is a diagonal inner matrix function of the form $\Theta := \operatorname{diag}(\theta_1, \cdots, \theta_N)$ (where each θ_i is a scalar inner function) then

$$\deg_B(\Theta) = \max \operatorname{card} \Big\{\sigma : \ \sigma \subseteq \{1, \cdots, N\}, \ \text{g.c.d.}\{\theta_i : i \in \sigma\} \ne 1\Big\}.$$

Proof. This follows at once from (67) and Theorem 4.6. □

Now Proposition 6.1 together with Theorem 4.6 gives a simple proof for "$\le$" in (67). Consequently, in (69), we may take "$=$" in place of "$\le$". However we were unable to derive a similar formula to (69) for *non-square* inner matrix function. Thus we would like to pose:

Problem 6.3. If Δ is an $n \times m$ inner matrix function, describe $\deg_B(\Delta)$ in terms of its entries (e.g., minors).

6.2. Spectra of model operators. We recall that if θ is a scalar inner function, then we may write

$$\theta(\zeta) = B(\zeta) \exp\left(-\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\mu(z)\right),$$

where B is a Blaschke product and μ is a singular measure on $\mathbb{T}$ and that the spectrum, $\sigma(\theta)$, of θ is defined by

$$\sigma(\theta) := \Big\{\lambda \in \operatorname{cl} \mathbb{D} : \ \frac{1}{\theta} \text{ can be continued analytically into a neighborhood of } \lambda\Big\}.$$

Then it was ([Ni1, p.63]) known that the spectrum $\sigma(\theta)$ of θ is given by

$$\sigma(\theta) = \operatorname{cl} \theta^{-1}(0) \bigcup \operatorname{supp} \mu. \tag{70}$$

It was also (cf. [Ni1, p.72]) known that if $T \equiv P_{\mathcal{H}(\Delta)} S_E|_{\mathcal{H}(\Delta)} \in C_0$, then

$$\sigma(T) = \sigma(m_\Delta). \tag{71}$$

In view of (71), we may ask what is the spectrum of the model operator $S_E^*|_{\mathcal{H}(\Delta)}$? Here is an answer.

Proposition 6.4. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ with values in $\mathcal{B}(D, E)$. If Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$ and ω_Δ is the pseudo-characteristic scalar inner function of Δ, then

$$\sigma(\omega_{\widetilde{\Delta}}) \subseteq \sigma(T). \tag{72}$$

Proof. If Δ_c is the complementary factor, with values in $\mathcal{B}(D', E)$, of Δ, then by the proof of Lemma 5.5, $[\Delta, \Delta_c]$ is two-sided inner and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Thus, by Proposition 2.34, $S_E^*|_{\mathcal{H}([\Delta,\Delta_c])}$ belongs to C_0. Then by the Model Theorem, we have

$$S_E^*|_{\mathcal{H}([\Delta,\Delta_c])} \cong P_{\mathcal{H}(\widetilde{[\Delta,\Delta_c]})} S_E|_{\mathcal{H}(\widetilde{[\Delta,\Delta_c]})}.$$

It thus follows from Lemma 5.5 and (71) that

$$\sigma(S_E^*|_{\mathcal{H}([\Delta,\Delta_c])}) = \sigma(m_{\widetilde{[\Delta,\Delta_c]}}) = \sigma(\omega_{\widetilde{\Delta}}). \tag{73}$$

On the other hand, observe

$$[\Delta, \Delta_c] H^2_{D\oplus D'} = \Delta H^2_D \oplus \Delta_c H^2_{D'},$$

and hence

$$\mathcal{H}(\Delta) = \mathcal{H}([\Delta, \Delta_c]) \oplus \Delta_c H^2_{D'}.$$

Thus we may write

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H}([\Delta,\Delta_c]) \\ \Delta_c H^2_{D'} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}([\Delta,\Delta_c]) \\ \Delta_c H^2_{D'} \end{bmatrix}. \tag{74}$$

Note that $T_1 = S_E^*|_{\mathcal{H}([\Delta,\Delta_c])}$. Since by (70) and (73), $\sigma(T_1)$ has no interior points, so that $\sigma(T_1) \cap \sigma(T_2)$ has no interior points. Thus we have $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$ because in the Banach space setting, the passage from $\sigma\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ to $\sigma(A) \cup \sigma(B)$ is the filling in certain holes in $\sigma\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, occurring in $\sigma(A) \cap \sigma(B)$ (cf. [HLL]). Therefore, by (73), we have $\sigma(\omega_{\widetilde{\Delta}}) \subseteq \sigma(T)$. $\square$

We would like to pose:

Problem 6.5. If $T := S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ having a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, describe the spectrum of T in terms of the pseudo-characteristic scalar inner function of Δ.

Appendix A: Strong L^2-functions

In this appendix we provide some properties of strong L^2-functions, $H^2_s(B(D, E))$-functions, strong H^2-functions, and connections between them in addition with $H^2_{\mathcal{B}(D,E)}$-functions, which we have not been able to find in the literature.

We first review a few essential facts concerning vector-valued L^p- and H^p-functions by using [DS], [Dur], [FF], [HP], [Hof], [Ni1], [Ni2], [Pel], [Sa2] as general references.

Let $(\Omega, \mathfrak{M}, \mu)$ be a positive σ-finite measure space and X be a complex Banach space. A function $f : \Omega \to X$ of the form $f = \sum_{k=1}^{\infty} x_k \chi_{\sigma_k}$ (where $x_k \in X$, $\sigma_k \in \mathfrak{M}$ and $\sigma_k \cap \sigma_j = \emptyset$ for $k \neq j$) is said to be *countable-valued*. A function $f : \Omega \to X$ is called *weakly measurable* if the map $s \mapsto \phi(f(s))$ is measurable for all $\phi \in X^*$ and is called *strongly measurable* if there exist countable-valued functions f_n such that $f(s) = \lim_n f_n(s)$ for almost all $s \in \Omega$. It is known that when X is separable,

(i) if f is weakly measurable, then $||f(\cdot)||$ is measurable;

(ii) f is strongly measurable if and only if it is weakly measurable.

A countable-valued function $f = \sum_{k=1}^{\infty} x_k \chi_{\sigma_k}$ is called (*Bochner*) *integrable* if

$$\int_\Omega ||f(s)|| d\mu(s) < \infty$$

and its integral is defined by

$$\int_\Omega f d\mu := \sum_{k=1}^{\infty} x_k \mu(\sigma_k).$$

A function $g : \Omega \to X$ is called *integrable* if there exist countable-valued integrable functions g_n such that $g(s) = \lim_n g_n(s)$ for almost all $s \in \Omega$ and $\lim_n \int_\Omega ||g - g_n|| d\mu = 0$. Then $\int_\Omega g d\mu \equiv \lim_n \int_\Omega g_n d\mu$ exists and $\int_\Omega g d\mu$ is called the (*Bochner*) *integral* of g. If $f : \Omega \to X$ is integrable, then we can see that

$$T\Big(\int_\Omega f d\mu\Big) = \int_\Omega (Tf) d\mu \quad \text{for each } T \in \mathcal{B}(X, Y). \tag{75}$$

Let m denote the normalized Lebesgue measure on $\mathbb{T}$. For a complex Banach space X and $1 \le p \le \infty$, let

$$L^p_X \equiv L^p(\mathbb{T}, X) := \big\{ f : \mathbb{T} \to X : f \text{ is strongly measurable and } ||f||_p < \infty \big\},$$

where

$$||f||_p \equiv ||f||_{L^p_X} := \begin{cases} \Big(\int_\mathbb{T} ||f(z)||_X^p dm(z) \Big)^{\frac{1}{p}} & (1 \le p < \infty); \\ \text{ess sup}_{z \in \mathbb{T}} ||f(z)||_X & (p = \infty). \end{cases}$$

Then we can see that L^p_X forms a Banach space. For $f \in L^1_X$, the n-th Fourier coefficient of f, denoted by $\widehat{f}(n)$, is defined by

$$\widehat{f}(n) := \int_\mathbb{T} \overline{z}^n f(z)\, dm(z) \quad \text{for each } n \in \mathbb{Z}.$$

Also, $H^p_X \equiv H^p(\mathbb{T}, X)$ is defined by the set of $f \in L^p_X$ with $\widehat{f}(n) = 0$ for $n < 0$. A function $f : \mathbb{D} \to X$ is (norm) analytic if f can be written as

$$f(\zeta) = \sum_{n=0}^{\infty} x_n \zeta^n \quad (\zeta \in \mathbb{D}, x_n \in X),$$

Let $\text{Hol}(\mathbb{D}, X)$ denote the set of all analytic functions $f : \mathbb{D} \to X$. Also we write $H^2(\mathbb{D}, X)$ for the set of all $f \in \text{Hol}(\mathbb{D}, X)$ satisfying

$$||f||_{H^2(\mathbb{D},X)} := \sup_{0<r<1} \Big(\int_\mathbb{T} ||f(rz)||_X^2 dm(z) \Big)^{\frac{1}{2}} < \infty.$$

Let E be a separable complex Hilbert space. As in the scalar-valued case, if $f \in H^2(\mathbb{D}, E)$, then there exists a "boundary function" $bf \in H^2_E$ such that

$$f(rz) = (bf * P_r)(z) \quad (r \in [0,1) \text{ and } z \in \mathbb{T})$$

(where P_r denotes the Poisson kernel) and

$$(bf)(z) = \lim_{rz \to z} f(rz) \quad \text{nontangentially a.e. on } \mathbb{T}.$$

Moreover, the mapping $f \mapsto bf$ is an isometric bijection (cf. [Ni2, Theorem 3.11.7]). We conventionally identify $H^2(\mathbb{D}, E)$ with $H^2_E \equiv H^2(\mathbb{T}, E)$. For $f, g \in L^2_E$ with a separable complex Hilbert space E, the inner product $\langle f,\ g\rangle$ is defined by

$$\langle f,\ g\rangle \equiv \big\langle f(z),\ g(z)\big\rangle_{L^2_E} := \int_{\mathbb{T}} \big\langle f(z), g(z)\big\rangle_E dm(z).$$

If $f, g \in L^2_X$ with $X = M_{n\times m}$, then $\langle f, g\rangle = \int_{\mathbb{T}} \mathrm{tr}\,(g^* f) dm$.

For a function $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$, write

$$\Phi^*(z) := \Phi(z)^* \quad \text{for } z \in \mathbb{T}.$$

A function $\Phi : \mathbb{T} \to \mathcal{B}(X, Y)$ is called *SOT measurable* if $z \mapsto \Phi(z)x$ is strongly measurable for every $x \in X$ and is called *WOT measurable* if $z \mapsto \Phi(z)x$ is weakly measurable for every $x \in X$. We can easily check that if $\Phi : \mathbb{T} \to \mathcal{B}(X, Y)$ is strongly measurable, then Φ is SOT-measurable and if D and E are separable complex Hilbert spaces then $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$ is SOT measurable if and only if Φ is WOT measurable.

We then have:

Lemma A.1. If $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$ is WOT measurable, then so is Φ^*.

Proof. Suppose that Φ is WOT measurable. Then the function

$$z \mapsto \overline{\big\langle \Phi^*(z)y,\ x\big\rangle} = \big\langle x,\ \Phi^*(z)y\big\rangle = \big\langle \Phi(z)x,\ y\big\rangle$$

is measurable for all $x \in D$ and $y \in E$. Thus the function $z \mapsto \big\langle \Phi^*(z)y,\ x\big\rangle$ is measurable for all $x \in D$ and $y \in E$. $\square$

Let $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$ be a WOT measurable function. Then Φ is called *WOT integrable* if $\big\langle \Phi(\cdot)x, y\big\rangle \in L^1$ for every $x \in D$ and $y \in E$, and there exists an operator $U \in \mathcal{B}(D, E)$ such that $\big\langle Ux, y\big\rangle = \int_{\mathbb{T}} \big\langle \Phi(z)x, y\big\rangle dm(z)$. Also Φ is called *SOT integrable* if $\Phi(\cdot)x$ is integrable for every $x \in D$. In this case, the operator $V : x \mapsto \int_{\mathbb{T}} \Phi(z)x dm(z)$ is bounded, i.e., $V \in \mathcal{B}(D, E)$. If $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$ is SOT integrable, then it follows from (75) that for every $x \in D$ and $y \in E$,

$$\left\langle \int_{\mathbb{T}} \Phi(z)x dm(z),\ y \right\rangle = \int_{\mathbb{T}} \big\langle \Phi(z)x,\ y\big\rangle dm(z), \tag{76}$$

which implies that Φ is WOT integrable and that the SOT integral of Φ is equal to the WOT integral of Φ.

We can say more:

Lemma A.2. For $\Phi \in L^1_{\mathcal{B}(D,E)}$, the Bochner integral of Φ is equal to the SOT integral of Φ, in the sense that

$$\left(\int_{\mathbb{T}} \Phi(z)dm(z)\right) x = \int_{\mathbb{T}} \Phi(z)xdm(z) \quad \text{for all } x \in D.$$

Proof. This follows from a straightforward calculation. □

Let $L^\infty(\mathcal{B}(D,E))$ be the space of all bounded (WOT) measurable $\mathcal{B}(D,E)$-valued functions on $\mathbb{T}$. For $\Psi \in L^\infty(\mathcal{B}(D,E))$, define

$$||\Psi||_\infty := \text{ess sup}_{z\in\mathbb{T}}||\Psi(z)||.$$

For $1 \le p < \infty$, we define the class $L^p_s(\mathcal{B}(D,E)) \equiv L^p_s(\mathbb{T},\mathcal{B}(D,E))$ as the set of all (WOT) measurable $\mathcal{B}(D,E)$-valued functions Φ on $\mathbb{T}$ such that $\Phi(\cdot)x \in L^p_E$. A function $\Phi \in L^p_s(\mathcal{B}(D,E))$ is called a *strong L^p-function.* We claim that

$$L^p_{\mathcal{B}(D,E)} \subseteq L^p_s(\mathcal{B}(D,E)) : \tag{77}$$

indeed if $\Phi \in L^p_{\mathcal{B}(D,E)}$, then for all $x \in D$ with $||x|| = 1$,

$$||\Phi(z)x||^p_{L^p_E} = \int_{\mathbb{T}} ||\Phi(z)x||^p_E dm(z) \le \int_{\mathbb{T}} ||\Phi(z)||^p_{\mathcal{B}(D,E)} dm(z) = ||\Phi||^p_{L^p_{\mathcal{B}(D,E)}},$$

which gives (77). Also we can easily check that

$$L^\infty_{\mathcal{B}(D,E)} \subseteq L^\infty(\mathcal{B}(D,E)) \subseteq L^p_s(\mathcal{B}(D,E)). \tag{78}$$

We define

$$H^\infty(\mathcal{B}(D,E)) \equiv H^\infty(\mathbb{T},\mathcal{B}(D,E)) := \big\{\Phi \in L^\infty(\mathcal{B}(D,E)) : \ \widehat{\Phi}(n) = 0 \text{ for } n < 0\big\}.$$

On the other hand, we define $H^\infty(\mathbb{D},\mathcal{B}(D,E))$ as the set of all analytic functions $\Phi : \mathbb{D} \to \mathcal{B}(D,E)$ satisfying

$$||\Phi||_{H^\infty} := \sup_{\zeta\in\mathbb{D}} ||\Phi(\zeta)||.$$

If D and E are separable Hilbert spaces, we conventionally identify $H^\infty(\mathbb{D},\mathcal{B}(D,E))$ with $H^\infty(\mathbb{T},\mathcal{B}(D,E))$ (cf. [Ni2, Theorem 3.11.10]).

