# HYPONORMALITY AND SPECTRA OF TOEPLITZ OPERATORS

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Abstract. This paper concerns algebraic and spectral properties of Toeplitz operators $T_\varphi$, on the Hardy space $H^2(\mathbb{T})$, under certain assumptions concerning the symbols $\varphi \in L^\infty(\mathbb{T})$. Among our algebraic results is a characterisation of normal Toeplitz opertors with polynomial symbols, and a characterisation of hyponormal Toeplitz operators with polynomial symbols of a prescribed form. The results on the spectrum are as follows. It is shown that by restricting the spectrum, a set-valued function, to the set of all Toeplitz operators, the spectrum is continuous at $T_\varphi$, for each quasicontinuous $\varphi$. Secondly, we examine under what conditions a classic theorem of H. Weyl, which has extensions to hyponormal and Toeplitz operators, holds for all analytic functions of a single Toeplitz operator with continuous symbol.

## Introduction

An elegant and useful theorem of C. Cowen [7] characterises the hyponormality of a Toeplitz operator $T_\varphi$ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} \subset \mathbb{C}$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. This result makes it possible to answer an algebraic question coming from operator theory – namely, is $T_\varphi$ hyponormal? – by studying the function $\varphi$ itself. In a recent paper [18] of T. Nakazi and K. Takahashi, Cowen's method is carried out to obtain substantial new information about hyponormal Toeplitz operators and their symbols. In the present paper we study the hyponormality of $T_\varphi$ in the cases where $\varphi$ is a trigonometric polynomial $\varphi(e^{i\theta}) = \sum_{-m}^{N} a_n e^{in\theta}$; the goal here is to find conditions on the coefficients $a_n$ that are necessary and sufficient for $T_\varphi$ to be hyponormal. This problem is still rather complicated in general, however in §1 we are able to offer necessary and sufficient conditions for the normality and hyponormality of $T_\varphi$ in the cases where the Fourier coefficients of $\varphi$ satisfy certain extremal and symmetry properties.

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In 1909 H. Weyl examined the spectra of all compact perturbations $A + K$ of a single hermitian operator $A$ and discovered that $\lambda \in \sigma(A + K)$ for every compact operator $K$ if and only if $\lambda$ is not an isolated eigenvalue of finite multiplicity in $\sigma(A)$. Today this result is known as Weyl's theorem, and it has been extended from hermitian operators $A$ to hyponormal operators and to Toeplitz operators by L. Coburn [4], and to seminormal operators by S. Berberian [1]. In §3 of this paper we determine properties of continuous functions $\varphi$ that imply that Weyl's theorem holds for all analytic functions of the Toeplitz operator $T_\varphi$. This analysis entails an interesting new fact, which seems to be absent from the literature, concerning the continuity of the spectrum: when restricted to the linear manifold of all Toeplitz operators, the spectrum is a continuous (set-valued) function at every Toeplitz operator $T_\varphi$ with quasicontinuous symbol $\varphi$. In fact, somewhat more general results are true, and these form the basis of our work in §2 of this paper.

Let $L(H)$ and $K(H)$ denote the algebra of bounded linear operators and the ideal of compact operators on a complex Hilbert space $H$, and let $\pi$ denote the canonical map $L(H) \to L(H)/K(H)$. If $T \in L(H)$ is a Fredholm operator, that is if $\pi(T)$ is invertible in $L(H)/K(H)$, then $\ker T$ and $\ker T^*$ are finite-dimensional and the index of $T$ is the integer

$$\operatorname{ind} T = \dim\, \ker T - \dim\, \ker T^* \ .$$

The subset of $\sigma(T)$ that is stable under compact perturbations is denoted by $w(T)$ and is called the Weyl spectrum of $T$. Those Fredholm operators that have index zero are called Weyl operators. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $w(T)$ are described succintly as follows [14],[15]:

$$\begin{aligned} \sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not a Fredholm operator}\} \\ w(T) &= \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not a Weyl operator}\} \ . \end{aligned}$$

Evidently $\sigma_e(T) \subseteq w(T) \subseteq \sigma(T)$, although unlike $\sigma_e$ and $\sigma$, the Weyl spectrum of $T$ need not satisfy the spectral mapping theorem. The most general result in this direction [12] states that if $f$ is an analytic function on an open set containing $\sigma(T)$, then

$$w(f(T)) \subseteq f(w(T)) \ ; \tag{0.1}$$

but if $T$ is hyponormal, then [17]

$$w(f(T)) = f(w(T)) \ . \tag{0.2}$$

It is of interest to know which classes of operators $T$ satisfy (0.2), and for some time it was thought that the Toeplitz operators with continuous symbols may be one of these classes. In §3 we will show that this conjecture is false; conditions on $\varphi \in C(\mathbb{T})$ will then be sought so that $w(f(T_\varphi)) = f(w(T_\varphi))$ for every function $f$ analytic on an open neighbourhood of

$\sigma(T_\varphi)$. We arrive at our results in §3 by comparing the spectra of the operators $T_{f\circ\varphi}$ and $f(T_\varphi)$.

We review here a few essential facts concerning Toeplitz operators with continuous symbols that we will need to begin with, using [8] as a general reference. The Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \dots\}$. An element $f \in L^2$ is referred to as analytic if $f \in H^2$ and coanalytic if $f \in L^2 \ominus H^2$. If $P$ denotes the projection operator $L^2 \to H^2$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operator $T_\varphi$ on $H^2$ defined by

$$T_\varphi g = P(\varphi g) \quad \text{for all } g \in H^2 \tag{0.3}$$

is called the Toeplitz operator with symbol $\varphi$. Every Toeplitz operator has connected spectrum and essential spectrum, and by [4],

$$\sigma(T_\varphi) = w(T_\varphi)\ . \tag{0.4}$$

The sets $C(\mathbb{T})$ of all continuous complex-valued functions on the unit circle $\mathbb{T}$ and $H^\infty(\mathbb{T}) = L^\infty \cap H^2$ are Banach algebras, and it is well-known that every Toeplitz operator with symbol $\varphi \in H^\infty$ is subnormal. The $C^*$-algebra $\mathfrak{A}$ generated by all Toeplitz operators $T_\varphi$ with $\varphi \in C(\mathbb{T})$ has an important property which is very useful for spectral theory: the commutator ideal of $\mathfrak{A}$ is the ideal $K(H^2)$ of compact operators on $H^2$. As $C(\mathbb{T})$ and $\mathfrak{A}/K(H^2)$ are $*$-isomorphic $C^*$-algebras, then for every $\varphi \in C(\mathbb{T})$,

(0.5) $T_\varphi$ is a Fredholm operator if and only if $\varphi$ is invertible

$$\operatorname{ind} T_\varphi = -wn(\varphi)\ , \tag{0.6}$$

$$\sigma_e(T_\varphi) = \varphi(\mathbb{T})\ , \tag{0.7}$$

where $wn(\varphi)$ denotes the winding number of $\varphi$ with respect to the origin. Finally, we make note that if $\varphi \in C(\mathbb{T})$ and if $f$ is an analytic function defined on an open set containing $\sigma(T_\varphi)$, then $f \circ \varphi \in C(\mathbb{T})$ and $f(T_\varphi)$ is well-defined by the analytic functional calculus.