On the other hand, by (77), we have $L^1_{\mathcal{B}(D,E)} \subseteq L^1_s(\mathcal{B}(D,E))$. Thus if $\Phi \in L^1_{\mathcal{B}(D,E)}$, then there are two definitions of the n-th Fourier coefficient of Φ. However, we can, by Lemma A.2, see that the n-th Fourier coefficient of Φ as an element of $L^1_{\mathcal{B}(D,E)}$ coincides with the n-th Fourier coefficient of Φ as an element of $L^1_s(\mathcal{B}(D,E))$.

We now denote by $H^2_s(\mathbb{D},\mathcal{B}(D,E))$ the set of all strong H^2-functions with values in $\mathcal{B}(D,E)$.

We then have:

Lemma A.3. $H^2(\mathbb{D},\mathcal{B}(D,E)) \subseteq H^2_s(\mathbb{D},\mathcal{B}(D,E))$.

Proof. Let $\Phi \in H^2(\mathbb{D}, \mathcal{B}(D,E))$. Then Φ can be written as

$$\Phi(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^n \quad (A_n \in \mathcal{B}(D,E)).$$

Thus for each $x \in D$,

$$\Phi(\zeta)x = \sum_{n=0}^{\infty} (A_n x)\zeta^n \in \mathrm{Hol}(\mathbb{D}, E).$$

Observe that

$$\begin{aligned} ||\Phi(\cdot)x||^2_{H^2(\mathbb{D},E)} &= \sup\nolimits_{0<r<1} \int_{\mathbb{T}} ||\Phi(rz)x||^2_E dm(z) \\ &\le ||\Phi||^2_{H^2(\mathbb{D},\mathcal{B}(D,E))} \cdot ||x||^2_D \\ &< \infty, \end{aligned}$$

which implies $\Phi \in H^2_s(\mathbb{D}, \mathcal{B}(D,E))$. $\square$

Theorem A.4. If $\dim D < \infty$, then

$$H^2(\mathbb{D}, \mathcal{B}(D,E)) = H^2_s(\mathbb{D}, \mathcal{B}(D,E)),$$

where the equality is set-theoretic.

Proof. By Lemma A.3, we have $H^2(\mathbb{D}, \mathcal{B}(D,E)) \subseteq H^2_s(\mathbb{D}, \mathcal{B}(D,E))$. For the reverse inclusion, suppose $\Phi \in H^2_s(\mathbb{D}, \mathcal{B}(D,E))$ and $\dim D = d < \infty$. Let $\{e_j : j = 1, 2, \cdots, d\}$ be an orthonormal basis of D. Then for each $j = 1, 2, \cdots, d$,

$$\phi_j(\zeta) \equiv \Phi(\zeta)e_j \in H^2(\mathbb{D}, E). \tag{79}$$

Thus we may write

$$\phi_j(\zeta) = \sum_{n=0}^{\infty} a_n^{(j)} \zeta^n \quad (a_n^{(j)} \in E).$$

For each $n = 0, 1, 2, \cdots$, define $A_n : D \to E$ by

$$A_n x := \sum_{j=1}^{d} \alpha_j a_n^{(j)} \quad \Big(\text{where } x := \sum_{j=1}^{d} \alpha_j e_j\Big).$$

Then $A_n \in \mathcal{B}(D,E)$. We claim that

$$\Phi(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^n \in \mathrm{Hol}(\mathbb{D}, \mathcal{B}(D,E)). \tag{80}$$

To prove (80), let $\epsilon > 0$ be arbitrary. For each $\zeta \in \mathbb{D}$, there exists $M > 0$ such that for all $j = 1, 2, \cdots, d$,

$$\Big|\Big| \sum_{n=M}^{\infty} a_n^{(j)} \zeta^n \Big|\Big|_E < \frac{\epsilon}{d}.$$

Let $x := \sum_{j=1}^{d} \alpha_j e_j$ with $||x||_D = 1$. Then we have

$$\begin{aligned}\left|\left|\left(\Phi(\zeta) - \sum_{n=0}^{M-1} A_n \zeta^n\right)x\right|\right|_E &= \left|\left|\sum_{n=M}^{\infty}\sum_{j=1}^{d} \alpha_j a_n^{(j)} \zeta^n\right|\right|_E \\ &\le \sum_{j=1}^{d}\left|\left|\sum_{n=M}^{\infty} a_n^{(j)}\zeta^n\right|\right|_E \\ &< \epsilon,\end{aligned}$$

which proves (80). For all $r \in [0, 1)$, we have that

$$\begin{aligned}||\Phi(rz)x||_E^2 &= \left|\left|\sum_{j=1}^{d} \alpha_j \Phi(rz)e_j\right|\right|_E^2 \\ &\le \left(\sum_{j=1}^{d} |\alpha_j| ||\Phi(rz)e_j||_E\right)^2 \\ &\le \sum_{j=1}^{d} ||\Phi(rz)e_j||_E^2.\end{aligned}$$

Thus $||\Phi(rz)||^2_{\mathcal{B}(D,E)} \le \sum_{j=1}^{d} ||\Phi(rz)e_j||_E^2$, and hence it follows from (79) that

$$\begin{aligned}||\Phi||_{H^2(\mathbb{D},\mathcal{B}(D,E))} &= \sup_{0<r<1}\int_{\mathbb{T}} ||\Phi(rz)||^2_{\mathcal{B}(D,E)} dm(z) \\ &\le \sup_{0<r<1}\int_{\mathbb{T}} \sum_{j=1}^{d} ||\Phi(rz)e_j||_E^2 dm(z) \\ &\le \sum_{j=1}^{d} ||\phi_j||^2_{H^2(\mathbb{D},E)} < \infty,\end{aligned}$$

which implies $\Phi \in H^2(\mathbb{D}, \mathcal{B}(D, E))$. This completes the proof. □

Remark A.5. Theorem A.4 may fail if the condition "$\dim D < \infty$ is dropped. For example, if Φ is defined on the unit disk $\mathbb{D}$ by

$$\Phi(\zeta) := \begin{bmatrix}\zeta & \zeta^2 & \zeta^3 & \cdots\end{bmatrix} : \ell^2 \to \mathbb{C} \qquad (\zeta \in \mathbb{D}),$$

then $\Phi(\zeta)$ is a bounded linear operator for each $\zeta \in \mathbb{D}$: indeed,

$$\begin{aligned}
||\Phi(\zeta)||_{\mathcal{B}(\ell^2,\mathbb{C})} &= \sup_{||x||=1} \big|\Phi(\zeta)x\big| \\
&= \sup_{||x||=1} \left|\sum_{n=1}^{\infty} \zeta^n x_n\right| \quad (x \equiv (x_n) \in \ell^2) \\
&= \sup_{||x||=1} \Big|\Big\langle (\zeta,\ \zeta^2,\ \zeta^3,\ \cdots),\ (\overline{x}_1,\ \overline{x}_2,\ \overline{x}_3,\ \cdots)\Big\rangle\Big| \\
&= \big|\big|(\zeta,\ \zeta^2,\ \zeta^3,\ \cdots)\big|\big|_{\ell^2} \\
&= \Big(\frac{|\zeta|^2}{1-|\zeta|^2}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Moreover, for each $x \equiv (x_n) \in \ell^2$,

$$\Phi(\zeta)x = \sum_{n=1}^{\infty} x_n \zeta^n \in H^2(\mathbb{D}, \mathbb{C}),$$

which says that $\Phi \in H^2_s(\mathbb{D}, \mathcal{B}(\ell^2, \mathbb{C}))$. However, we have $\Phi \notin H^2(\mathbb{D}, \mathcal{B}(\ell^2, \mathbb{C}))$: indeed, for $\zeta = rz \in \mathbb{D}$,

$$||\Phi(\zeta)||^2_{\mathcal{B}(\ell^2,\mathbb{C})} = ||\Phi(\zeta)\Phi(\zeta)^*||_{\mathcal{B}(\ell^2,\mathbb{C})} = \frac{r^2}{1-r^2},$$

so that

$$\begin{aligned}
\sup_{0<r<1} \int_{\mathbb{T}} ||\Phi(rz)||^2_{\mathcal{B}(\ell^2,\mathbb{C})} dm(z) &= \sup_{0<r<1} \int_{\mathbb{T}} \frac{r^2}{1-r^2} dm(z) \\
&= \sup_{0<r<1} \frac{r^2}{1-r^2} \\
&= \infty.
\end{aligned}$$

In general, the boundary values of strong H^2-functions do not need to be bounded linear operators (defined almost everywhere on $\mathbb{T}$). Thus we do not guarantee that the boundary value of a strong H^2-function belongs to $H^2_s(\mathbb{T}, \mathcal{B}(D, E))$. For example, if Φ is defined on the unit disk $\mathbb{D}$ by

$$\Phi(\zeta) = \begin{bmatrix} 1 & \zeta & \zeta^2 & \zeta^3 & \cdots \end{bmatrix} : \ell^2 \to \mathbb{C} \quad (\zeta \in \mathbb{D}),$$

then by Remark A.5, Φ is a strong H^2-function with values in $\mathcal{B}(\ell^2, \mathbb{C})$. However, the boundary value

$$\Phi(z) = \begin{bmatrix} 1 & z & z^2 & z^3 & \cdots \end{bmatrix} : \ell^2 \to \mathbb{C} \quad (z \in \mathbb{T})$$

is not bounded for all $z \in \mathbb{T}$ because for any $z_0 \in \mathbb{T}$, if we let

$$x_0 := \Big(1, \overline{z}_0, \frac{\overline{z}_0^2}{2}, \frac{\overline{z}_0^3}{3}, \cdots\Big)^t \in \ell^2,$$

then

$$\Phi(z_0)x_0 = 1 + \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

which shows that $\Phi \notin H^2_s(\mathbb{T}, \mathcal{B}(D,E))$.

In spite of it, there are useful relations between the set $H^2_s(\mathbb{D}, \mathcal{B}(D,E))$ and the set $H^2_s(\mathbb{T}, \mathcal{B}(D,E))$. To see this, let $\Phi \in H^2_s(\mathbb{T}, \mathcal{B}(D,E))$. Then $\Phi(z) \in \mathcal{B}(D,E)$ for almost all $z \in \mathbb{T}$ and $\Phi(z)x \in H^2_E$ for each $x \in D$. We now define a (function-valued with domain D) function $p\Phi$ on the unit disk $\mathbb{D}$ by the Poisson integral in the strong sense:

$$\begin{aligned} p\Phi(re^{i\theta})x &:= \left(\Phi(\cdot)x * P_r\right)(e^{i\theta}) \quad (x \in D) \\ &= \int_0^{2\pi} P_r(\theta - t)\Phi(e^{it})x \, dm(t) \in E, \end{aligned}$$

where $P_r(\cdot)$ is the Poisson kernel. Then $p\Phi(\zeta)x \in H^2(\mathbb{D}, E)$. Thus, for all $\zeta \in \mathbb{D}$, $p\Phi(\zeta)$ can be viewed as a function from D into E. A straightforward calculation shows that $p\Phi(\zeta)$ is a linear map for each $\zeta \in \mathbb{D}$. Since $p\Phi(\zeta)x \in H^2(\mathbb{D}, E)$ is the Poisson integral of $\Phi(z)x \in H^2_E$, we will conventionally identify $\Phi(z)x$ and $p\Phi(\zeta)x$ for each $x \in D$. From this viewpoint, we will also regard $\Phi \in H^2_s(\mathbb{T}, \mathcal{B}(D,E))$ as an (linear, but not necessarily bounded) operator-valued function defined on the unit disk $\mathbb{D}$.

We thus have:

Lemma A.6. The following inclusion holds:

$$H^2_{\mathcal{B}(D,E)} \cup H^\infty(\mathcal{B}(D,E)) \subseteq H^2_s(\mathbb{D}, \mathcal{B}(D,E)).$$

Proof. Note that by (77) and (78), $H^2_{\mathcal{B}(D,E)} \cup H^\infty(\mathcal{B}(D,E)) \subseteq H^2_s(\mathbb{T}, \mathcal{B}(D,E))$. Thus in view of the preceding remark, it suffices to show $\Phi(\zeta) \in \mathcal{B}(D,E)$ for all $\zeta \in \mathbb{D}$. To see this we first claim that there exists $M > 0$ such that

$$\sup\Big\{||\Phi(\cdot)x||_{L^1_E} : x \in D \text{ with } ||x|| = 1\Big\} < M, \tag{81}$$

To see this, if $\Phi \in H^2_{\mathcal{B}(D,E)}$, then for all $x \in D$ with $||x|| = 1$,

$$\begin{aligned} ||\Phi(\cdot)x||_{L^1_E} &\le ||\Phi(\cdot)x||_{L^2_E} \\ &\le \left(\int_{\mathbb{T}} ||\Phi(z)||^2_{\mathcal{B}(D,E)} dm(z)\right)^{\frac{1}{2}} \\ &= ||\Phi||_{L^2_{\mathcal{B}(D,E)}}. \end{aligned}$$

If instead $\Phi \in H^\infty(\mathcal{B}(D,E))$, then for all $x \in D$ with $||x|| = 1$,

$$||\Phi(\cdot)x||_{L^1_E} = \int_{\mathbb{T}} ||\Phi(z)x||_E dm(z) \le ||\Phi(z)||_\infty,$$

which proves the claim (81). Now, let $\zeta = re^{i\theta} \in \mathbb{D}$ and $x \in D$ with $||x|| = 1$. Then for

$y \in E$ with $||y|| \le 1$,

$$\begin{aligned}\left|\left\langle \Phi(re^{i\theta})x,\ y\right\rangle_E\right| &= \left|\left\langle \int_0^{2\pi} P_r(\theta - t)\Phi(e^{it})x dm(t),\ y\right\rangle_E\right| \\ &= \left|\int_0^{2\pi} \left\langle P_r(\theta - t)\Phi(e^{it})x,\ y\right\rangle_E dm(t)\right| \quad \text{(by (76))} \\ &\le \frac{1+r}{1-r}\int_0^{2\pi} \left|\left\langle \Phi(e^{it})x,\ y\right\rangle_E\right| dm(t),\end{aligned}$$

which implies, by our assumption,

$$\begin{aligned}||\Phi(\zeta)x||_E &\le \frac{1+r}{1-r}\int_0^{2\pi} \left|\left|\Phi(e^{it})x\right|\right|_E dm(t) \\ &= \frac{1+r}{1-r}||\Phi(\cdot)x||_{L^1_E} \\ &< \infty,\end{aligned}$$

which shows that $\Phi(\zeta) \in \mathcal{B}(D, E)$ for all $\zeta \in \mathbb{D}$. Thus we have $\Phi \in H^2_s(\mathbb{D}, \mathcal{B}(D, E))$. $\square$

We now recall a notion from classical Banach space theory, about regarding a vector as an operator acting on the scalars. This notion is important as motivation for the study of strong L^2-functions. Let E be a separable complex Hilbert space. For a function $f : \mathbb{T} \to E$, define $[f] : \mathbb{T} \to \mathcal{B}(\mathbb{C}, E)$ by

$$[f](z)\alpha := \alpha f(z) \quad (\alpha \in \mathbb{C}). \tag{82}$$

If $g : \mathbb{T} \to E$ is a countable-valued function of the form

$$g = \sum_{k=1}^{\infty} x_k \chi_{\sigma_k} \quad (x_k \in E),$$

then for each $\alpha \in \mathbb{C}$,

$$\left(\sum_{k=1}^{\infty} [x_k]\chi_{\sigma_k}\right)\alpha = \sum_{k=1}^{\infty} \alpha x_k \chi_{\sigma_k} = \alpha g = [g]\alpha,$$

which implies that $[g]$ is a countable-valued function of the form $[g] = \sum_{k=1}^{\infty}[x_k]\chi_{\sigma_k}$.

We then have:

Lemma A.7. Let E be a separable complex Hilbert space and $1 \le p \le \infty$. Define $\Gamma : L^p_E \to L^p_{\mathcal{B}(\mathbb{C},E)}$ by

$$\Gamma(f)(z) = [f](z),$$

where $[f](z) : \mathbb{C} \to E$ is given by $[f](z)\alpha := \alpha f(z)$. Then

(a) Γ is unitary, and hence $L^p_E \cong L^p_{\mathcal{B}(\mathbb{C},E)}$;

(b) $L^p_{\mathcal{B}(\mathbb{C},E)} = L^p_s(\mathcal{B}(\mathbb{C},E))$ for $1 \le p < \infty$;

(c) $\widehat{[f]}(n) = [\widehat{f}(n)]$ for $f \in L^p_E$ and $n \in \mathbb{Z}$.

In particular, $H^p_E \cong H^p_{\mathcal{B}(\mathbb{C},E)} = H^p_s(\mathcal{B}(\mathbb{C},E))$ for $1 \le p < \infty$.

Proof. (a) Let $f \in L^p_E$ $(1 \le p \le \infty)$ be arbitrary. We first show that $[f] \in L^p_{\mathcal{B}(\mathbb{C},E)}$. Since f is strongly measurable, there exist countable-valued functions f_n such that $f(z) = \lim_n f_n(z)$ for almost all $z \in \mathbb{T}$. Observe that for almost all $z \in \mathbb{T}$,

$$||[f](z)||_{\mathcal{B}(\mathbb{C},E)} = \sup_{|\alpha|=1}||[f](z)\alpha||_E = ||f(z)||_E.$$

Thus we have that

$$\big|\big|[f_n](z) - [f](z)\big|\big|_{\mathcal{B}(\mathbb{C},E)} = \big|\big|f_n(z) - f(z)\big|\big|_E \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $[f]$ is strongly measurable and $||[f]||_{L^p_{\mathcal{B}(\mathbb{C},E)}} = ||f||_{L^p_E}$. Thus Γ is an isometry. For $h \in L^p_{\mathcal{B}(\mathbb{C},E)}$, let $g(z) := h(z)1 \in L^p_E$. Then for all $\alpha \in \mathbb{C}$, we have

$$\Gamma(g)(z)\alpha = \alpha h(z)1 = h(z)\alpha,$$

which implies that Γ is a surjection from L^p_E onto $L^p_{\mathcal{B}(\mathbb{C},E)}$. Thus Γ is unitary, so that $L^p_E \cong L^p_{\mathcal{B}(\mathbb{C},E)}$. This proves (a).

(b) Suppose $h \in L^p_s(\mathcal{B}(\mathbb{C},E))$ $(1 \le p < \infty)$. If $g(z) := h(z)1 \in L^p_E$, then $h = [g] \in L^p_{\mathcal{B}(\mathbb{C},E)}$. The converse is clear.

(c) Let $f \in L^p_E$. Then for all $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$\widehat{[f]}(n)\alpha = \int_{\mathbb{T}} \overline{z}^n [f](z)\alpha dm = \alpha \int_{\mathbb{T}} \overline{z}^n f(z) dm = \alpha \widehat{f}(n) = [\widehat{f}(n)]\alpha,$$

which gives (c).

The last assertion follows at once from (b) and (c). □

For $\mathcal{X}$ a closed subspace of D, $P_{\mathcal{X}}$ denotes the orthogonal projection from D onto $\mathcal{X}$. Then we have:

Lemma A.8. If $\dim D < \infty$, then

(a) $L^2_s(\mathbb{T}, \mathcal{B}(D,E)) = L^2_{\mathcal{B}(D,E)}$;

(b) $H^2_s(\mathbb{T}, \mathcal{B}(D,E)) = H^2_{\mathcal{B}(D,E)}$,

where the equalities are set-theoretic.

Proof. (a) Let $d := \dim D < \infty$. It follows from (77) that $L^2_{\mathcal{B}(D,E)} \subseteq L^2_s(\mathcal{B}(D,E))$. For the reverse inclusion, let $\{e_j\}_{j=1}^d$ be an orthonormal basis of D. Suppose $\Phi \in L^2_s(\mathcal{B}(D,E))$. Then

$$\phi_j(z) \equiv \Phi(z)e_j \in L^2_E \qquad (j = 1, 2, \cdots, d).$$

It thus follows from Lemma A.7 that $[\phi_j] \in L^2_{\mathcal{B}(\mathbb{C},E)}$. For $j = 1, 2, \cdots, d$, define $\Phi_j : \mathbb{T} \to \mathcal{B}(D,E)$ by

$$\Phi_j := [\phi_j]P_{D_j} \quad \Bigl(\mathbb{C} \cong D_j := \bigvee e_j\Bigr).$$

Since $[\phi_j]$ is strongly measurable, it is easy to show that Φ_j is strongly measurable for each $j = 1, 2, \cdots$. It follows from Lemma A.7 that

$$\begin{aligned} ||\Phi_j||^2_{L^2_{\mathcal{B}(D,E)}} &= \int_{\mathbb{T}} \bigl|\bigl|\Phi_j(z)\bigr|\bigr|^2_{\mathcal{B}(D,E)} dm(z) \\ &= \int_{\mathbb{T}} \bigl|\bigl|[\phi_j](z)\bigr|\bigr|^2_{\mathcal{B}(\mathbb{C},E)} dm(z) \\ &= \bigl|\bigl|[\phi_j]\bigr|\bigr|^2_{L^2_{\mathcal{B}(\mathbb{C},E)}} \\ &= ||\phi_j||^2_{L^2_E} \\ &< \infty. \end{aligned}$$

Thus $\Phi_j \in L^2_{\mathcal{B}(D,E)}$, and hence $\Phi = \sum_{j=1}^d \Phi_j \in L^2_{\mathcal{B}(D,E)}$. This proves (a).

(b) This follows from Lemma A.2 and (a). □

To proceed, we define a "boundary function" $b\Phi$ for each function $\Phi \in H^2_s(\mathbb{D}, \mathcal{B}(D,E))$ with dim $D < \infty$. In this case, we may assume that $D = \mathbb{C}^d$.

Let $\Phi \in H^2_s(\mathbb{D}, \mathcal{B}(D,E))$ and $\{e_j\}_{j=1}^d$ be the canonical basis for $\mathbb{C}^d$. Then $\phi_j(\zeta) \equiv \Phi(\zeta)e_j \in H^2(\mathbb{D}, E)$. Thus we have

$$\phi_j(z) \equiv (b\phi_j)(z) := \lim_{rz \to z} \phi_j(rz) \in H^2_E. \tag{83}$$

It follows from Lemma A.7 that for each $j = 1, 2, 3, \cdots, d$,

$$[\phi_j] \in H^2_{\mathcal{B}(\mathbb{C},E)} = H^2_s(\mathbb{T}, \mathcal{B}(\mathbb{C}, E)),$$

where $[\phi_j](z)\alpha := \alpha\phi_j(z)$ for all $\alpha \in \mathbb{C}$. Note that there exists a subset $\sigma \subset \mathbb{T}$ with $m(\sigma) = 0$ such that

$$\phi_j(z) \in E \quad \text{for each } z \in \mathbb{T}_0 := \mathbb{T} \setminus \sigma. \tag{84}$$

Define a function b on $H^2_s(\mathbb{D}, \mathcal{B}(D,E))$ by

$$(b\Phi)(z) := \bigl[[\phi_1](z), [\phi_2](z), \cdots, [\phi_d](z)\bigr] \quad (z \in \mathbb{T}_0). \tag{85}$$

Then we have that for all $x \in D$,

$$(b\Phi)(z)x = \lim_{rz \to z} \Phi(rz)x \in E \ \ (z \in \mathbb{T}_0). \tag{86}$$

A straightforward calculation shows that $(b\Phi)(z)$ is a linear mapping from D into E for almost all $z \in \mathbb{T}$.

We thus have:

Theorem A.9. If $\dim D < \infty$, then the function b defined by (85) is a linear bijection from $H_s^2(\mathbb{D}, \mathcal{B}(D,E))$ onto $H_s^2(\mathbb{T}, \mathcal{B}(D,E))$.

Proof. Let $d := \dim D < \infty$. Then we may assume that $D = \mathbb{C}^d$. Let $\{e_j\}_{j=1}^d$ be the canonical basis for $\mathbb{C}^d$ and $\mathbb{T}_0$ be defined as the above.

(1) b is well-defined: Let $\Phi \in H_s^2(\mathbb{D}, \mathcal{B}(\mathbb{C}^d, E))$. Then it follows from (84) that for each $z_0 \in \mathbb{T}_0$,

$$||(b\Phi)(z_0)||_{\mathcal{B}(\mathbb{C}^d,E)} \leq \sum_{n=1}^{d} ||\phi_j(z_0)||_E < \infty$$

which implies that $(b\Phi)(z_0)$ is bounded for each $z_0 \in \mathbb{T}_0$. If $x \equiv (x_1, x_2, \cdots, x_d)^t \in \mathbb{C}^d$, then

$$(b\Phi)(z)x = \sum_{n=1}^{d} x_j \phi_j(z) \in H_E^2,$$

which implies that $b\Phi \in H_s^2(\mathcal{B}(\mathbb{C}^d, E))$, and hence b is well-defined.

(2) b is linear: Immediate from a direct calculation.

(3) b is one-one: Let $\Phi, \Psi \in H_s^2(\mathbb{D}, \mathcal{B}(\mathbb{C}^d, E))$. If $b\Phi = b\Psi$, then it follows that for each $x \in \mathbb{C}^d$ and $rz \in \mathbb{D}$,

$$\begin{aligned}\Phi(rz)x &= ((b\Phi)x * P_r)(z)\\ &= \int_0^{2\pi} P_r(\theta - t)(b\Phi)(e^{it})x dm(t)\\ &= \int_0^{2\pi} P_r(\theta - t)(b\Psi)(e^{it})x dm(t)\\ &= \Psi(rz)x \quad (z = e^{i\theta}),\end{aligned}$$

which gives the result.

(4) b is onto: Let $A \in H_s^2(\mathbb{T}, \mathcal{B}(\mathbb{C}^d, E))$. Then $A(z)e_j \in H_E^2$ for all $j = 1, 2, \cdots, d$. For each $j = 1, 2, \cdots, d$, let

$$\phi_j(rz) := (Ae_j * P_r)(z) \in H^2(\mathbb{D}, E)$$

and define

$$\Phi(\zeta) := [\phi_1(\zeta), \phi_2(\zeta), \cdots, \phi_d(\zeta)] \quad (\zeta := rz).$$

Then $\Phi \in H_s^2(\mathbb{D}, \mathcal{B}(\mathbb{C}^d, E))$. It follows from (86) that for all $x = (x_1, x_2, \cdots, x_d)^t \in \mathbb{C}^d$ and for almost all $z \in \mathbb{T}$,

$$\begin{aligned}(b\Phi)(z)x &= \lim_{rz \to z} \Phi(rz)x \\ &= \lim_{rz \to z} \sum_{j=1}^{d} x_j \phi_j(rz) \\ &= \sum_{j=1}^{d} x_j A(z) e_j \\ &= A(z)x,\end{aligned}$$

which implies that b is onto. This completes the proof. □

We thus have:

Corollary A.10. If $\dim D < \infty$, then the function b defined by (85) is an isometric bijection from $H^2(\mathbb{D}, \mathcal{B}(D, E))$ onto $H^2_{\mathcal{B}(D,E)}$.

Proof. By Theorem A.9 together with Theorem A.4 and Lemma A.8, the function b defined by (85) is a linear bijection from $H^2(\mathbb{D}, \mathcal{B}(D, E))$ onto $H^2_{\mathcal{B}(D,E)}$. In view of the Banach space-valued version of the usual Hardy space theory (cf. [Ni2, Theorem 3.11.6]), it suffices to show that

$$\Phi(re^{it}) = (b\Phi * P_r)(e^{it}). \tag{87}$$

Indeed, if $x \in D$, then

$$\begin{aligned}(b\Phi * P_r)(e^{it})x &= \left(\int_0^{2\pi} P_r(\theta - t)(b\Phi)(e^{it}) dm(t) \right) x \\ &= \int_0^{2\pi} P_r(\theta - t)(b\Phi)(e^{it}) x dm(t) \quad \text{(by Lemma } A.2) \\ &= \Phi(re^{it})x,\end{aligned}$$

which gives (87). □

According to the convention of the usual Hardy space theory, we will identify $b\Phi$ with $\Phi \in H^2(\mathbb{D}, \mathcal{B}(D, E))$. In this sense, we eventually have:

Corollary A.11. If dim $D < \infty$, then

$$H_s^2(\mathbb{D}, \mathcal{B}(D, E)) = H^2(\mathbb{D}, \mathcal{B}(D, E)) = H^2_{\mathcal{B}(D,E)} = H_s^2(\mathbb{T}, \mathcal{B}(D, E)),$$

where the first and last equalities are set-theoretic, while the second equality establishes an isometric isomorphism.