## 1. Hyponormality of Toeplitz operators with trigonometric polynomial symbols

An operator $T$ is said to be hyponormal if its selfcommutator $[T^*, T] = T^*T - TT^*$ is positive (semidefinite). Normal Toeplitz operators were characterised by a property of their symbol in the early 1960's by A. Brown and P.R. Halmos [3], and so it is somewhat of a surprise that 25 years passed before the exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the positivity of the selfcommutator $[T_\varphi^*, T_\varphi]$ was understood (via Cowen's theorem [7]). As Cowen notes in his survey paper [6], the intensive study of subnormal

Toeplitz operators in the 1970's and early 80's is one explanation for the relatively late appearence of the sequel to the Brown-Halmos work. The characterisation of hyponormality in [7] requires one to solve a certain functional equation in the unit ball of $H^\infty$ (see below); in this section we solve this functional equation for trigonometric polynomials $\varphi$ under certain assumptions about the coefficients of $\varphi$. Very recently K. Zhu has studied the same problem from a different point of view [27]; in his work he reformulates Cowen's theorem, in the case of a polynomial $\varphi$, so that the hyponormality of $T_\varphi$ can be decided by applying Schur's algorithm to the Schur function $\Phi_N$. Our results here are more readily applicable, but they apply to only a subclass of all possible trigonometric polynomials $\varphi$ inducing hyponormal Toeplitz operators. The case of arbitrary trigonometric polynomials $\varphi$, though solved in principle by Cowen's theorem or Schur's algorithm, is in practice very complicated. Indeed it may not even be possible to find tractable necessary and sufficient conditions for the hyponormality of $T_\varphi$ in terms of the Fourier coefficients of a trigonometric polynomial $\varphi$ unless certain assumptions are made about $\varphi$.

For each $\varphi \in L^\infty$ let $\mathcal{E}(\varphi) = \{k \in H^\infty : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}$. Cowen's theorem can be stated as follows (see [18; Lemma 1]): a Toeplitz operator $T_\varphi$ is hyponormal if and only if the subset $\mathcal{E}(\varphi)$ of $H^\infty$ is nonempty. Suppose that $\varphi$ is the trigonometric polynomial $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$, where $a_N \ne 0$. If a function $k \in H^\infty$ satisfies $\varphi - k\overline{\varphi} \in H^\infty$, then $k$ necessarily satisfies

$$k \sum_{n=1}^{N} \overline{a}_n e^{-in\theta} - \sum_{n=1}^{N} a_{-n} e^{-in\theta} \in H^\infty \ . \tag{1.0}$$

From (1.0) one computes the Fourier coefficients $\hat{k}(0), \dots, \hat{k}(N-1)$ of $k$ to be $\hat{k}(n) = c_n$, for $n = 0, 1, \dots, N-1$, where $c_0, c_1, \dots, c_{N-1}$ are determined uniquely from the coefficients of $\varphi$ by the recurrence relation

$$\begin{aligned} c_0 &= \frac{a_{-N}}{\overline{a_N}} \\ c_n &= (\overline{a_N})^{-1}\Big(a_{-N+n} - \sum_{j=0}^{n-1} c_j \overline{a_{N-n+j}}\Big) \ , \text{ for } n = 1, \dots, N-1 \ . \end{aligned} \tag{1.1}$$

Therefore if $k_1, k_2 \in \mathcal{E}(\varphi)$, then $c_n = \hat{k}_1(n) = \hat{k}_2(n)$ for all $n = 0, 1, \dots, N-1$, and $k_p(z) = \sum_{j=0}^{N-1} c_j z^j$ is the unique (analytic) polynomial of degree less than $N$ satisfying $\varphi - k\overline{\varphi} \in H^\infty$. Conversely, if $k_p$ is the polynomial $k_p(z) = \sum_{j=0}^{N-1} c_j z^j$, where $c_0, c_1, \dots, c_{N-1}$ are determined from the recurrence relation (1.1), then for every integer $n > 0$, the Fourier coefficients $\widehat{\varphi - k\overline{\varphi}}(-n)$ of $\varphi - k\overline{\varphi}$ satisfy

$$\begin{aligned}\widehat{\varphi - k\overline{\varphi}}(-n) &= a_{-n} - \sum_{j=0}^{N-n} c_j \overline{a_{n+j}} \\ &= \big(a_{-n} - \sum_{j=0}^{N-n-1} c_j \overline{a_{n+j}}\big) - c_{N-n}\overline{a_N} \\ &= 0\ ,\end{aligned}$$

which implies that $\varphi - k_p\overline{\varphi} \in H^2$. But since $\varphi - k_p\overline{\varphi}$ is a polynomial, it follows that $\varphi - k_p\overline{\varphi} \in H^\infty$. However despite the fact that the recurrence relation (1.1) can always be solved uniquely to produce an analytic polynomial $k_p$ satisfying $\varphi - k_p\overline{\varphi} \in H^\infty$, the polynomial $k_p$ need not be contained in the set $\mathcal{E}(\varphi)$, even if $\mathcal{E}(\varphi)$ is known to be nonempty; the problem here is that it is possible for the norm $||k_p||_\infty > 1$. Consider, for example, the trigonometric polynomial $\varphi(e^{i\theta}) = e^{-i2\theta} + 2e^{-i\theta} + e^{i\theta} + 2e^{i2\theta}$. Solving the recurrence relation (1.1) produces the polynomial $k_p(z) = \frac{1}{2} + \frac{3}{4}z$, which has norm $||k_p||_\infty = \frac{5}{4} > 1$, making $k_p$ ineligible for membership in $\mathcal{E}(\varphi)$. On the other hand, a straightforward calculation shows that the linear fractional transformation

$$b(z) = \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z}$$

satisfies $\varphi - b\overline{\varphi} \in H^\infty$; as $b$ maps the unit circle onto itself, $b$ has norm $||b||_\infty = 1$. Thus $b \in \mathcal{E}(\varphi)$ and so $T_\varphi$ is hyponormal. We note here that the Fourier series of $b$, namely

$$\begin{aligned}b(e^{i\theta}) &\sim \frac{1}{2} + \frac{3}{4}e^{i\theta} - \frac{3}{2}\sum_{j=2}^{\infty}(\frac{-1}{2})^n e^{in\theta} \\ &= k_p(e^{i\theta}) + h(e^{i\theta})\ ,\end{aligned}$$

converges uniformly on $\mathbb{T}$ to $b$, and that $b$ is a finite Blaschke product. (The existence of a such a Blaschke product in $\mathcal{E}(\varphi)$ is predicted by Theorem 10 of [18].)

The discussion above views the solution $c_0, \ldots, c_{N-1}$ to the recurrence relation (1.1) as the Fourier coefficients of every possible candidate $k$ for membership in $\mathcal{E}(\varphi)$. In [27], Zhu applies the Schur functions $\Phi_N$ to the $N$ complex numbers $c_0, \ldots, c_{N-1}$ to obtain his formulation of Cowen's theorem (for trigonometeric polynomials).

Before continuing further, we record here a condition that $\varphi$ must necessarily satisfy in order for $T_\varphi$ to be a hyponormal operator.

**A Condition Necessary for Hyponormality.** *Suppose that $\varphi$ is a trigonometric polynomial of the form $\varphi(e^{i\theta}) = \sum_{n=-m}^{N} a_n e^{in\theta}$, where $a_{-m}$ and $a_N$ are nonzero. If $T_\varphi$ is hyponormal, then $m \le N$ and $|a_{-m}| \le |a_N|$.*

*Proof.* Proofs that $m \le N$ can be found in [18;Cor5] and [27;Cor2]. Let $c_0, \ldots, c_{N-1}$ be the solution to the recurrence relation (1.1); because $|a_N| \ne 0$, we have $|c_{N-m}| = |a_{-m}|/|a_N|$. There is a function $k \in \mathcal{E}(\varphi)$ such that $\hat{k}(N-m) = c_{N-m}$; thus $1 \ge ||k||_\infty \ge |c_{N-m}| = |a_{-m}|/|a_N|$, which implies that $|a_{-m}| \le |a_N|$. $\square$

The necessary condition above shows that the cases where $|a_{-m}| = |a_N|$ are, in some sense, extremal among all possibilites for hyponormality. Theorem 1.4 treats such cases, and the result will show that one further feature, namely a symmetry property, is also present.

Proposition 1.1 shows that under strong enough conditions, the polynomial $k_p$ will be an element of $\mathcal{E}(\varphi)$.

**Proposition 1.1.** *If $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$, where $a_N \ne 0$, and if $c_0, c_1, \ldots, c_{N-1} \in \mathbb{C}$ are obtained from the coefficients of $\varphi$ by solving the recurrence relation* (1.1)*, then the Toeplitz operator $T_\varphi$ is hyponormal whenever*

$$\sum_{j=0}^{N-1} |c_j| \le 1 \ . \tag{1.1.1}$$

*Proof.* As we know, the polynomial $k_p(z) = \sum_{j=0}^{N-1} c_j z^j$ satisfies $\varphi - k_p\overline{\varphi} \in H^\infty$. From $||k_p||_\infty \le \sum_{j=0}^{N-1} |c_j| \le 1$ we have that $k_p \in \mathcal{E}(\varphi)$ and so $T_\varphi$ is hyponormal. $\square$

**Remark 1.2.** If $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$, where $|a_j| \le |a_N|$, for all $j = 2, \ldots, N-1$, then from the recurrence relation (1.1) we have that

$$\sum_{j=0}^{N-1} |c_j| \le |c_0| + |a_N|^{-2} \sum_{n=1}^{N-1} 2^{n-1} |D_n| \,,$$

where $D_n = \det \begin{pmatrix} a_{-n} & a_{-N} \\ \overline{a_n} & \overline{a_N} \end{pmatrix}$. Therefore if

$$\sum_{n=1}^{N-1} 2^{n-1} |D_n| \, + \, |a_{-N} a_N| \le |a_N|^2 \,, \tag{1.2.1}$$

then by Proposition 1.1, $T_\varphi$ is hyponormal. Because the left-hand side of (1.2.1) depends on $\overline{a_{-N}}$ and $a_N$ and the right-hand side depends on $|a_N|^2$, it follows that $T_\varphi$ is hyponormal whenever $|a_N|$ is sufficiently large. In particular, the Toeplitz operator with symbol $\varphi + \lambda e^{iN\theta}$ is hyponormal whenever $\lambda \in \mathbb{C}$ is such that

$$|\lambda| \geq \sum_{n=1}^{N-1} 2^{n-1}(|a_{-n}| + |a_n|) \, + \, |a_{-N}| + |a_N| \, .$$

**Remark 1.3.** If $a_{-N} = \cdots = a_{-2} = 0$, then the solution to the recurrence relation (1.1) is $c_0 = \cdots = c_{N-2} = 0$ and $c_{N-1} = a_{-1}/\overline{a_N}$; thus the analytic polynomial $k_p \in H^\infty$ is $k_p(z) = (a_{-1}/\overline{a_N})z^{N-1}$. Therefore the norm of every $k \in H^\infty$ that satisfies $\varphi - k\overline{\varphi} \in H^\infty$ is such that

$$||k||_\infty \geq \left| \frac{a_{-1}}{\overline{a_N}} \right| = ||k_p||_\infty \ .$$

Therefore, $T_\varphi$ is hyponormal if and only if $|a_{-1}| \leq |a_N|$ (which was shown earlier in [11]).

The following theorem and its corollary concern the extremal cases: $|a_{-m}| = |a_N| \neq 0$. Equations (1.4.1) and (1.5.1) below emphasize the symmetry underlying the hyponormality and normality of these operators.

**Theorem 1.4.** *Suppose that $\varphi(e^{i\theta}) = \sum_{n=-m}^{N} a_n e^{in\theta}$, where $m \leq N$ and $|a_{-m}| = |a_N| \neq 0$, and let $\mathcal{E}(\varphi) \subset H^\infty$ be the subset of all $k \in H^\infty$ for which $||k||_\infty \leq 1$ and $\varphi - k\overline{\varphi} \in H^\infty$. The following statements are equivalent.*

1. *The Toeplitz operator $T_\varphi$ is hyponormal.*
2. *For all $k = 1, \ldots, N-1$,* $\det \begin{pmatrix} a_{-(m-k)} & a_{-m} \\ \overline{a_{(N-k)}} & \overline{a_N} \end{pmatrix} = 0$.
3. *The following equation in $\mathbb{C}^m$ holds:*

$$\overline{a_N} \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \overline{a_{N-m+1}} \\ \overline{a_{N-m+2}} \\ \vdots \\ \\ \vdots \\ \overline{a_N} \end{pmatrix} . \tag{1.4.1}$$

4. $\mathcal{E}(\varphi) = \{a_{-m}(\overline{a_N})^{-1}z^{N-m}\}$.

*Moreover, if $T_\varphi$ is hyponormal, then the rank of $[T_\varphi^*, T_\varphi]$ is $N - m$.*

*Proof.* Let $c_0, \ldots, c_{N-1}$ be the solution to the recurrence relation (1.1); because $|a_{-m}| = |a_N| \neq 0$, we have $|c_{N-m}| = 1$. Note that if $m < N$, then $c_0 = \cdots = c_{N-m-1} = 0$.

If a function $k \in H^\infty$ satisfies $\varphi - k\overline{\varphi} \in H^\infty$, then the Fourier series expansion of $k$ is

$$k = \sum_{j=0}^{N-1} c_j e^{ij\theta} + \sum_{n=N}^{\infty} b_n e^{in\theta} \qquad \text{for some set of } b_n \in \mathbb{C} \, .$$

From the fact that $||k||_\infty \geq ||k||_2$ we have that $||k||_\infty \geq |c_{N-m}| = 1$; if for some $j > (N-m)$ or $n \geq N$ there is a nonzero Fourier coefficient $c_j$ or $b_n$ of $k$, then

$$||k||_\infty \geq \max\left\{\sqrt{|c_{N-m}|^2 + |c_j|^2}, \sqrt{|c_{N-m}|^2 + |b_n|^2}\right\} > 1 \ .$$

Thus $||k||_\infty = 1$ if and only if $c_{N-m}$ is the only nonzero Fourier coefficient of $k$. Therefore $\mathcal{E}(\varphi)$ can have at most one element: namely $c_{N-m}z^{N-m}$. Hence, statements (1) and (4) are equivalent. We now proceed to prove the equivalence of statements (1) and (2); obviously (2) and (3) are the exact same statement.