Proof. This follows from Theorem A.4, Lemma A.8, and Corollary A.10. □

Lemma A.12. If $\Phi \in L^\infty(\mathcal{B}(D,E))$, then $\Phi^* \in L^\infty(\mathcal{B}(E,D))$. In this case,

$$\widehat{\Phi^*}(-n) = \widehat{\widetilde{\Phi}}(n) = \widehat{\Phi}(n)^* \quad (n \in \mathbb{Z}). \tag{88}$$

In particular, $\Phi \in H^\infty(\mathcal{B}(D,E))$ if and only if $\widetilde{\Phi} \in H^\infty(\mathcal{B}(E,D))$.

Proof. Suppose $\Phi \in L^\infty(\mathcal{B}(D,E))$. Then

$$\text{ess sup}_{z\in\mathbb{T}}||\Phi^*(z)|| = \text{ess sup}_{z\in\mathbb{T}}||\Phi(z)|| < \infty,$$

which together with Lemma A.1 implies $\Phi^* \in L^\infty(\mathcal{B}(E,D))$. The first equality of the assertion (88) comes from the definition. For the second equality, observe that for each $x \in D$, $y \in E$ and $n \in \mathbb{Z}$,

$$\begin{aligned}\big\langle \widehat{\Phi}(n)x,\ y\big\rangle &= \left\langle \int_{\mathbb{T}} \overline{z}^n \Phi(z)x dm(z),\ y \right\rangle\\ &= \int_{\mathbb{T}} \langle \overline{z}^n \Phi(z)x,\ y\rangle dm(z) \quad \text{(by (76))}\\ &= \int_{\mathbb{T}} \langle x,\ \overline{z}^n \widetilde{\Phi}(z)y\rangle dm(z)\\ &= \big\langle x,\ \widehat{\widetilde{\Phi}}(n)y\big\rangle.\end{aligned}$$

□

Lemma A.13. Let $1 \le p < \infty$. If $\Phi \in L^\infty(\mathcal{B}(D,E))$, then $\Phi L^p_s(\mathcal{B}(E',D)) \subseteq L^p_s(\mathcal{B}(E',E))$. Also, if $\Phi \in H^\infty(\mathcal{B}(D,E))$, then $\Phi H^2_s(\mathcal{B}(E',D)) \subseteq H^2_s(\mathcal{B}(E',E))$.

Proof. Suppose that $\Phi \in L^\infty(\mathcal{B}(D,E))$ and $A \in L^p_s(\mathcal{B}(E',D))$. Let $x \in E'$ be arbitrary. Then we have $A(z)x \in L^p_D$. Let $\{d_k\}_{k\ge1}$ be an orthonormal basis for D. Thus we may write

$$A(z)x = \sum_{k\ge1} \langle A(z)x,\ d_k\rangle d_k \quad \text{for almost all } z \in \mathbb{T}. \tag{89}$$

Thus it follows that for all $y \in E$,

$$\big\langle \Phi(z)A(z)x,\ y\big\rangle = \sum_{k\ge1} \big\langle A(z)x,\ d_k\big\rangle\big\langle \Phi(z)d_k, y\big\rangle,$$

which implies that ΦA is WOT measurable. On the other hand, since $\Phi \in L^\infty(\mathcal{B}(D,E))$, it follows that

$$\int_{\mathbb{T}} ||(\Phi A)(z)x||^p_E dm(z) \le ||\Phi||^p_\infty \int_{\mathbb{T}} ||A(z)x||^p_D dm(z) < \infty \quad (x \in E'),$$

which implies that $\Phi A \in L^p_s(\mathcal{B}(E',E))$. This proves the first assertion. For the second assertion, suppose $\Phi \in H^\infty(\mathcal{B}(D,E))$ and $A \in H^2_s(\mathcal{B}(E',D))$. Then $\Phi A \in L^2_s(\mathcal{B}(E',E))$. Assume to the contrary that $\Phi A \notin H^2_s(\mathcal{B}(E',E))$. Thus, there exists $n_0 > 0$ such that $\widehat{\Phi A}(-n_0) \ne 0$. Thus for some $x_0 \in E'$,

$$\int_{\mathbb{T}} z^{n_0} \Phi(z)A(z)x_0 dm(z) \ne 0. \tag{90}$$

Then by (76), there exists a nonzero $y_0 \in E$ such that

$$0 \neq \left\langle \int_{\mathbb{T}} z^{n_0} \Phi(z) A(z) x_0 dm(z),\ y_0 \right\rangle = \int_{\mathbb{T}} \big\langle A(z)x_0,\ \overline{z}^{n_0} \Phi^*(z) y_0 \big\rangle dm(z). \tag{91}$$

On the other hand, since $\Phi \in H^\infty(\mathcal{B}(D,E))$, it follows from Lemma A.12 that $\widehat{\Phi^*}(n_0) = \widehat{\Phi}(-n_0)^* = 0$. Thus it follows from (76) that

$$0 = \big\langle \widehat{\Phi^*}(n_0) y_0,\ A(z)x_0 \big\rangle = \int_{\mathbb{T}} \big\langle \overline{z}^{n_0} \Phi^*(z) y_0,\ A(z)x_0 \big\rangle dm(z),$$

a contradiction. □

Corollary A.14. Let $1 \leq p < \infty$. If $\Phi \in L^\infty(\mathcal{B}(D,E))$, then $\Phi L^p_D \subseteq L^p_E$. Also, if $\Phi \in H^\infty(\mathcal{B}(D,E))$, then $\Phi H^2_D \subseteq H^2_E$.

Proof. Suppose that $\Phi \in L^\infty(\mathcal{B}(D,E))$. For $f \in L^P_D$, we can see that $[\Phi f] = \Phi[f]$. The result thus follows from Lemma A.7 and Lemma A.13. □

We then have:

Corollary A.15. Let Δ be an inner function with values in $\mathcal{B}(D,E)$. Then $f \in \mathcal{H}(\Delta)$ if and only if $f \in H^2_E$ and $\Delta^* f \in L^2_D \ominus H^2_D$.

Proof. Let $f \in H^2_E$. By Lemma A.12 and Corollary A.14, $\Delta^* f \in L^2_D$. Then $f \in \mathcal{H}(\Delta)$ if and only if $\big\langle f, \Delta g \big\rangle = 0$ for all $g \in H^2_D$ if and only if $\big\langle \Delta^* f,\ g \big\rangle = 0$ for all $g \in H^2_D$, which gives the result. □

Appendix B: Spectral multiplicity of model operators

The theory of spectral multiplicity for C_0-operators has been well developed in terms of their characteristic functions (cf. [Ni1, Appendix 1]). However this theory is not applied directly to $C_{0\,\cdot}$-operators, in which cases their characteristic functions need not be two-sided inner. In this appendix we show that if the characteristic function of a $C_{0\,\cdot}$-operator T has a finite-dimensional domain and a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then its spectral multiplicity can be computed by that of the C_0-operator induced by T. The main theorem of this appendix is as follows: Given an inner function Δ with values in $\mathcal{B}(E', E)$, with dim $E' < \infty$, let $T := S^*_E|_{\mathcal{H}(\Delta)}$. If Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then

$$\mu_T = \mu_{T_s}, \tag{92}$$

where T_s is a C_0-contraction of the form $T_s := S^*_{E'}|_{\mathcal{H}(\Delta_s)}$ with $\Delta_s := (\widetilde{\Delta})^i$. Hence in particular, $\mu_T \leq \dim E'$. (Here $(\cdot)^i$ means the inner part of the inner-outer factorization of the given H^∞-function.) (see Theorem B.8).

In Theorem B.8, we note that $\Delta_s \equiv \widetilde{(\widetilde{\Delta})^i}$ is a two-sided inner function (see Lemma B.5) (and hence, T_s belongs to the class C_0). Therefore (92) shows that the spectral multiplicity of a $C_{0\bullet}$-operator can be determined by the induced C_0-operator if its characteristic function has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. On the other hand, it was known (cf. [Ni1, p. 41]) that if $T := S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ with values in $\mathcal{B}(E', E)$, with $\dim E' < \dim E$, then

$$\mu_T \leq \dim E' + 1; \tag{93}$$

if further $\dim E' = \dim E < \infty$, then

$$\mu_T \leq \dim E'. \tag{94}$$

Thus, the equation (92) shows that (94) still holds without the assumption $\dim E' = \dim E$.

We first observe:

Lemma B.1. If $\Phi \in L^2_{\mathcal{B}(D,E)}$ and $f \in H^\infty_D$, then $\Phi f \in L^2_E$.

Proof. Suppose $\Phi \in L^2_{\mathcal{B}(D,E)}$ and $f \in H^\infty_D$. Since f is strongly measurable, there exist countable valued functions $f_n = \sum_{k=1}^\infty d_k^{(n)} \chi_{\sigma_k^{(n)}}$ such that $f(z) = \lim_n f_n(z)$ for almost all $z \in \mathbb{T}$. For all $e \in E$ and $n = 1, 2, 3, \cdots$,

$$\big\langle \Phi(z) f_n(z),\ e \big\rangle_E = \sum_{k=1}^\infty \chi_{\sigma_k^{(n)}}(z) \cdot \big\langle \Phi(z) d_k^{(n)},\ e \big\rangle_D. \tag{95}$$

But since Φ is WOT measurable, by (95), Φf_n is weakly measurable and in turn, $\Phi f : \mathbb{T} \to E$ is weakly measurable, and hence it is strongly measurable. Observe that

$$\int_{\mathbb{T}} ||\Phi(z) f(z)||_E^2 dm(z) \leq ||f||_\infty \int_{\mathbb{T}} ||\Phi(z)||^2 dm(z) < \infty,$$

which implies that $\Phi f \in L^2_E$. This completes the proof. $\square$

Lemma B.2. Let $\Phi \in L^2_{\mathcal{B}(D,E)}$ and let $A : H^2_D \to H^2_E$ be a densely defined operator, with domain $H^\infty_D \subset H^2_D$, defined by

$$Af := JP_-(\Phi f) \quad (f \in H^\infty_D).$$

Then

$$\ker A^* = \ker H^*_\Phi.$$

Proof. Let $\Phi \in L^2_{\mathcal{B}(D,E)} \subseteq L^2_s(\mathcal{B}(D,E))$. Since the domain of H_Φ is a subset of the domain of A, it follows that the domain of A^* is a subset of the domain of H^*_Φ, so that $\ker A^* \subseteq \ker H^*_\Phi$. For the reverse inclusion, suppose $g \in \ker H^*_\Phi$. Then

$$\big\langle H_\Phi p,\ g \big\rangle_{L^2_E} = 0 \quad \text{for all } p \in \mathcal{P}_D. \tag{96}$$

Let $f \in H^\infty_D$ be arbitrary. Then we may write

$$f(z)=\sum_{k=0}^{\infty} a_k z^k \quad (a_k \in D).$$

Let

$$p_n(z):=\sum_{k=0}^{n} a_k z^k \in \mathcal{P}_D.$$

Then it follows from (96) that

$$0=\lim_{n\to\infty}\big\langle H_\Phi p_n,\ g\big\rangle_{L^2_E}=\lim_{n\to\infty}\big\langle p_n,\ \Phi^* Jg\big\rangle_{L^2_D}=\big\langle \Phi f,\ Jg\big\rangle_{L^2_E}=\big\langle Af,\ g\big\rangle_{L^2_E},$$

which implies that $g \in \ker A^*$, so that $\ker H^*_\Phi \subseteq \ker A^*$. This completes the proof. $\square$

Corollary B.3. If $\Phi \in H^2_{\mathcal{B}(D,E)}$, then

$$E^*_{\{\Phi\}}=\mathrm{cl}\Big\{JP_-\big(\overline{z}\breve{\Phi}h\big): h \in H^\infty_D\Big\}.$$

Proof. Define $A: H^2_D \to H^2_E$ by $Af := JP_-(\overline{z}\breve{\Phi}h)$ $(h \in H^\infty_D)$. By Lemma B.2, $\ker H^*_{\overline{z}\breve{\Phi}} = \ker A^*$. By Lemma 2.9, we have

$$E^*_{\{\Phi\}}=\mathrm{cl}\,\mathrm{ran}\,H_{\overline{z}\breve{\Phi}}=\mathrm{cl}\,\mathrm{ran}\,A=\mathrm{cl}\Big\{JP_-\big(\overline{z}\breve{\Phi}h\big): h \in H^\infty_D\Big\}.$$

$\square$

We thus have:

Lemma B.4. Suppose Δ is a two-sided inner function and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Let $F \equiv \{f_1, f_2, \cdots, f_p\} \subseteq \mathcal{H}(\Delta)$. Then

$$E^*_F=\bigvee\Big\{P_+(\breve{h}_j f_j): h_j \in H^\infty \cap \mathcal{H}(\widetilde{\omega_\Delta}),\ j=1,2,\cdots p\Big\},$$

where ω_Δ is the pseudo-characteristic scalar inner function of Δ.