Suppose that $T_\varphi$ is hyponormal. Then there exists $k \in \mathcal{E}(\varphi)$ and by the discussion above, $k(z) = c_{N-m}z^{N-m}$. Hence, for every $k = 1, \ldots, m-1$,

$$\begin{aligned} 0 = |c_{N-m+k}| &= \left|\frac{1}{\overline{a_N}}(a_{-(m-k)} - c_{N-m}\overline{a_{N-k}})\right| \\ &= \left|\frac{1}{\overline{a_N}}\right|^2 \left|\det\begin{pmatrix} a_{-(m-k)} & a_{-m} \\ \overline{a_{(N-k)}} & \overline{a_N}\end{pmatrix}\right| \ . \end{aligned}$$

Conversely, if $\det\begin{pmatrix} a_{-(m-k)} & a_{-m} \\ \overline{a_{(N-k)}} & \overline{a_N}\end{pmatrix} = 0$ for all $k = 1, \ldots, N-1$, then

$$|c_{N-m+1}| = \left|\frac{1}{\overline{a_N}}(a_{-(m-1)} - c_{N-m}\overline{a_{N-1}})\right| = \left|\frac{1}{\overline{a_N}}\right|^2 \left|\det\begin{pmatrix} a_{-(m-1)} & a_{-m} \\ \overline{a_{(N-1)}} & \overline{a_N}\end{pmatrix}\right| = 0$$

and hence

$$\begin{aligned} |c_{N-m+2}| &= |\frac{1}{\overline{a_N}}(a_{-(m-2)} - c_{N-m}\overline{a_{N-2}} - c_{N-m+1}\overline{a_{N-1}})| \\ &= \left|\frac{1}{\overline{a_N}}\right|^2 \left|\det\begin{pmatrix} a_{-(m-2)} & a_{-m} \\ \overline{a_{(N-2)}} & \overline{a_N}\end{pmatrix}\right| = 0 \ . \end{aligned}$$

Inductively, we obtain $c_k = 0$ for all $k = 1, \ldots, N-1$. As $c_0 = \cdots = c_{N-m-1} = 0$ if $m < N$, and $|c_{N-m}| = 1$, we have that the analytic polynomial $k_p(z) = \sum_{j=0}^{N-1}$ is of the form $k_p(z) = c_{N-m}z^{N-m}$ and therefore $k_p \in \mathcal{E}(\varphi)$. This completes the proof that statements (1) and (2) are equivalent.

Lastly, if $T_\varphi$ is hyponormal, then $\mathcal{E}(\varphi) = \{\frac{a_{-m}}{\overline{a_N}}z^{N-m}\}$. Because the selfcommutator $[T_\varphi^*, T_\varphi]$ has finite rank, Theorem 10 of [18] states that there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$ of degree equal to the rank of $[T_\varphi^*, T_\varphi]$. In our case there is only one element in $\mathcal{E}(\varphi)$: $b(z) = \frac{a_{-m}}{\overline{a_N}}z^{N-m}$, which is a finite Blaschke product of degree $N-m$. □

The following necessary and sufficient condition for normality is to be expected, given that every normal Toeplitz operator is a translation and rotation of a hermitian Toeplitz operator [3].

**Corollary 1.5.** *If $\varphi(e^{i\theta}) = \sum_{n=-m}^{N} a_n e^{in\theta}$, then $T_\varphi$ is normal if and only if $m = N$, $|a_{-N}| = |a_N|$, and*

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \\ \vdots \\ \overline{a_N} \end{pmatrix}. \tag{1.5.1}$$

*Proof.* If $m = N$, $|a_{-m}| = |a_N|$, and $\det\begin{pmatrix} a_{-(m-k)} & a_{-m} \\ \overline{a_{(N-k)}} & \overline{a_N} \end{pmatrix} = 0$ for all $k = 1, \ldots, N-1$, then by Theorem 1.4, $T_\varphi$ is hyponormal and rank $[T_\varphi^*, T_\varphi] = N - m = 0$; that is, $T_\varphi$ is normal. Conversely, if $T_\varphi$ is normal, then by the Brown-Halmos theorem [3], there are scalars $\alpha, \beta \in \mathbb{C}$ and a real-valued $\psi \in L^\infty$ such that $T_\varphi = \alpha T_\psi + \beta 1$. As $T_\psi$ is a hermitian Toeplitz operator, the Fourier coefficients of $\psi$ satisfy $\hat{\psi}(n) = \overline{\hat{\psi}(-n)}$ for all $n$; in particular $|\alpha|\,|a_N| = |\hat{\psi}(N)| = |\hat{\psi}(-N)| = |\alpha|\,|a_{-N}|$, showing that $|a_{-N}| = |a_N|$. Thus, $N = m$ and (1.5.1) holds. □

**Remark 1.6.** For trigonometric polynomials $\varphi$ satisfying the assumptions of Theorem 1.4, the question of whether or not the Toeplitz operator $T_\varphi$ is hyponormal is completely independent of the values the coefficients $a_0, \ldots, a_{N-m}$ of $\varphi$. This interesting fact does not appear to be a coincidence, for it is noted as well by Zhu [27] under weaker assumptions.

**Example 1.7.** Consider the following two trigonometric polynomials:

$$\begin{aligned} \varphi_1(e^{i\theta}) &= e^{-i2\theta} + e^{i3\theta} + e^{i4\theta} \\ \varphi_2(e^{i\theta}) &= e^{-i2\theta} + e^{-i\theta} + e^{i3\theta} + e^{i4\theta} \end{aligned}$$

Intuition suggests that $\varphi_2$ is less likely than $\varphi_1$ to induce a hyponormal Toeplitz operator, as $\varphi_2$ is "less analytic" in that the (coanalytic) term $e^{-i\theta}$ is present in $\varphi_2$ but not in $\varphi_1$. However the opposite is true: Theorem 1.4 shows that $T_{\varphi_2}$ is hyponormal (with rank-2 selfcommutator) whereas $T_{\varphi_1}$ is not.

With the following result, we relax the condition that $|a_{-N}| = |a_N|$, however we retain some symmetry. In the case where $N = 2$ below, the result reduces to Theorem 1 of [10].

**Theorem 1.8.** *Suppose that $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$, where $N \geq 2$, $|a_N| \neq 0$, and the* *coefficients of* $\varphi$ *satisfy*

$$\overline{a_N}\begin{pmatrix} a_{-2} \\ a_{-3} \\ \vdots \\ \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N}\begin{pmatrix} \overline{a_2} \\ \overline{a_3} \\ \vdots \\ \\ \vdots \\ \overline{a_N} \end{pmatrix}. \tag{1.8.1}$$

*Then* $T_\varphi$ *is hyponormal if and only if*

$$|a_N|^2 - |a_{-N}|^2 \geq \sqrt{\left|\det\begin{pmatrix} a_{-1} & a_{-N} \\ \overline{a_1} & \overline{a_N} \end{pmatrix}\right|^2 + d^2} - d\,, \tag{1.8.2}$$

*where* $d = \frac{1}{2}(1 - |a_{-N}|^2|a_N|^{-2})\sum_{n=2}^{N-1}|a_n|^2$ *and* $d$ *is taken to be* $0$ *of* $N = 2$.

*Proof.* To begin with, assume that inequality (1.8.2) holds; we are to prove that $T_\varphi$ is hyponormal. Solving the recurrence relation (1.1) under the condition (1.8.1) produces the analytic polynomial $k_p(z) = c_0 + c_{N-1}z^{N-1}$, where

$$c_0 = a_{-N}/\overline{a_N} \qquad \text{and} \qquad c_{N-1} = (\overline{a_N})^{-2}\det\begin{pmatrix} a_{-1} & a_{-N} \\ \overline{a_1} & \overline{a_N} \end{pmatrix}.$$

The inequality (1.8.2) implies that

$$1 - |c_0|^2 \geq \sqrt{|c_{N-1}|^2 + d^2|a_N|^{-4}} - d|a_N|^{-2}\,. \tag{1.8.3}$$

The right-hand side of (1.8.3) is nonnegative and so $|c_0| \leq 1$. Now if $|c_0| = 1$, then $|c_{N-1}| = 0$ and $T_\varphi$ is normal; assume, therefore, that $|c_0| < 1$. Let $k \in H^2$ be the function with Fourier series expansion

$$k = k_p(e^{i\theta}) + c_{N-1}\sum_{n=1}^{\infty}(-1)^n\left(\frac{c_{N-1}\overline{c}_0}{|c_{N-1}|}\right)^n e^{i(N-1)(n+1)\theta}\,.$$

As $\hat{k}(n) = c_n$ for $n = 0, \ldots, N-1$, it remains only to prove that $k$ is in the unit ball of $H^\infty$.

Let $\alpha = \frac{\overline{c_{N-1}}c_0}{|c_{N-1}|}$, which is a complex number of modulus $|\alpha| = |c_0| < 1$. Then

$$\begin{aligned}
k(z) &= c_0 + c_{N-1}z^{N-1} + c_{N-1}\sum_{n=1}^{\infty}(-1)^n\left(\frac{c_{N-1}\overline{c}_0}{|c_{N-1}|}\right)^n (z^{N-1})^{n+1} \\
&= \frac{c_{N-1}}{1-|c_0|^2}\left(\frac{c_0\overline{c_{N-1}}}{|c_{N-1}|} + (1-|c_0|^2)z^{N-1}\right) + \left(1 - \frac{|c_{N-1}|}{1-|c_0|^2}\right)c_0 \\
&\quad + \frac{c_{N-1}}{1-|c_0|^2}\sum_{n=1}^{\infty}\left(-\frac{\overline{\overline{c_{N-1}}c_0}}{|c_{N-1}|}\right)^n (1-|c_0|^2)(z^{N-1})^{n+1} \\
&= \frac{c_{N-1}}{1-|\alpha|^2}\left(-\alpha + \sum_{n=0}^{\infty}\overline{\alpha}^n(1-|\alpha|^2)(z^{N-1})^{n+1}\right) + \left(1 - \frac{|c_{N-1}|}{1-|\alpha|^2}\right)c_0 \\
&= \frac{c_{N-1}}{1-|\alpha|^2}\left(\frac{z^{N-1}-\alpha}{1-\overline{\alpha}z^{N-1}}\right) + \left(1 - \frac{|c_{N-1}|}{1-|\alpha|^2}\right)c_0\,.
\end{aligned}$$

Because the function $w \mapsto (w-\alpha)(1-\overline{\alpha}w)^{-1}$ is a linear fractional transformation, mapping $\mathbb{T}$ onto itself, we obtain the estimate

$$\begin{aligned}
||k||_\infty &\le \frac{|c_{N-1}|}{1-|\alpha|^2}\left|\frac{z^{N-1}-\alpha}{1-\overline{\alpha}z^{N-1}}\right| + \left(1 - \frac{|c_{N-1}|}{1-|\alpha|^2}\right)|c_0| \\
&\le \frac{|c_{N-1}|}{1-|\alpha|^2} + \left(1 - \frac{|c_{N-1}|}{1-|\alpha|^2}\right) \\
&= 1\,,
\end{aligned}$$

which proves that $k \in \mathcal{E}(\varphi)$.

Conversely, suppose now that $T_\varphi$ is hyponormal. With repsect to the orthonormal basis $\{z^n : n = 0, 1\dots\}$ of $H^2$, the selfcommutator of $T_\varphi$ is a matrix with $(\mu,\nu)$-entry given by

$$\alpha_{\mu\,\nu} = \sum_{j=0}^{\infty}(\overline{a_{j-\mu}}a_{j-\nu} - a_{\mu-j}\overline{a_{\nu-j}})\,, \qquad \text{where } \mu,\nu = 0,1,2,\dots\,.$$

Thus, in particular,

$$\begin{aligned}
\alpha_{0\,0} &= \sum_{n=1}^{N}(|a_n|^2 - |a_{-n}|^2) \\
\alpha_{N-1\,N-1} &= |a_N|^2 - |a_{-N}|^2 \\
\alpha_{0\,N-1} &= \overline{\alpha_{N-1\,0}} = \overline{a_N}a_1 - a_{-N}\overline{a_{-1}}\,.
\end{aligned}$$

The operator $[T_\varphi^*, T_\varphi]$ is positive and, therefore, so is its $2\times 2$ principal submatrix

$$\begin{pmatrix} \alpha_{0\,0} & \alpha_{0\,N-1} \\ \alpha_{N-1\,0} & \alpha_{N-1\,N-1} \end{pmatrix}.$$

Hence $\alpha_{0\,0}$ and $\alpha_{N-1\,N-1}$ are nonnegative and

$$\begin{aligned} 0 \le \det \begin{pmatrix} \alpha_{0\,0} & \alpha_{0\,N-1} \\ \alpha_{N-1\,0} & \alpha_{N-1\,N-1} \end{pmatrix} &= \alpha_{0\,0}\alpha_{N-1\,N-1} - |\alpha_{0\,N-1}|^2 \\ &= \left(\sum_{n=1}^{N} (|a_n|^2 - |a_{-n}|^2)\right)(|a_N|^2 - |a_{-N}|^2) - |\overline{a_N}a_1 - a_{-N}\overline{a_{-1}}|^2 . \end{aligned}$$

The symmetry condition (1.8.1) yields $|a_{-n}| = |a_{-N}/a_N|\,|a_n|$ for $n = 2, \ldots, N-1$. Direct computation reveals that

$$(|a_1|^2 - |a_{-1}|^2)(|a_N|^2 - |a_{-N}|^2) + |\overline{a_N}a_{-1} - a_{-N}\overline{a_1}|^2 = |\overline{a_N}a_1 - a_{-N}\overline{a_{-1}}|^2 ,$$

and so

$$\begin{aligned} 0 &\le \det \begin{pmatrix} \alpha_{0\,0} & \alpha_{0\,N-1} \\ \alpha_{N-1\,0} & \alpha_{N-1\,N-1} \end{pmatrix} \\ &\le (|a_N|^2 - |a_{-N}|^2)^2 + (|a_1|^2 - |a_{-1}|^2)(|a_N|^2 - |a_{-N}|^2) \\ &\qquad - |\overline{a_N}a_1 - a_{-N}\overline{a_{-1}}|^2 + (|a_N|^2 - |a_{-N}|^2)\sum_{n=2}^{N-1}(|a_n|^2 - |a_{-n}|^2) \\ &= (|a_N|^2 - |a_{-N}|^2)^2 - |\overline{a_N}a_{-1} - a_{-N}\overline{a_1}|^2 \\ &\qquad + (|a_N|^2 - |a_{-N}|^2)(1 - \left|\frac{a_{-N}}{a_N}\right|^2)\sum_{n=2}^{N-1}|a_n|^2 . \end{aligned}$$

Therefore,

$$|a_N|^2 - |a_{-N}|^2 \ge \sqrt{\left|\det\begin{pmatrix} a_{-1} & a_{-N} \\ \overline{a_1} & \overline{a_N}\end{pmatrix}\right|^2 + d^2} - d ,$$

where $d = \frac{1}{2}(1 - |a_{-N}|^2|a_N|^{-2})\sum_{n=2}^{N-1}|a_n|^2$. □

Theorem 1.8 can be applied to show that the Toeplitz operator with symbol

$$\varphi(e^{i\theta}) = e^{-i5\theta} - e^{-i4\theta} + e^{-i2\theta} + e^{-i\theta} + 2e^{i2\theta} - 2e^{i4\theta} + 2e^{i5\theta} ,$$

whose coefficents satisfy the symmetric relation (1.8.1), but for which there is no symmetry involving $a_{-1}$ and $a_1$, is hyponormal. However with full symmetry, meaning that $a_{-1}$ and $a_1$ are related as well, we obtain the following interesting necessary and sufficient condition for hyponormality, which is, in some sense, dual to Theorem 1.4 but comparable to Corollary 1.5. The result is a generalisation of the fact that $U^* + \lambda U$, where $U$ is the unilateral shift operator, is hyponormal if and only if $|\lambda| \ge 1$.