Proof. Suppose Δ is a two-sided inner function with values in $\mathcal{B}(E)$ and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Let $F \equiv \{f_1, f_2, \cdots, f_p\} \subseteq \mathcal{H}(\Delta)$. Write $[F] := \big[[f_1],[f_2],\cdots,[f_p]\big]$ and $\theta := \omega_\Delta$. Then, by Lemma A.7, $[f_j] \in H^2_{\mathcal{B}(\mathbb{C},E)}$ for each $j=1,2,\cdots,p$, so that $F \in H^2_{\mathcal{B}(\mathbb{C}^p,E)}$. We first claim that

$$E^*_F=\mathrm{cl}\Big\{JP_-(\overline{z}[\breve{F}]h): h \in H^\infty_{\mathbb{C}^p} \cap \mathcal{H}(\widetilde{\theta} I_p)\Big\}. \tag{97}$$

By Corollary B.3 we have

$$E^*_F=\mathrm{cl}\Big\{JP_-(\overline{z}[\breve{F}]h): h \in H^\infty_{\mathbb{C}^p}\Big\} \supseteq \mathrm{cl}\Big\{JP_-(\overline{z}[\breve{F}]h): h \in H^\infty_{\mathbb{C}^p} \cap \mathcal{H}(\widetilde{\theta} I_p)\Big\}.$$

For the reverse inclusion, it suffices to show that

$$P_-(\overline{z}[\breve{F}]\widetilde{\theta}h) = 0 \quad \text{for all } h \in H^\infty_{\mathbb{C}^p}. \tag{98}$$

By Lemma 2.28, we may write

$$\Delta = \theta A^* \quad \text{for some } A \in H^\infty(\mathcal{B}(E)).$$

Since Δ is two-sided inner, it follows that $I_E = \Delta\Delta^* = A^*A$, so that $\theta H^2_E = \Delta A H^2_E \subseteq \Delta H^2_E$. We thus have

$$\mathcal{H}(\Delta) \subseteq \mathcal{H}(\theta I_E).$$

Thus $f_j \in \mathcal{H}(\theta I_E)$ $(j = 1, \cdots, p)$. By Corollary A.15, $\overline{\theta} f_j \in L^2_E \ominus H^2_E$. Hence for all $h \in H^\infty_{\mathbb{C}^p}$, by Lemma B.1, we have $\overline{\theta}[F]\breve{h} \in L^2_E \ominus H^2_E$, so that $\overline{z}[\breve{F}]\widetilde{\theta}h \in H^2_E$, and hence $P_-(\overline{z}[\breve{F}]\widetilde{\theta}h) = 0$, which gives (98). This proves (97). Write $h = (h_1, h_2, \cdots h_p)^t \in H^\infty_{\mathbb{C}^p} \cap \mathcal{H}(\widetilde{\theta}I_p)$, and hence $h_j \in H^\infty \cap \mathcal{H}(\widetilde{\theta})$. Thus it follows from (97) that

$$\begin{aligned} E^*_F &= \mathrm{cl}\Big\{JP_-(\overline{z}[\breve{F}]h) : h \in H^\infty_{\mathbb{C}^p} \cap \mathcal{H}(\widetilde{\theta}I_p)\Big\} \\ &= \bigvee \Big\{JP_-(\overline{z}[\breve{f}_j]h_j) : h_j \in H^\infty \cap \mathcal{H}(\widetilde{\theta}),\ j = 1, 2, \cdots p\Big\} \\ &= \bigvee \Big\{P_+(\breve{h}_j f_j) : h_j \in H^\infty \cap \mathcal{H}(\widetilde{\theta}),\ j = 1, 2, \cdots p\Big\}. \end{aligned}$$

This completes the proof. □

Lemma B.5. Let Δ be an inner function with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$. If $\widetilde{\Delta} = (\widetilde{\Delta})^i(\widetilde{\Delta})^e$ is the inner-outer factorization of $\widetilde{\Delta}$, then we have:

(a) $(\widetilde{\Delta})^i$ is a two-sided inner function with values in $\mathcal{B}(E')$;

(b) $\widetilde{(\widetilde{\Delta})^e}$ is an inner function with values in $\mathcal{B}(E', E)$.

Proof. Let $\dim E' = r$. Then the inner part $(\widetilde{\Delta})^i$ is an $r \times p$ inner matrix function for some $p \le r$. Thus we have

$$p = \mathrm{Rank}\,(\widetilde{\Delta})^i \ge \mathrm{Rank}\,\widetilde{\Delta} = \mathrm{Rank}\,\Delta = r,$$

which proves (a). For (b), observe $\Delta = \widetilde{(\widetilde{\Delta})^e}\widetilde{(\widetilde{\Delta})^i}$. Since Δ is inner, we have that

$$I_r = \Delta^*\Delta = \widetilde{(\widetilde{\Delta})^i}^{\,*}\,\widetilde{(\widetilde{\Delta})^e}^{\,*}\,\widetilde{(\widetilde{\Delta})^e}\widetilde{(\widetilde{\Delta})^i}.$$

But since $(\widetilde{\Delta})^i$ is two-sided inner, so is $\widetilde{(\widetilde{\Delta})^i}$. Thus it follows that

$$\widetilde{(\widetilde{\Delta})^e}^{\,*}\,\widetilde{(\widetilde{\Delta})^e} = \widetilde{(\widetilde{\Delta})^i}\,\widetilde{(\widetilde{\Delta})^i}^{\,*} = I_r,$$

which implies that $\widetilde{(\widetilde{\Delta})^e}$ is an inner function. This proves (b). □

Lemma B.6. Suppose Δ is an inner function with values in $\mathcal{B}(E', E))$, with $\dim E' < \infty$ and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Write

$$\Delta_1 := \widetilde{(\widetilde{\Delta})^e}.$$

Then Δ_1 has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$.

Proof. Let $\dim E' = r$ and let $\widetilde{\Delta} = (\widetilde{\Delta})^i (\widetilde{\Delta})^e$ be the inner-outer factorization of $\widetilde{\Delta}$. Then

$$\Delta = \widetilde{(\widetilde{\Delta})^e}\, \widetilde{(\widetilde{\Delta})^i} \equiv \Delta_1 \Delta_{\rm s} \quad \Big(\text{where } \Delta_1 \equiv \widetilde{(\widetilde{\Delta})^e} \text{ and } \Delta_{\rm s} \equiv \widetilde{(\widetilde{\Delta})^i}\Big).$$

By Lemma B.5, $\Delta_{\rm s} \in H^\infty_{M_r}$ is square inner and $\Delta_1 \in H^\infty(\mathcal{B}(\mathbb{C}^r, E))$ is inner. Since Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, it follows from Lemma 2.28 that there exists a scalar inner function θ such that $\theta H^2_E \subseteq \ker H_{\Delta^*} = \ker H_{\Delta_{\rm s}^* \Delta_1^*}$. Thus we have

$$\Delta_{\rm s}^* \Delta_1^* \theta H^2_E \subseteq H^2_{\mathbb{C}^r}. \tag{99}$$

Since $\Delta_{\rm s}$ is square inner, it follows from (99) that $\Delta_1^* \theta H^2_E \subseteq \Delta_{\rm s} H^2_{\mathbb{C}^r} \subseteq H^2_{\mathbb{C}^r}$, so that $\theta H^2_E \subseteq \ker H_{\Delta_1^*}$, which implies, by Lemma 2.28, that Δ_1 has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. This completes the proof. $\square$

Lemma B.7. Let Δ_1 be an inner function with values in $\mathcal{B}(D, E)$ and Δ_2 be a two-sided inner function with values in $\mathcal{B}(D)$. Then,

$$\mathcal{H}(\Delta_1 \Delta_2) = \mathcal{H}(\Delta_1) \bigoplus \Delta_1 \mathcal{H}(\Delta_2).$$

Proof. This follows from a straightforward calculation together with Corollary A.14 and Corollary A.15. $\square$

We are ready for:

Theorem B.8. (The spectral multiplicity of model operators) Given an inner function Δ with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$, let $T := S_E^*|_{\mathcal{H}(\Delta)}$. If Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then

$$\mu_T = \mu_{T_s}, \tag{100}$$

where T_s is a C_0-contraction of the form $T_s := S_{E'}^*|_{\mathcal{H}(\Delta_s)}$ with $\Delta_s := \widetilde{(\widetilde{\Delta})^i}$. Hence in particular, $\mu_T \le \dim E'$.

Proof. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$. Suppose Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Let $\Delta_{\rm s} \equiv \widetilde{(\widetilde{\Delta})^i}$ and write

$$T_s := S_{E'}^*|_{\mathcal{H}(\Delta_{\rm s})}.$$

If Δ is two-sided inner, then $\Delta = \Delta_{\rm s}$, so that $\mu_T = \mu_{T_{\rm s}}$. Suppose that Δ is not two-sided inner. Without loss of generality, we may assume that $E' = \mathbb{C}^r$. By Lemma B.5,

$\Delta_s \in H^\infty_{M_r}$ is square inner. Thus by (24) and Proposition 2.34, we have that $T_s \in C_0$. We will prove that

$$\mu_T = \mu_{T_s}. \tag{101}$$

Write

$$\Delta_1 \equiv \widetilde{(\widetilde{\Delta})^e}.$$

Then it follows from Lemma B.5 and Lemma B.6 that Δ_1 is an inner function having a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Let

$$\theta := \omega_{\Delta_1}\omega_{\Delta_s}. \tag{102}$$

Let $p := \mu_{T_s}$. In view of (94), we have $p \le r$. Then there exists a set $F \equiv \{f_1, f_2, \cdots, f_p\} \subseteq \mathcal{H}(\Delta_s)$ such that $E^*_F = \mathcal{H}(\Delta_s)$. Since by (102), $\mathcal{H}(\widetilde{\omega_{\Delta_s}}) \subseteq \mathcal{H}(\widetilde{\theta})$, it follows from Lemma B.4 that

$$\mathcal{H}(\Delta_s) = \bigvee\Big\{P_+(\breve{h}_j f_j) : h_j \in H^\infty \cap \mathcal{H}(\widetilde{\theta}),\ j = 1, 2, \cdots p\Big\}. \tag{103}$$

Write

$$\Omega := (\Delta_1)_c \in H^\infty(E'', E) \quad (E'' \text{ is a subspace of } E).$$

Since $\widetilde{\Delta_1}$ is outer, it follows from Lemma 5.14 that $\Delta_1 = \Omega_c$. Choose a cyclic vector g of $S^*_{E''}$. Then it follows from Lemma 5.5, Lemma 5.7 and Lemma B.4 that

$$\mathcal{H}(\Delta_1) = E^*_{\Omega g} = \mathrm{cl}\,\big\{P_+\big(\breve{h}\Omega g\big) : h \in H^\infty\big\}. \tag{104}$$

Let

$$\gamma_1 := \theta\Omega g + \Delta_1 f_1 \quad \text{and} \quad \gamma_j := \Delta_1 f_j \quad (j = 2, 3, \cdots, p).$$

Now we will show that

$$\mathcal{H}(\Delta) = \bigvee\Big\{P_+(\breve{\eta}_j\gamma_j) : \eta_j \in H^\infty,\ j = 1, 2, \cdots p\Big\}. \tag{105}$$

Let $\xi \in \mathcal{H}(\Delta)$ and $\epsilon > 0$ be arbitrary. Then, by Lemma B.7, we may write

$$\xi = \xi_1 + \Delta_1\xi_2 \quad \big(\xi_1 \in \mathcal{H}(\Delta_1),\ \xi_2 \in \mathcal{H}(\Delta_s)\big).$$

By (103), there exist $h_j \in H^\infty \cap \mathcal{H}(\widetilde{\theta})$ $(j = 1, 2, \cdots, p)$ such that

$$\left|\left| \sum_{j=1}^p P_+(\breve{h}_j f_j) - \xi_2 \right|\right|_{L^2_{\mathbb{C}^r}} < \frac{\epsilon}{2}. \tag{106}$$

For each $j = 1, 2, \cdots, p$, observe that

$$\begin{aligned} P_+(\breve{h}_j\Delta_1 f_j) &= P_+(\Delta_1\breve{h}_j f_j) \\ &= \Delta_1 P_+(\breve{h}_j f_j) + P_+\big(\Delta_1 P_-(\breve{h}_j f_j)\big), \end{aligned} \tag{107}$$

and

$$\Delta_1 P_+(\breve{h}_j f_j) \in \Delta_1\mathcal{H}(\Delta_s) \quad \text{and} \quad P_+\big(\Delta_1 P_-(\breve{h}_j f_j)\big) \in \mathcal{H}(\Delta_1). \tag{108}$$

Since $\ker(\theta\Omega)^* = \ker \Omega^*$, we have $(\theta\Omega)_c = \Omega_c$. Thus by (104), $P_+(\breve{h}_1\theta\Omega g)$ belongs to $\mathcal{H}(\Delta_1)$. Thus it follows from (108) that

$$\xi_0 \equiv \xi_1 - \sum_{j=1}^{p} P_+\big(\Delta_1 P_-(\breve{h}_j f_j)\big) - P_+(\breve{h}_1\theta\Omega g) \in \mathcal{H}(\Delta_1).$$

Thus by (104), there exists $h_0 \in H^\infty$ such that

$$\Big|\Big| P_+\big(\breve{h}_0\Omega g\big) - \xi_0 \Big|\Big|_{L^2_E} < \frac{\epsilon}{2}. \tag{109}$$

Let

$$\eta_1 := \widetilde{\theta} h_0 + h_1 \quad \text{and} \quad \eta_j := h_j \quad (j = 2, 3, \cdots, p).$$

It follows from Lemma 2.28 that

$$\Delta_{\rm s} = \omega_{\Delta_{\rm s}} A^* \quad (A \in H^\infty_{M_r}).$$

It thus follows that $\overline{\omega_{\Delta_{\rm s}}} f_1 = A^*\Delta_{\rm s}^* f_1 \in L^2_{\mathbb{C}^r} \ominus H^2_{\mathbb{C}^r}$. Thus we have

$$\overline{\theta}\breve{h}_0 \Delta_1 f_1 = \breve{h}_0 \overline{\omega_{\Delta_1}} \Delta_1 \overline{\omega_{\Delta_{\rm s}}} f_1 \in L^2_E \ominus H^2_E,$$

which implies $P_+(\overline{\theta}\breve{h}_0\Delta_1 f_1) = 0$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{p} P_+(\breve{\eta}_j\gamma_j) &= P_+\big((\overline{\theta}\breve{h}_0 + \breve{h}_1)(\theta\Omega g + \Delta_1 f_1)\big) + \sum_{j=2}^{p} P_+(\breve{h}_j \Delta_1 f_j)\\
&= P_+\big(\breve{h}_0\Omega g\big) + P_+(\breve{h}_1\theta\Omega g) + \sum_{j=1}^{p} P_+(\breve{h}_j\Delta_1 f_j).
\end{aligned}$$

Since Δ_1 is inner, it follows from (106), (107) and (109) that

$$\begin{aligned}
\Big|\Big| \sum_{j=1}^{p} P_+(\breve{\eta}_j\gamma_j) - \xi \Big|\Big|_{L^2_E} &\le \Big|\Big| P_+\big(\breve{h}_0\Omega g\big) - \xi_0 \Big|\Big|_{L^2_E} + \Big|\Big| \sum_{j=1}^{p} \Delta_1 P_+(\breve{h}_j f_j) - \Delta_1\xi_2 \Big|\Big|_{L^2_E}\\
&< \epsilon.
\end{aligned}$$