**Corollary 1.9.** *If $\varphi(e^{i\theta}) = \sum_{n=-N}^{N} a_n e^{in\theta}$ is such that*

$$\overline{a_N} \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \\ \vdots \\ a_{-N} \end{pmatrix} = a_{-N} \begin{pmatrix} \overline{a_1} \\ \overline{a_2} \\ \vdots \\ \\ \vdots \\ \overline{a_N} \end{pmatrix}, \tag{1.9.1}$$

*then $T_\varphi$ is hyponormal if and only if $|a_{-N}| \le |a_N|$.*

## 2. Spectral variation within the manifold $\mathfrak{T}$ of Toeplitz operators.

Let $\mathcal{K}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. The spectrum can be viewed as function $\sigma : L(H) \to \mathcal{K}$, mapping each operator $T$ to its spectrum $\sigma(T)$. It is well-known that the function $\sigma$ is upper-semicontinuous and that $\sigma$ does have points of discontinuity. Of interest, therefore, is the identification of points of spectral continuity, as in [5], and of classes $\mathfrak{C}$ of operators for which $\sigma$ becomes continuous when restricted to $\mathfrak{C}$. Perhaps the most accessible result in the latter direction is the one of J. Newburgh [19]: when restricted to the set of normal operators, $\sigma$ is a continuous function. As noted in Solution 104 of [13], Newburgh's argument uses the fact that normal operators have normal resolvents and that normal operators are *normaloid* (i.e., the spectral radius is the same as the norm). Although Toeplitz operators are normaloid, their inverses need not, in general, be normaloid. Of course, if $\varphi$ is analytic or coanalytic, and if $T_\varphi$ is invertible, then its inverse $T_\varphi^{-1}$ is also a Toeplitz operator $T_{\frac{1}{\varphi}}$ [25; Theorem II] and, hence, normaloid. In this case, the arguments of Newburgh apply to show that $\sigma$ is continuous when restricted to the manifolds of analytic Toeplitz operators and co-analytic Toeplitz operators.

Let $\mathfrak{T}$ denote the subset of $L(H^2)$ consisting of all Toeplitz operators. In this section we study the continuity properties of $\sigma$ as a function $\sigma : \mathfrak{T} \to \mathcal{K}$; that is, we restrict $\sigma$ to the set of Toeplitz operators. Although it is open in regards to whether or not the function $\sigma : \mathfrak{T} \to \mathcal{K}$ is continuous, we are able to establish points of spectral continuity at Toeplitz operators with quasicontinuous symbols. In fact we shall demonstrate that under fairly general assumptions on $\varphi \in L^\infty$, the operator $T_\varphi$ is a point of continuity for the spectral function $\sigma : \mathfrak{T} \to \mathcal{K}$.

We require the use of certain closed subspaces and subalgebras of $L^\infty(\mathbb{T})$, which are described in further detail in [9] and Appendix 4 of [20]. Recall that the subspace $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is a closed subalgebra of $L^\infty$. The elements of the closed selfadjoint subalgebra $QC$, which is defined to be

$$QC = \big(H^\infty(\mathbb{T}) + C(\mathbb{T})\big) \cap \overline{\big(H^\infty(\mathbb{T}) + C(\mathbb{T})\big)},$$

are called quasicontinuous functions. The subspace $PC$ is the closure in $L^\infty(\mathbb{T})$ of the set of all piecewise continuous functions on $\mathbb{T}$. Thus $\varphi \in PC$ if and only if it is right continuous and has both a left- and right-hand limit at every point. There are certain algebraic relations among Toeplitz operators whose symbols come from these classes, including

$$T_\psi T_\varphi - T_{\psi\varphi} \in K(H^2) \ \text{ for every } \varphi \in H^\infty(\mathbb{T}) + C(\mathbb{T}) \text{ and } \psi \in L^\infty(\mathbb{T})\,, \tag{2.0.1}$$

and

$$\text{the commutator } [T_\varphi, T_\psi] \ \text{ is compact for every } \varphi, \psi \in PC\,. \tag{2.0.2}$$

We add to these relations the following one.

**Lemma 2.1.** *If $T_\varphi$ is a Toeplitz operator with quasicontinuous symbol $\varphi$, and if $f$ is an analytic function on an open set containing $\sigma(T_\varphi)$, then $T_{f\circ\varphi} - f(T_\varphi)$ is a compact operator.*

*Proof.* Assume that $\varphi \in QC$. Recall from [8;p.188] that if $\psi \in H^\infty + C(\mathbb{T})$, then $T_\psi$ is Fredholm if and only if $\psi$ is invertible in $H^\infty + C(\mathbb{T})$. Therefore for every $\lambda \notin \sigma(T_\varphi)$, both $\varphi - \lambda$ and $\overline{\varphi - \lambda}$ are invertible in $H^\infty + C(\mathbb{T})$; hence, $(\varphi - \lambda)^{-1} \in QC$. Using this fact together with (2.0.1) we have that, for $\psi \in L^\infty$ and $\lambda, \mu \in \mathbb{C}$,

$$T_{\varphi-\mu} T_\psi T_{(\varphi-\lambda)^{-1}} - T_{(\varphi-\mu)\psi(\varphi-\lambda)^{-1}} \in K(H^2) \quad \text{whenever } \lambda \notin \sigma(T_\varphi)\,.$$

The arguments above extend to rational functions to yield: if $r$ is any rational function with all of its poles outside of $\sigma(T_\varphi)$, then $r(T_\varphi) - T_{r\circ\varphi} \in K(H^2)$. Suppose that $f$ is an analytic function on an open set containing $\sigma(T_\varphi)$. By Runge's theorem there exists a sequence of rational functions $r_n$ such that the poles of each $r_n$ lie outside of $\sigma(T_\varphi)$ and $r_n \to f$ uniformly on $\sigma(T_\varphi)$. Thus $r_n(T_\varphi) \to f(T_\varphi)$ in the norm-topology of $L(H^2)$. Furthermore, because $r_n \circ \varphi \to f \circ \varphi$ uniformly, we have $T_{r_n\circ\varphi} \to T_{f\circ\varphi}$ in the norm-topology. Hence, $T_{f\circ\varphi} - f(T_\varphi) = \lim \big(T_{r_n\circ\varphi} - r_n(T_\varphi)\big)$, which is compact. □

Lemma 2.1 does not extend to piecewise continuous symbols $\varphi \in PC$, as we cannot guarantee that $T_\varphi^n - T_{\varphi^n} \in K(H^2)$ for each $n \in \mathbb{Z}^+$. For example, if $\varphi(e^{i\theta}) = \chi_{\mathbb{T}_+} - \chi_{\mathbb{T}_-}$, where $\chi_{\mathbb{T}_+}$ and $\chi_{\mathbb{T}_-}$ are characteristic functions of, respectively, the upper semicircle and the lower semicircle, then $T_\varphi^2 - I$ is not compact.

**Corollary 2.2.** *If $T_\varphi$ is a Toeplitz operator with quasicontinuous symbol $\varphi$, then for every analytic function $f$ on an open set containing $\sigma(T_\varphi)$,*

1. *$w(f(T_\varphi)) = \sigma(T_{f\circ\varphi})$, and*
2. *$f(T_\varphi)$ is essentially normal and is unitarily equivalent to a compact perturbation of $f(T_\varphi) \oplus M_{f\circ\varphi}$, where $M_{f\circ\varphi}$ is the operator of multiplication by $f \circ \varphi$ on $L^2(\mathbb{T})$.*

*Proof.* Because the Weyl spectrum is stable under the compact perturbations, it follows from Lemma 2.1 that $w(f(T_\varphi)) = w(T_{f\circ\varphi}) = \sigma(T_{f\circ\varphi})$, which proves statement (1). To prove (2), observe that because $QC$ is a closed algebra, the composition of the analytic function $f$ with $\varphi \in QC$ produces a quasicontinuous function $f \circ \varphi \in QC$. Moreover, by (2.0.1), every Toeplitz operator with quasicontinuous symbol is essentially normal. The (normal) Laurent operator $M_{f\circ\varphi}$ on $L^2(\mathbb{T})$ has its spectrum contained within the spectrum of the (essentially normal) Toeplitz operator $T_{f\circ\varphi}$. Thus there is the following relationship involving the essentially normal operators $f(T_\varphi)$ and $M_{f\circ\varphi} \oplus f(T_\varphi)$:

$$\sigma_e\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big) = \sigma_e(f(T_\varphi)) \quad \text{and} \quad \mathcal{SP}(f(T_\varphi)) = \mathcal{SP}\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big)\,,$$

where $\mathcal{SP}(T)$ denotes the spectral picture of an operator $T$. (The spectral picture $\mathcal{SP}(T)$ is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the Fredholm indices associated with these holes and pseudoholes.) Thus it follows from the Brown-Douglas-Fillmore theorem [23] that $f(T_\varphi)$ is compalent to $f(T_\varphi) \oplus M_{f\circ\varphi}$, in the sense that there exists a unitary operator $W$ and a compact operator $K$ such that $W\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big)W^* + K = f(T_\varphi)$. $\square$

Corollary 2.2 (1) can be viewed as saying that $\sigma(f(T_\varphi)) \setminus \sigma(T_{f\circ\varphi})$ consists of holes with winding number zero.

Continuity modulo the compact operators will be a key to our study of spectral variation. The first result is an easy application of a theorem of Newburgh.

**Lemma 2.3.** ([19, Theorem 4]) *If $\{T_n\}_n$ is a sequence of operators converging to an operator $T$ and such that $[T_n, T]$ is compact for each $n$, then $\lim \sigma_e(T_n) = \sigma_e(T)$.*

*Proof.* Newburgh's theorem is stated as follows: if in a Banach algebra $A$, $\{a_i\}_i$ is a sequence of elements commuting with $a \in A$ and such that $a_i \to a$, then $\lim\, \sigma(a_i) = \sigma(a)$. If $\pi$ denotes the canonical homomorphism of $L(H)$ onto the Calkin algebra $L(H)/K(H)$, then the assumptions give that $\pi(T_n) \to \pi(T)$ and $[\pi(T_n), \pi(T)] = 0$ for each $n$. Hence, $\lim\, \sigma(\pi(T_n)) = \sigma(\pi(T))$; that is, $\lim \sigma_e(T_n) = \sigma_e(T)$. $\square$

**Theorem 2.4.** *Suppose that $T, T_n \in L(H)$, for $n \in \mathbb{Z}^+$, are such that $T_n$ converges to $T$. Suppose that $f$ is any analytic function whose domain is an open set $V$ containing $\sigma(T)$. If $[T_n, T] \in K(H)$ for each $n$, then*

$$\lim\, w(f(T_n)) = w(f(T)). \tag{2.4.1}$$

*Remark.* Because $T_n \to T$, by the upper-semicontinuity of the spectrum, there is a positive integer $N$ such that $\sigma(T_n) \subseteq V$ whenever $n > N$. Thus, in the left-hand side of (2.4.1) it is to be understood that $n > N$.

*Proof of Theorem 2.4.* If $T_n$ and $T$ commute modulo the compact operators then, whenever $T_n^{-1}$ and $T^{-1}$ exist, $T_n, T, T_n^{-1}$ and $T^{-1}$ all commute modulo the compact operators.

Therefore $r(T_n)$ and $r(T)$ also commute modulo $K(H)$ whenever $r$ is a rational function with no poles in $\sigma(T)$ and $n$ is sufficiently large. Using Runge's theorem we can approximate $f$ on compact subsets of $V$ by rational functions $r$ who poles lie off of $V$. So there exists a sequence of rational functions $r_i$ whose poles lie outside of $V$ and $r_i \to f$ uniformly on compact subsets of $V$. If $n > N$, then by the functional calculus,

$$f(T_n)f(T) - f(T)f(T_n) = \lim_i \big(r_i(T_n)r_i(T) - r_i(T)r_i(T_n)\big),$$

which is compact for each $n$. Furthermore,

$$\begin{aligned} ||f(T_n) - f(T)|| &= ||\frac{1}{2\pi i}\int_\Gamma f(\lambda)\big((\lambda - T_n)^{-1} - (\lambda - T)^{-1}\big)\,d\lambda|| \\ &\le \frac{1}{2\pi i}\,\ell(\Gamma)\max_{\lambda\in\Gamma}|f(\lambda)|\cdot\max_{\lambda\in\Gamma}||(\lambda - T_n)^{-1} - (\lambda - T)^{-1}||\,, \end{aligned}$$

where $\Gamma$ is the boundary of $V$ and $\ell(\Gamma)$ denotes the arc length of $\Gamma$. The right-hand side of the above inequality converges to 0, and so $f(T_n) \to f(T)$. By Lemma 2.3, lim $\sigma_e(f(T_n)) = \sigma_e(f(T))$. The arguments used by J.B. Conway and B.B. Morrel in Proposition 3.11 of [5] can now be used here to obtain the conclusion lim $w(f(T_n)) = w(f(T))$. □

We now are ready to prove the main result in this section.

**Theorem 2.5.** *The restriction of* $\sigma$ *to the manifold* $\mathfrak{T}$ *of all Toeplitz operators is continuous at every Toeplitz operator with quasicontinuous symbol. Moreover, if* $\varphi \in QC$, $\varphi_n \in L^\infty$, *and* $||T_{\varphi_n} - T_\varphi|| \to 0$, *then* lim $w(f(T_{\varphi_n})) = \sigma(T_{f\circ\varphi})$.