This proves (105). Let $\Gamma := \{\gamma_1, \gamma_2, \cdots, \gamma_p\}$. It thus follows from Lemma B.4 and (105) that

$$E^*_\Gamma = \bigvee \Big\{ P_+(\breve{\eta}_j\gamma_j) : \eta_j \in H^\infty,\ j = 1, 2, \cdots p \Big\} = \mathcal{H}(\Delta),$$

which implies that $\mu_T \le \mu_{T_s}$. For the reverse inequality, let $q \equiv \mu_T < \infty$. Then there exists a set $F \equiv \{f_1, f_2, \cdots, f_q\} \subseteq \mathcal{H}(\Delta)$ such that $E^*_F = \mathcal{H}(\Delta)$. For each $j = 1, 2 \cdots q$, by Lemma B.7, we can write

$$f_j = g_j + \Delta_1\gamma_j \qquad (g_j \in \mathcal{H}(\Delta_1),\ \gamma_j \in \mathcal{H}(\Delta_{\rm s})).$$

Now we will show that

$$E^*_\Gamma = \mathcal{H}(\Delta_{\rm s}) \quad (\Gamma \equiv \{\gamma_j : j = 1, 2, \cdots, q\}). \tag{110}$$

Clearly, $E_\Gamma^* \subseteq \mathcal{H}(\Delta_s)$. On the other hand, since $E_F^* = \mathcal{H}(\Delta)$ and $\mathcal{H}(\Delta_1)$ is an invariant subspace for S_E^*, it follows from Lemma B.4, (107) and (108) that

$$\begin{aligned}
\Delta_1 \mathcal{H}(\Delta_s) &= \bigvee \Big\{ P_{\Delta_1 \mathcal{H}(\Delta_s)} \big(S_E^{*n} \Delta_1 \gamma_j \big) : j = 1, 2, \cdots q, \ n = 0, 1, 2 \cdots \Big\} \\
&= \bigvee \Big\{ P_{\Delta_1 \mathcal{H}(\Delta_s)} (\breve{h}_j \Delta_1 \gamma_j) : h_j \in H^\infty, \ j = 1, 2, \cdots p \Big\} \\
&= \bigvee \Big\{ \Delta_1 P_+ (\breve{h}_j \gamma_j) : h_j \in H^\infty, \ j = 1, 2, \cdots p \Big\} \\
&= \Delta_1 E_\Gamma^*.
\end{aligned}$$

This proves (110). Thus we have that $\mu_{T_s} \le q = \mu_T$. This proves (101). The last assertion follows at once from (94) since Δ_s is square-inner. This completes the proof. □

Corollary B.9. Suppose Δ is an $n \times r$ inner matrix function whose flip $\breve{\Delta}$ is of bounded type. If $T := S_E^*|_{\mathcal{H}(\Delta)}$, then $\mu_T \le r$.

Proof. It follows from Corollary 2.32 and Theorem B.8. □

On the other hand, we were unable to find an example showing that Theorem B.8 may fail if the condition "Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$" is dropped. Thus we would like to pose:

Problem B.10. Find an example of the operator $T \equiv S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ with values in $\mathcal{B}(E', E)$, with dim $E' < \infty$, satisfying $\mu_T = \text{dim } E' + 1$.

Appendix C: Miscellanea

In this appendix, by using the preceding results, we analyze left and right coprimeness, the model operator, and an interpolation problem for operator-valued functions.

C.1. Left and right coprime-ness. In this subsection we consider conditions for the equivalence of left coprime-ness and right coprime-ness.

If δ is a scalar inner function, a function $A \in H^\infty(\mathcal{B}(E))$ is said to have a *scalar inner multiple* δ if there exists a function $G \in H^\infty(\mathcal{B}(E))$ such that

$$GA = AG = \delta I_E.$$

We write mul$\,(A)$ for the set of all scalar inner multiples of A, and we define

$$m_A := \text{g.c.d.} \big\{ \delta : \delta \in \text{mul}\,(A) \big\}. \tag{111}$$

We note that if Δ is a two-sided inner function then by Lemma 2.28 and (55), the following are equivalent:

(a) Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$;

(b) Δ has a scalar inner multiple.

Thus if $\Delta \in H^\infty(\mathcal{B}(E))$ is two-sided inner and has a scalar multiple, then m_Δ defined in (111) coincides with the characteristic function of Δ. This justifies the use of the notation m_A for (111).

On the other hand, we may ask:

Question C.1. If $A \in H^\infty(\mathcal{B}(D,E))$ has a scalar inner multiple, does it follow that $m_A \in \operatorname{mul}(A)$?

If A is two-sided inner with values in $\mathcal{B}(E)$, then the answer to Question C.1 is affirmative: indeed, by (55),

$$m_A H^2_E = \bigvee \bigl\{\delta H^2_E : \delta \in \operatorname{mul}(A)\bigr\} \subseteq \ker H_{A^*},$$

which implies, again by (55), that

$$m_A \in \operatorname{mul}(A). \tag{112}$$

Lemma C.2. If $A \in H^\infty(\mathcal{B}(E))$ is an outer function having a scalar inner multiple, then $1 \in \operatorname{mul}(A)$, i.e., A is invertible in $H^\infty(\mathcal{B}(E))$.

Proof. Suppose that $A \in H^\infty(\mathcal{B}(E))$ is an outer function having a scalar inner multiple δ. Then

$$AG = GA = \delta I_E \quad \text{for some } G \in H^\infty(\mathcal{B}(E)). \tag{113}$$

We claim that

$$AH^2_E = \operatorname{cl} AH^2_E. \tag{114}$$

To see this, suppose $f \in \operatorname{cl} AH^2_E$. Then there exists a sequence (g_n) in H^2_E such that $||Ag_n - f||_{L^2_E} \longrightarrow 0$. Thus we have that

$$||GAg_n - Gf||_{L^2_E} \le ||G||_\infty ||Ag_n - f||_{L^2_E} \longrightarrow 0. \tag{115}$$

It thus follows from (113) and (115) that

$$||g_n - \overline{\delta} Gf||_{L^2_E} = ||\delta g_n - Gf||_{L^2_E} = ||GAg_n - Gf||_{L^2_E} \longrightarrow 0$$

But since H^2_E is a closed subspace of L^2_E, we have $g \equiv \overline{\delta} Gf \in H^2_E$. Since $A \in H^\infty(\mathcal{B}(E))$, it follows that

$$||Ag_n - Ag||_{L^2_E} \le ||A||_\infty ||g_n - g||_{L^2_E} \longrightarrow 0,$$

which implies that $f = Ag \in AH^2_E$. This proves (114). Since A is an outer function, it follows from (114) that

$$AH^2_E = \operatorname{cl} AH^2_E \supseteq \operatorname{cl} A\mathcal{P}_E = H^2_E,$$

so that

$$AH^2_E = H^2_E. \tag{116}$$

We thus have that

$$H_E^2 = (\overline{\delta}G)AH_E^2 = \overline{\delta}GH_E^2,$$

which implies that $G_1 := \overline{\delta}G \in H^\infty(\mathcal{B}(E))$. It thus follows from (113) that

$$AG_1 = G_1A = I_E,$$

which gives the result. $\square$

We are tempted to guess that (116) holds for every outer function A in $H^\infty(\mathcal{B}(E))$. However, the following example shows that this is not such a case.

Example C.3. Let $A := \text{diag}(\frac{1}{n}) \in H^\infty(\mathcal{B}(\ell^2))$. Then $(1, \frac{1}{2}, \frac{1}{3}, \cdots)^t \notin AH_{\ell^2}^2$, so that

$$AH_{\ell^2}^2 \neq H_{\ell^2}^2.$$

It is easy to show that A is an outer function.

Lemma C.4. If $A \in H^\infty(\mathcal{B}(E))$ has a scalar inner multiple, then

(a) A^i is two-sided inner and has a scalar inner multiple with $\text{mul}\,(A) \subseteq \text{mul}\,(A^i)$;

(b) $1 \in \text{mul}\,(A^e)$.

Proof. Suppose that $A \in H^\infty(\mathcal{B}(E))$ has a scalar inner multiple δ, i.e., $\delta \in \text{mul}\,(A)$. Then there exist a function $G \in H^\infty(\mathcal{B}(E))$ such that

$$AG = GA = \delta I_E. \tag{117}$$

Thus $A(z)$ and $G(z)$ are invertible for almost all $z \in \mathbb{T}$. Write

$$A = A^iA^e \quad \text{(inner-outer factorization)}.$$

Since $A(z)$ is invertible for almost all $z \in \mathbb{T}$, $A^i(z)$ is onto, so that $A^i(z)$ is an unitary operator for almost all $z \in \mathbb{T}$. We thus have that

$$A^i(A^eG) = AG = \delta I_E = (A^eG)A^i,$$

which implies that A^i has a scalar inner multiple δ, i.e., $\delta \in \text{mul}\,(A^i)$. This proves (a). Also observe that

$$(GA^i)A^e = \delta I_E = A^e(GA^i),$$

which implies that A^e has a scalar inner multiple. Thus by Lemma C.2, $1 \in \text{mul}\,(A^e)$. This proves (b). $\square$

Lemma C.5. If $A \in H^\infty(\mathcal{B}(E))$ has a scalar inner multiple, then

$$\text{mul}\,(A) = \text{mul}\,(A^i).$$

Proof. In view of Lemma C.4 (a), it suffices to show that $\text{mul}\,(A^i) \subseteq \text{mul}\,(A)$. To see this, let $\delta \in \text{mul}\,(A^i)$. Then

$$A^i G = G A^i = \delta I_E \quad \text{for some } G \in H^\infty(\mathcal{B}(E)).$$

Put $G_0 := (A^e)^{-1} G$. Then by Lemma C.4, $G_0 \in H^\infty(\mathcal{B}(E))$ and

$$A G_0 = A^i A^e (A^e)^{-1} G = \delta I_E.$$

But since A has a scalar inner multiple, $A(z)$ is invertible for almost all $z \in \mathbb{T}$. Thus we have $\delta \in \text{mul}\,(A)$. This proves $\text{mul}\,(A^i) \subseteq \text{mul}\,(A)$. This completes the proof. $\square$

The following corollary gives an affirmative answer to Question C.1.

Corollary C.6. If $A \in H^\infty(\mathcal{B}(E))$ has a scalar inner multiple then

$$m_A \in \text{mul}\,(A).$$

Proof. By Lemma C.4, A^i is two-sided inner. By (112), $m_{A^i} \in \text{mul}\,(A^i)$. Thus it follows from Lemma C.5 that

$$m_A = m_{A^i} \in \text{mul}\,(A^i) = \text{mul}\,(A).$$

$\square$

The following lemma is elementary.

Lemma C.7. Let E be a complex Hilbert space. If θ and δ are scalar inner functions, then

$$\text{left-g.c.d.}\,\{\theta I_E, \delta I_E\} = \text{g.c.d.}\,\{\theta, \delta\} I_E.$$

Proof. Let

$$\Omega := \text{left-g.c.d.}\,\{\theta I_E, \delta I_E\} \quad \text{and} \quad \omega := \text{g.c.d.}\,\{\theta, \delta\}.$$

Then we can write

$$\theta = \omega \theta_1 \quad \text{and} \quad \delta = \omega \delta_1,$$

where θ_1 and δ_1 are coprime inner functions. Thus we have

$$\Omega H^2_E = \theta H^2_E \bigvee \delta H^2_E = \omega\theta_1 H^2_E \bigvee \omega\delta_1 H^2_E = \omega\Big(\theta_1 H^2_E \bigvee \delta_1 H^2_E\Big) = \omega H^2_E,$$

which implies that $\Omega = \omega I_E$. This completes the proof. $\square$

Lemma C.8. Let $A \in H^\infty(\mathcal{B}(E))$ have a scalar inner multiple and θ be a scalar inner function. Suppose that m_A is not an inner divisor of θ. If $\delta_0 \in \text{mul}\,(A)$ is such that A and $\omega I_E \equiv \text{g.c.d.}\{\theta, \delta_0\} I_E$ are left coprime, then $\delta_0 \overline{\omega} \in \text{mul}\,(A)$.

Proof. Let $A \in H^\infty(\mathcal{B}(E))$ have a scalar inner multiple and θ be a scalar inner function. Suppose that m_A is not an inner divisor of θ. Then we should have $1 \notin \mathrm{mul}\,(A)$. Thus, by Lemma C.2, A is not an outer function, so that A^i is not a unitary operator. Let $\delta_0 \in \mathrm{mul}\,(A)$ be such that A and $\omega I_E \equiv \text{g.c.d.}\{\theta, \delta_0\} I_E$ are left coprime. Then, by Lemma C.7, we may write

$$\theta = \omega\theta_1 \quad \text{and} \quad \delta_0 = \omega\delta_1, \tag{118}$$

where θ_1 and δ_1 are coprime scalar inner functions. On the other hand, since $\delta_0 \in \mathrm{mul}\,(A)$, we have that

$$\delta_0 I_E = GA = AG \quad \text{for some } G \in H^\infty(\mathcal{B}(E)). \tag{119}$$

Thus by (118) and (119), we have that

$$G(\overline{\omega} I_E)A = (\overline{\omega} I_E)GA = \delta_1 I_E \in H^\infty(\mathcal{B}(E)),$$

which implies that

$$AH^2_E \subseteq \ker H_{G(\overline{\omega} I_E)} \equiv \Theta H^2_{E'}. \tag{120}$$

Thus Θ is a left inner divisor of A. Since also $\omega H^2_E \subseteq \ker H_{G\overline{\omega} I_E} = \Theta H^2_{E'}$, Θ is a left inner divisor of ωI_E. Thus Θ is a common left inner divisor of A and ωI_E, so that, by our assumption, Θ is a unitary operator. Thus

$$\ker H_{G\overline{\omega} I_E} = \Theta H^2_{E'} = H^2_E,$$

which implies that $\overline{\omega} I_E G \in H^\infty(\mathcal{B}(E))$. On the other hand, by (118) and (119), we have

$$\delta_1 I_E = (\overline{\omega}\delta_0) I_E = (\overline{\omega} I_E G)A = A(\overline{\omega} I_E G),$$

which implies that $\delta_1 = \delta_0 \overline{\omega} \in \mathrm{mul}\,(A)$. This completes the proof. $\square$

We then have:

Theorem C.9. Let $A \in H^\infty(\mathcal{B}(E))$ and θ be a scalar inner function. If A has a scalar inner multiple, then the following are equivalent:

(a) θ and m_A are coprime;

(b) θI_E and A are left coprime;

(c) θI_E and A are right coprime.

Proof. Let $A \in H^\infty(\mathcal{B}(E))$ have a scalar inner multiple. Write

$$A = A^i A^e \quad \text{(inner-outer factorization)}.$$

(a) $\Rightarrow$ (b): Suppose that θI_E and A are not left coprime. Then

$$\theta H^2_E \bigvee A^i H^2_E \neq H^2_E.$$

By Corollary C.6, there exists $G \in H^\infty(\mathcal{B}(E))$ such that $GA = AG = m_A I_E$. Thus we have that

$$\text{left-g.c.d.}\,\{\theta I_E, m_A I_E\} H^2_E = \theta H^2_E \bigvee AGH^2_E \subseteq \theta H^2_E \bigvee A^i H^2_E \neq H^2_E,$$

which implies that θI_E and $m_A I_E$ are not left coprime. Thus by Lemma C.7, θ and m_A are not coprime.

(b) $\Rightarrow$ (a): Suppose that θ and m_A are not coprime. If m_A is an inner divisor of θ, then by Corollary C.6 and Lemma C.7, we may write

$$\theta I_E = m_A \theta_1 I_E = A^i A^e G \theta_1 I_E \quad (G \in H^\infty(\mathcal{B}(E)),\ \theta_1 \text{ is a scalar inner}).$$

Thus, A^i is a common left inner divisor of θI_E and A. If A^i is a unitary operator, then A is an outer function. It thus follows from Lemma C.2 that $m_A = 1$, so that θ and m_A are coprime, a contradiction. Therefore A^i is not a unitary operator, and hence θI_E and A are not left coprime. Suppose instead that m_A is not an inner divisor of θ. Write $\omega \equiv \text{g.c.d.}\{\theta, m_A\} \neq 1$. We then claim that

$$A \text{ and } \omega I_E \text{ are not left coprime.} \tag{121}$$

Towards (121), we assume to the contrary that A and ωI_E are left coprime. Then it follows from Corollary C.6 and Lemma C.8 that $\overline{\omega} m_A \in \text{mul}\,(A)$, which contradicts the definition of m_A. This proves (121). But since ω is an inner divisor of θ, it follows from Lemma C.7 that A and θI_E is not left coprime.

(b) $\Leftrightarrow$ (c). Since $\delta \in \text{mul}\,(A)$ if and only if $\widetilde{\delta} \in \text{mul}\,(\widetilde{A})$, it follows that $\widetilde{m_A} = m_{\widetilde{A}}$. It thus follows from (a) $\Leftrightarrow$ (b). This completes the proof. $\square$

Corollary C.10. Let Δ be an inner function with values in $\mathcal{B}(D, E)$ and θ be a scalar inner function. If Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then the following are equivalent:

(a) θ and ω_Δ are coprime;

(b) θI_E and $[\Delta, \Delta_c]$ are left coprime;

(c) θI_E and $[\Delta, \Delta_c]$ are right coprime.

Proof. Suppose that Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then by Lemma 2.27, $\breve{\Delta}$ is of bounded type, so that by Corollary 2.25, $[\Delta, \Delta_c]$ is two-sided inner. Thus the result follows from Theorem C.9 and Lemma 5.5. $\square$

Example C.11. Let

$$\Delta := \begin{bmatrix} b_\alpha & 0 \\ 0 & b_\beta \\ 0 & 0 \end{bmatrix} \quad (\alpha \neq 0, \beta \neq 0).$$

Then zI_3 and Δ are not left coprime because $zH^2_{\mathbb{C}^3} \bigvee \Delta H^2_{\mathbb{C}^2} \neq H^2_{\mathbb{C}^3}$. But zI_3 and $[\Delta, \Delta_c]$ are left coprime, so that, by Corollary C.10, z and ω_Δ are coprime. Indeed, we note that $\ker H_{\Delta^*} = [\Delta, \Delta_c] H^2_{\mathbb{C}^3}$, and hence $\omega_\Delta = b_\alpha b_\beta$.

The following example shows that if the condition "A has a scalar inner multiple" is dropped in Theorem C.9, then Theorem C.9 may fail.

Example C.12. Let
$$\Delta(z) = S_E \quad (E = \ell^2(\mathbb{Z}_+))$$
Then Δ is an inner function (not two-sided inner, an isometric operator) with values in $\mathcal{B}(E)$. For $f \in H^2_E$, we can write
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_n \in E).$$
We thus have that
$$(\widetilde{\Delta} f)(z) = S^*\Big(\sum_{n=0}^{\infty} a_n z^n\Big) = \sum_{n=0}^{\infty} (S^* a_n) z^n.$$
Thus $\widetilde{\Delta} H^2_E = H^2_E$, so that Δ and θI_E are right coprime for all scalar inner function θ. Let $\theta(z) = z\theta_1$ (θ_1 a scalar inner). Then
$$(\Delta f)(z) = S\Big(\sum_{n=0}^{\infty} a_n z^n\Big) = \sum_{n=0}^{\infty} (S a_n) z^n.$$
We thus have
$$\Delta H^2_E \bigvee \theta H^2_E = \Delta H^2_E \bigvee z\theta_1 H^2_E \subseteq \Delta H^2_E \bigvee z H^2_E \neq H^2_E,$$
which implies that θI_E and Δ are not left coprime. Note that Δ has no scalar inner multiple.

On the other hand, since $\ker H_{\Delta^*} = H^2_E$, we have $\omega_\Delta = 1$. Thus, it follows from Corollary C.10 that θI_E and $[\Delta, \Delta_c]$ are left (and right) coprime for all scalar inner function θ. □

Lemma C.13. If $\Delta \in H^\infty_{M_n}$ is an inner function then
$$\theta \text{ and } m_\Delta \text{ are coprime} \Longleftrightarrow \theta \text{ and } \det \Delta \text{ are coprime.} \tag{122}$$

Proof. If $\Delta \in H^\infty_{M_n}$ is inner, then $m_\Delta \in \operatorname{mul}(\Delta)$, so that we may write
$$m_\Delta I_n = \Delta G \text{ for some inner function } G \in H^\infty_{M_n}.$$
Thus, $\det \Delta \, \det G = m_\Delta^n$. If θ and m_Δ are coprime, then θ and m_Δ^n are coprime, so that θ and $\det \Delta$ are coprime. Conversely, suppose that θ and $\det \Delta$ are coprime. Since $(\det \Delta) I_n = (\operatorname{adj}\Delta)\Delta$, it follows that $\det \Delta \in \operatorname{mul}(\Delta)$. Thus, m_Δ is an inner divisor of $\det \Delta$, and hence θ and m_Δ are coprime. This proves (122). □

We can recapture [CHL3, Theorem 4.16].

Corollary C.14. Let $A \in H^2_{M_n}$ and θ be a scalar inner function. Then the following are equivalent:

(a) θ and $\det A$ are coprime;

(b) θI_n and A are left coprime;

(c) θI_n and A are right coprime.

Proof. If $\Delta \in H^\infty_{M_n}$ is inner then by Theorem C.9 and Lemma C.13, we have

$$\theta I_n \text{ and } \Delta \text{ are left coprime} \Longleftrightarrow \theta \text{ and } \det \Delta \text{ are coprime.} \tag{123}$$

We now write

$$A = A^i A^e \quad \text{(inner-outer factorization).}$$

Now we will show that if (b) or (c) holds, then A^i is two-sided inner: indeed if (b) or (c) holds, then by [CHL3, Lemma 4.15], $\det A \neq 0$, so that $A(z)$ is invertible, and hence $A^i(z)$ is onto for almost all $z \in \mathbb{T}$. Thus A^i is two-sided inner. Then by the Helson-Lowdenslager Theorem (cf. [Ni1, p.22]) we have that

$$\det A = \det A^i \cdot \det A^e \quad \text{(inner-outer factorization)}$$

It thus follows from (123) that

$$\begin{aligned} \theta I_n \text{ and } A \text{ are left coprime} &\Longleftrightarrow \theta I_n \text{ and } A^i \text{ are left coprime} \\ &\Longleftrightarrow \theta \text{ and } \det A^i \text{ are coprime} \\ &\Longleftrightarrow \theta \text{ and } \det A \text{ are coprime} \end{aligned}$$

For right coprime-ness, we apply the above result and the fact that $\det \widetilde{A} = \widetilde{\det A}$. $\square$

C.2. The model operator. We recall that the model theorem (p. 7) states that if $T \in \mathcal{B}(\mathcal{H})$ is a contraction such that $\lim_{n\to\infty} T^n x = 0$ for each $x \in \mathcal{H}$ (i.e., $T \in C_{0\cdot}$), then there exists a unitary imbedding $V : \mathcal{H} \to H^2_E$ with

$$E = \text{cl ran}\,(I - T^*T). \tag{124}$$

such that $V\mathcal{H} = \mathcal{H}(\Delta)$ for some inner function Δ with values in $\mathcal{B}(E', E)$ and

$$T = V^*\Big(S_E^*|_{\mathcal{H}(\Delta)}\Big)V. \tag{125}$$

We may now ask what is a necessary and sufficient condition for dim $E' < \infty$ in the Model Theorem. In this subsection, we give a necessary condition for the finite-dimensionality of E'.

For an inner function Δ with values in $\mathcal{B}(E', E)$, define

$$H_0 := \big\{f \in \mathcal{H}(\Delta) : \lim_{n\to\infty} P_{\mathcal{H}(\Delta)} S_E^n f = 0\big\}. \tag{126}$$

Then H_0 is a closed subspace of $\mathcal{H}(\Delta)$ and in this case, write

$$E_0(\Delta) := \mathcal{H}(\Delta) \ominus H_0.$$

Then $E_0(\Delta)$ is an invariant subspace of S_E^*, so that there exists an inner function $\Delta^s \in H^\infty(\mathcal{B}(E_1, E))$ such that

$$E_0(\Delta) = \mathcal{H}(\Delta^s). \tag{127}$$

We then have:

Lemma C.15. Let Δ be an inner function with values in $\mathcal{B}(E', E)$. Then

$$\Delta = \Delta^s \Delta_1$$

for some two-sided inner function Δ_1 with values in $\mathcal{B}(E', E_1)$.

Proof. Observe that $H^2_E = \Delta H^2_{E'} \oplus E_0(\Delta) \oplus H_0$. Thus,

$$\Delta H^2_{E'} \subseteq H^2_E \ominus E_0(\Delta) = \Delta^s H^2_{E_1},$$

which implies that $\Delta = \Delta^s \Delta_1$ for some inner function Δ_1 with values in $\mathcal{B}(E', E_1)$. We must show that Δ_1 is two-sided. We first claim that

$$f \in \Delta^s H^2_{E_1} \Longleftrightarrow ||f||_{L^2_E} = ||\Delta^* f||_{L^2_{E'}} : \tag{128}$$

indeed, since $\lim_{n\to\infty} ||(I_E - P_+)\Delta^* S^n_E f||_{L^2_{E'}} = 0$ for each $f \in H^2_E$, a straightforward calculation shows that

$$\lim_{n\to\infty} ||P_{\mathcal{H}(\Delta)} S^n_E f||^2_{L^2_E} = ||f||^2_{L^2_E} - ||\Delta^* f||^2_{L^2_{E'}},$$

giving (128). Thus for all $x \in E_1$ with $||x|| = 1$,

$$1 = ||\Delta^s x||_{L^2_E} = ||\Delta^* \Delta^s x||_{L^2_{E'}} = ||\Delta_1^* x||_{L^2_{E'}},$$

which says that

$$\int_{\mathbb{T}} ||\Delta_1^*(z)x||^2 dm(z) = 1.$$

But since $||\Delta_1^*(z)x|| \le 1$, it follows that $||\Delta_1^*(z)x|| = 1$ a.e. on $\mathbb{T}$, so that $\Delta_1^*(z)$ is isometry for almost all $z \in \mathbb{T}$ and therefore Δ_1 is two-sided inner. This completes the proof. □

We then have:

Theorem C.16. Let $T \in \mathcal{B}(H)$ be a contraction such that $\lim_{n\to\infty} T^n x = 0$ for each $x \in H$ and have a characteristic function Δ with values in $\mathcal{B}(E', E)$. Then,

$$\sup_{\zeta\in\mathbb{D}} \dim\big\{f(\zeta) : f \in H_0\big\} \le \dim E',$$

where H_0 is defined by (126). In particular, if $\dim E' < \infty$, then $\max_{\zeta\in\mathbb{D}} \dim\big\{f(\zeta) : f \in H_0\big\}$ is finite.

Proof. It follows from (127) that

$$H_0 = \mathcal{H}(\Delta) \ominus E_0(\Delta) = \mathcal{H}(\Delta) \ominus \mathcal{H}(\Delta^s) \subseteq \Delta^s H^2_{E'}.$$

Thus, by Lemma C.15, we have

$$\sup_{\zeta\in\mathbb{D}} \dim\big\{f(\zeta) : f \in H_0\big\} \le \sup_{\zeta\in\mathbb{D}} \dim\big\{\Delta^s(\zeta) g(\zeta) : g \in H^2_{E'}\big\} = \dim E'.$$

□

Until now, we were unable to determine E' in terms of spectral properties of T as in (124). In Theorem C.16, we give a necessary condition for "dim $E' < \infty$." Thus, we would like to pose:

Problem C.17. Let $T \in C_0.$ and $\Delta \in H^\infty(\mathcal{B}(E', E))$ be the characteristic function of T. For which operator T, we have dim $E' < \infty$?

C.3. An interpolation problem. In the literature, many authors have considered the special cases of the following (scalar-valued or operator-valued) interpolation problem (cf. [Co1], [CHL2], [CHL3], [FF], [Gar], [Gu], [GHR], [HKL], [HL1], [HL2], [NT], [Zhu]).

Problem C.18. For $\Phi \in L^\infty(\mathcal{B}(E))$, when does there exist a function $K \in H^\infty(\mathcal{B}(E))$ with $||K||_\infty \le 1$ satisfying

$$\Phi - K\Phi^* \in H^\infty(\mathcal{B}(E))\,? \tag{129}$$

If Φ is a matrix-valued rational function, this question reduces to the classical Hermite-Fejér interpolation problem.

For notational convenience, we write, for $\Phi \in L^\infty(\mathcal{B}(E))$,

$$\mathcal{C}(\Phi) := \Big\{K \in H^\infty(\mathcal{B}(E)) : \ \Phi - K\Phi^* \in H^\infty(\mathcal{B}(E))\Big\}.$$

We then have:

Theorem C.19. Let $\Phi \equiv \breve{\Phi}_- + \Phi_+ \in L^\infty(\mathcal{B}(E))$. If $\mathcal{C}(\Phi)$ is nonempty then

$$\ker H^*_{\breve{\Phi}_+} \subseteq \ker H^*_{\Phi^*_-}.$$

In particular,

$$\text{nc}\{\Phi_+\} \le \text{nc}\{\widetilde{\Phi_-}\}.$$

Proof. Suppose $\mathcal{C}(\Phi) \neq \emptyset$. Then there exists a function $K \in H^\infty(\mathcal{B}(E))$ such that $\Phi - K\Phi^* \in H^\infty(\mathcal{B}(E))$, then $H_\Phi = T^*_{\widetilde{K}} H_{\Phi^*}$, which implies that $\ker H_{\Phi^*} \subseteq \ker H_\Phi$. But since $\Phi \equiv \breve{\Phi}_- + \Phi_+ \in L^\infty(\mathcal{B}(E))$, it follows that

$$H_{\Phi^*} = H_{\Phi^*_+} = H^*_{\breve{\Phi}_+} \quad \text{and} \quad H_\Phi = H_{\breve{\Phi}_-} = H^*_{\Phi^*_-}.$$

We thus have

$$\ker H^*_{\breve{\Phi}_+} \subseteq \ker H^*_{\Phi^*_-}.$$

On the other hand, it follows from Lemma 2.6 that

$$\Omega H^2_{E'} = \ker H^*_{\breve{\Phi}_+} \subseteq \ker H^*_{\Phi^*_-} = \ker H^*_{\breve{\breve{\Phi}}_-} = \Delta H^2_{E''} \tag{130}$$

for some inner functions Ω and Δ with values in $\mathcal{B}(E', E)$ and $\mathcal{B}(E'', E)$, respectively. Thus Δ is a left inner divisor of Ω, so that we have $\dim E' \le \dim E''$, which implies, by Theorem 2.13, that $\text{nc}\{\Phi_+\} \le \text{nc}\{\widetilde{\Phi_-}\}$. $\square$

Corollary C.20. Let $\Phi \equiv \breve{\Phi}_- + \Phi_+ \in L^\infty(\mathcal{B}(E))$ and $\mathcal{C}(\Phi) \neq \emptyset$. If $\breve{\Phi}_+$ is of bounded type, then Φ_-^* is of bounded type.

Proof. Suppose that $\Phi \equiv \breve{\Phi}_- + \Phi_+ \in L^\infty(\mathcal{B}(E))$. Then by Lemma A.12, $\Phi^* = (\breve{\Phi}_-)^* + (\Phi_+)^* \in L^\infty(\mathcal{B}(E))$. Thus $(\breve{\Phi}_-)^*$ is a strong L^2-function and so is Φ_-^*. Assume that $\mathcal{C}(\Phi) \neq \emptyset$ and $\breve{\Phi}_+$ is of bounded type. Then it follows from Theorem C.19 and Lemma 2.6 that

$$\Omega H_E^2 = \ker H^*_{\breve{\Phi}_+} \subseteq \ker H^*_{\Phi_-^*} = \Delta H^2_{E''} \tag{131}$$

for some two-sided inner function Ω with values in $\mathcal{B}(E)$ and an inner function Δ with values in $\mathcal{B}(E'', E)$. Thus, Δ is a left inner divisor of Ω and hence, by Lemma 2.2, Δ is two-sided inner, so that Φ_-^* is of bounded type. $\square$

For $\Phi \in L^\infty(\mathcal{B}(E))$, write

$$\mathcal{E}(\Phi) := \Big\{ K \in H^\infty(\mathcal{B}(E)) : \ \Phi - K\Phi^* \in H^\infty(\mathcal{B}(E)) \ \text{ and } \ ||K||_\infty \leq 1 \Big\},$$

i.e., $\mathcal{E}(\Phi) = \{K \in \mathcal{C}(\Phi) : \ ||K||_\infty \leq 1\}$ (cf. p.86). If dim $E = 1$ and $\Phi \equiv \varphi$ is a scalar-valued function then an elegant theorem of C. Cowen (cf. [Co1], [NT], [CL]) says that $\mathcal{E}(\varphi)$ is nonempty if and only if T_φ is hyponormal, i.e., the self-commutator $[T_\varphi^*, T_\varphi]$ is positive semi-definite. Cowen's Theorem is to recast the operator-theoretic problem of hyponormality into the problem of finding a solution of an interpolation problem. In [GHR], it was shown that the Cowen's theorem still holds for a Toeplitz operator T_Φ with a matrix-valued *normal* (i.e., $\Phi^*\Phi = \Phi\Phi^*$) symbol $\Phi \in L^\infty_{M_n}$.

Problem C.21. Extend Cowen's theorem for a Toeplitz operator with an operator-valued normal symbol $\Phi \in L^\infty(\mathcal{B}(E))$.

We recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called *subnormal* if T has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant for N. In 1979, P.R. Halmos posed the following problem, listed as Problem 5 in his Lecture "Ten problems in Hilbert space" ([Ha2], [Ha3]): Is every subnormal Toeplitz operator T_φ with symbol $\varphi \in L^\infty$ either normal or analytic (i.e., $\varphi \in H^\infty$) ? In 1984, C. Cowen and J. Long [CoL] have answered this question in the negative. To date, a characterization of subnormality of Toeplitz operators T_φ in terms of the symbols φ has not been found. The best partial answer to Halmos' Problem 5 was given by M.B. Abrahamse: If $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then T_φ is either normal or analytic; this is called Abrahamse's Theorem. Very recently, in [CHL3, Theorem 7.3], Abrahamse's Theorem was extended to the cases of Toeplitz operators T_Φ with matrix-valued symbols Φ under some constraint on the symbols Φ; concretely, when "Φ has a tensored-scalar singularity."

We would like to pose:

Problem C.22. Extend Abrahamse's Theorem to Toeplitz operators T_Φ with operator-valued symbols $\Phi \in L^\infty(\mathcal{B}(E))$.

Acknowledgment. The authors are deeply grateful to the referee for a very careful reading of the manuscript, and for many helpful suggestions which helped improve both the content and the presentation.

References

[Abr] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[AD] M.B. Abrahamse and R.G. Douglas, *A class of subnormal operators related to multiply-connected domais*, Adv. Math. **19** (1976), 106–148.

[ARS] A. Aleman, S. Richter, and C. Sundberg, *Beurling's theorem for the Bergman space*, Acta Math. **177(2)** (1996), 275-310.

[ADR] D. Alpay, A. Dijksma, and J. Rovnyak, *A theorem on Beurling-Lax type for Hilbert spaces of functions analytic in the unit ball*, Integral Equations Operator Theory **47** (2003), 251–274.

[AS] D. Alpay and I. Sabodini, *Beurling-Lax type theorems in the complex and quaternionic setting*, Linear Alg. Appl. **530** (2017), 15–46.

[BaB] J.A. Ball and V. Bolotnikov, *Weighted Hardy spaces: shift invariant and coinvariant subspaces, linear systems and operator model theory*, Acta Sci. Math. (Szeged) **79** (2013), 623–686.

[BH1] J.A. Ball and J.W. Helton, *A Beurling-Lax theorem for the Lie group $U(m,n)$ which contains most classical interpolation theory*, J. Operator Theory **9** (1983), 107–142.

[BH2] J.A. Ball and J.W. Helton, *Beurling-Lax representations using classical Lie groups with many applications. II. $GL(n,\mathbb{C})$ and Wiener-Hopf factorization*, Integral Equations Operator Theory **7(3)** (1984), 291–309.

[BH3] J.A. Ball and J.W. Helton, *Beurling-Lax representations using classical Lie groups with many applications. III. Groups preserving two bilinear forms*, Amer. J. Math. **108** (1986), 95–174.

[BH4] J.A. Ball and J.W. Helton, *Shift invariant manifolds and nonlinear analytic function theory*, Integral Equations Operator Theory **11** (1988), 615–725.

[BR] J.A. Ball and M.W. Raney, *Discrete-time dichotomous well-posed linear systems and generalized Schur-Nevanlinna-Pick interpolation*, Complex Anal. Oper. Theory **1(1)** (2007), 1–54.

[Ber] H. Bercovici, *Operator Theory and Arithmetic in H^∞*, Mathematical Surveys and Monographs, vol. 26, Amer. Math. Soc., Providence, 1988.

[Beu] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1949), 239-255.

[BoB] S. Bochner and H.F. Bohnenblust, *Analytic functions with almost periodic coefficients*, Ann. Math. **35** (1934), 152–161.

[BS] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Springer, Berlin-Heidelberg, 2006.

[Car] M. Carlsson, *On the Beurling-Lax theorem for domains with one hole*, New York J. Math. **17A** (2011), 193–212.

[Con] J.B. Conway, *A Course in Functional Analysis*, Springer, New York, 1990.

[Co1] C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), 809–812.

[Co2] C. Cowen, *Hyponormal and subnormal Toeplitz operators*, Surveys of Some Recent Results in Operator Theory, I (J.B. Conway and B.B. Morrel, eds.), Pitman Research Notes in Mathematics, Vol **171** (1988), 155–167.

[CoL] C. Cowen and J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351** (1984), 216–220.

[CHKL] R.E. Curto, I.S. Hwang, D. Kang and W.Y. Lee, *Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols*, Adv. Math. **255** (2014), 561–585.

[CHL1] R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. **263(8)** (2012), 2333–2354.

[CHL2] R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230** (2012), 2094–2151.

[CHL3] R.E. Curto, I.S. Hwang and W.Y. Lee, *Matrix functions of bounded type: An interplay between function theory and operator theory*, Mem. Amer. Math. Soc. **260** (2019), no. 1253, vi+100.

[CL] R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Mem. Amer. Math. Soc. **150** (2001), no. 712, x+65.

[dBR] L. de Branges and J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York-Toronto, 1966.

[Do1] R.G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[Do2] R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, CBMS vol. 15, Amer. Math. Soc., Providence, 1973.

[DH] R.G. Douglas and J.W. Helton, *Inner dilations of analytic matrix functions and Darlington synthesis*, Acta Sci. Math. (Szeged) **34** (1973), 61–67.

[DSS] R.G. Douglas, H. Shapiro, and A. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier(Grenoble) **20** (1970), 37–76.

[DS] N. Dunford and J. Schwartz, *Linear Operators. General theory*, Part I., New York, Wiley, 1958.

[Dur] P.L. Duren, *Theory of H^p spaces*, New York, Academic Press, 1970.

[FF] C. Foiaş and A. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, Oper. Th. Adv. Appl. vol. 44, Birkhäuser, Boston, 1993.

[FB] A. Frazho and W. Bhosri, *An Operator Perspective on Signals and Systems*, Oper. Th. Adv. Appl. vol. 204, Birkhäuser, Basel, 2010.

[Fu1] P. Fuhrmann, *On Hankel operator ranges, meromorphic pseudo-continuation and factorization of operator-valued analytic functions*, J. London Math. Soc.(2), **13** (1975), 323–327.

[Fu2] P. Fuhrmann, *Linear Systems and Operators in Hilbert Spaces*, McGraw-Hill, New York, 1981.

[Gar] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[Gol] S. Goldberg, *Unbounded Linear Operators*, Dover, New York, 2006.

[Gu] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124** (1994), 135–148.

[GHR] C. Gu, J. Hendricks and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (2006), 95–111.

[Ha1] P.R. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.

[Ha2] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.

[Ha3] P.R. Halmos, *Ten years in Hilbert space*, Integral Equations Operator Theory **2** (1979), 529–564.

[Ha4] P.R. Halmos, *A Hilbert Space Problem Book*, Springer, New York, 1982.

[HLL] J.K. Han, H.Y. Lee, and W.Y. Lee, *Invertible completions of* 2×2 *triangular operator matrices*. Proc. Amer. Math. Soc. **128**, 119–123.

[Hed] H. Hedenmalm, *A factorization theorem for square area-integrable analytic functions*, J. Reine Angew. Math. **422** (1991), 45–68.

[Hel] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[HP] E. Hille and R.S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Coll. Publ. **31**, Providence, 1957.

[Hof] K. Hoffman, *Banach Spaces of Analytic functions*, Englewood Cliffs, Prentice-Hall, 1962.

[HKL] I.S. Hwang, I.H. Kim, and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols*, Math. Ann. **313(2)** (1999), 247–261.

[HL1] I.S. Hwang and W.Y. Lee, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math. Soc. **354** (2002), 2461–2474.

[HL2] I.S. Hwang and W.Y. Lee, *Hyponormality of Toeplitz operators with rational symbols*, Math. Ann. **335** (2006), 405–414.

[Lax] P.D. Lax, *Translation invariant subspaces*, Acta Math. **101** (1959), 163–178.

[MT] S. McCullough and T.T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. **178(1)** (2000), 226-249.

[MR] R. A. Martínez-Avendaño and P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Space*, Springer, New York, 2007.

[MN] B. Moore III and E.A. Nordgren, *On quasi-equivalence and quasi-similarity*, Acta. Sci. Math. (Szeged) **34** (1973), 311–316.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), 753–767.

[Ni1] N.K. Nikolskii, *Treatise on the Shift Operator*, Springer, New York, 1986.

[Ni2] N.K. Nikolskii, *Operators, Functions, and Systems: An Easy Reading Volume I: Hardy, Hankel, and Toeplitz*, Mathematical Surveys and Monographs, vol. 92, Amer. Math. Soc., Providence, 2002.

[Nor] E.A. Nordgren, *On quasi-equivalence of matrices over H^∞*, Acta. Sci. Math. (Szeged) **34** (1973), 301–310.

[Pel] V.V. Peller, *Hankel Operators and Their Applications*, Springer, New York, 2003.

[Pop] G. Popescu, *Multi-analytic operators and some factorization theorems*, Indiana Univ. Math. J. **38** (1989), 693–710.

[Ric] S. Richter, *A representation theorem for cyclic analytic two-isometries*, Trans. Amer. Math. Soc. **328(1)** (1991), 325–349.

[Sa1] D. Sarason, *The H^p spaces of an annulus*, Mem, Amer. Math. Soc. **56** (1965), 78pp.

[Sa2] D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley & Sons, Inc., New York, 1994.

[Sha] H.S. Shapiro, *Generalized analytic continuation*, In: Symposia on Theoretical Physics and Mathematics, vol. 8, pp. 151–163, Plenum Press, 1968.

[Shi] S. Shimorin, *Wold-type decomposition and wandering subspaces for operators close to isometries*, J. Reine Angew. Math. **531** (2001), 147–189.

[SFBK] B. Sz.-Nagy, C. Foiaş, H. Bercovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, Springer, New York, 2010.

[VN] V.I. Vasyunin and N.K. Nikolskii, *Classification of H^2-functions according to the degree of their cyclicity*, Math. USSR Izvestiya **23(2)** (1984), 225–242.

[Wog] W.R. Wogen, *On some operators with cyclic vectors*, Indiana Univ. Math. J. **27** (1978), 163–171.

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21** (1996), 376–381.

Raúl E. Curto
Department of Mathematics, University of Iowa, Iowa City, IA 52242, U.S.A.
E-mail: raul-curto@uiowa.edu

In Sung Hwang
Department of Mathematics, Sungkyunkwan University, Suwon 16419, Korea
E-mail: ihwang@skku.edu

Woo Young Lee
Department of Mathematics and RIM, Seoul National University, Seoul 08826, Korea
E-mail: wylee@snu.ac.kr