*Proof.* Suppose $\varphi \in QC$, $\varphi_n \in L^\infty$, and $||T_{\varphi_n} - T_\varphi|| \to 0$. Then by (2.0.1), $[T_{\varphi_n}, T_\varphi] \in K(H^2)$. Therefore by Theorem 2.4, $\lim w(T_{\varphi_n}) = w(T_\varphi)$, and hence $\lim \sigma(T_{\varphi_n}) = \sigma(T_\varphi)$. Also, because $f\circ\varphi \in QC$ and $f\circ\varphi_n \to f\circ\varphi$, it follows from Lemma 2.1 that $\lim w(f(T_{\varphi_n})) = \lim \sigma(T_{f\circ\varphi_n}) = \sigma(T_{f\circ\varphi})$. □

The argument of Theorem 2.5 is limited to quasicontinuous symbols, as we need to ensure that $[T_{\varphi_n}, T_\varphi]$ is compact for every $n$. If one imposes more requirements on the functions $\varphi_n \in L^\infty$, then Theorem 2.5 can be made more general. This occurs, in particular, if each $\varphi_n$ is an element of $PC$.

**Corollary 2.6.** *The restriction of* $\sigma$ *to* $\mathfrak{T}_{PC}$ *is continuous, where* $\mathfrak{T}_{PC}$ *is the set of all Toeplitz operators having symbols that are uniform limits of piecewise continuous functions.*

*Proof.* This follows from (2.0.2) and Theorem 2.4. □

With a piecewise continuous function $\varphi$, one can obtain a continuous curve $\varphi^\#$ by joining $\varphi(e^{i\theta-0})$ and $\varphi(e^{i\theta})$ $(0 \le \theta < 2\pi)$ by the line segment $[\varphi(e^{i\theta-0}), \varphi(e^{i\theta})]$. Widom [25] showed that for every $\varphi \in PC$, $\sigma_e(T_\varphi) = \varphi^\#(\mathbb{T})$ and $\sigma(T_\varphi)$ consists of $\varphi^\#(\mathbb{T})$ together with some of its holes. This work is described in [8] and [20] as well. In a footnote on page

23 of his monograph [9], R.G. Douglas observes that the results he had been developing for Toeplitz operators with piecewise continuous symbols in fact hold, more generally, for symbols $\varphi \in L^\infty(\mathbb{T})$ having the property that

$$V_{\lambda_0}(\varphi) = \bigcap_{\epsilon > 0} \mathrm{cl}\big[\varphi(\lambda_0 - \epsilon, \lambda_0 + \epsilon)\big] \tag{2.7.1}$$

is contained in some line segment $L_{\lambda_0}$ for each $\lambda_0 \in \mathbb{T}$. In this case,

$$\sigma_e(T_\varphi) = \bigcup_{\lambda_0 \in \mathbb{T}} \mathrm{conv}\, V_{\lambda_0}(\varphi). \tag{2.7.2}$$

We shall call functions $\varphi$ satisfying (2.7.1) *Douglas functions*; let $D(\mathbb{T})$ denote the set of all Douglas functions in $L^\infty(\mathbb{T})$. Our aim is to extend Corollary 2.6 so that $\mathfrak{T}_{PC}$ is replaced by a more general class in which every Toeplitz operator with symbol $\varphi \in D(\mathbb{T})$ is a point of spectral continuity (see Theorem 2.12). This general class of operators will be the Toeplitz operators whose symbols are pseudo-piecewise continuous functions.

**Definition 2.8.** Let $\hat{}: L^\infty \to C(\widetilde{\partial} H^\infty)$ denote the Gelfand transform, where $\widetilde{\partial} H^\infty$ is the Šilov boundary of $H^\infty(\mathbb{T})$ (i.e. $\widetilde{\partial} H^\infty$ is the maximal ideal space of $L^\infty$). If $\varphi \in L^\infty$, then by the Gelfand theory, $\hat{\varphi}(\widetilde{\partial} H^\infty)$ is the spectrum of $\varphi$, as an element of $L^\infty$; namely, $\hat{\varphi}(\widetilde{\partial} H^\infty)$ is the closure of the essential range $\text{ess-ran}\,\varphi$ of $\varphi$. Now given $\varphi \in L^\infty(\mathbb{T})$, let $V_{\lambda_0}(\varphi)$ be as in (2.7.1). If $\varphi$ has the property that $\partial\, \mathrm{conv}\, V_{\lambda_0}(\varphi) \subseteq \hat{\varphi}(\widetilde{\partial} H^\infty)$, or that $\partial\, \mathrm{conv}\, V_{\lambda_0}(\varphi)$ is contained in some line segment $L_{\lambda_0}$, for each $\lambda_0 \in \mathbb{T}$, then $\varphi$ will be called *pseudo-piecewise continuous*. Write $PPC$ for the set of all pseudo-piecewise continuous functions in $L^\infty$.

For every $\lambda_0 \in \mathbb{T}$ and $\varphi \in D(\mathbb{T})$, $\mathrm{conv}\, V_{\lambda_0}(\varphi) = \partial \mathrm{conv}\, V_{\lambda_0}(\varphi)$, and so $D(\mathbb{T}) \subseteq PPC$. If $\varphi \in PPC$, then (2.7.2) (together with the fact that $T_\varphi$ is not a Fredholm operator whenever $\varphi$ cannot be inverted in $L^\infty(\mathbb{T})$) gives

$$\bigcup_{\lambda_0 \in \mathbb{T}} \partial\, \mathrm{conv}\, V_{\lambda_0}(\varphi) \subseteq \sigma_e(T_\varphi)\,. \tag{2.8.1}$$

The following example shows that the inclusion $D(\mathbb{T}) \subseteq PPC$ is proper.

**Example 2.9.** *There exists $\varphi \in L^\infty(\mathbb{T})$ such that $\varphi \in PPC \setminus D(\mathbb{T})$.*

*Proof.* Set

$$\varphi(e^{i\theta}) = \begin{cases} e^{i\pi(1+\frac{1}{2}\sin\frac{1}{\theta})} & (0 < \theta < \frac{2}{3\pi}) \\ (\frac{1}{\pi} + i) + \frac{1}{\pi} e^{i\frac{3\pi^2}{6\pi^2-8}(2\pi - \frac{2}{\pi} - \theta)} & (\frac{2}{3\pi} \le \theta \le 2\pi - \frac{2}{\pi}) \\ 2\pi - \theta + i \sin\frac{1}{2\pi - \theta} & (2\pi - \frac{2}{\pi} < \theta < 2\pi) \end{cases}$$

At $\lambda_0 = 0$, the graphs of $\varphi(\mathbb{T})$ and $V_0(\varphi)$ are in Figure 1. Therefore $\operatorname{conv} V_0(\varphi)$ is contained in no line segment and hence $\varphi \notin D(\mathbb{T})$. But evidently, $\partial \operatorname{conv} V_{\lambda_0}(\varphi) = V_{\lambda_0}(\varphi)$ for each $\lambda_0 \in \mathbb{T}$. In fact,

$$\bigcup_{\lambda_0 \in \mathbb{T}} V_{\lambda_0}(\varphi) = \{\hat{\varphi}(\gamma) : \gamma \in \widetilde{\partial} H^\infty\}\,.$$

Therefore $\varphi \in PPC$. $\square$

Figure 1

**Definition 2.10.** The map $\partial\sigma : L(H) \to \mathcal{K}$ sends every operator $T \in L(H)$ to the topological boundary $\partial\sigma(T)$ of its spectrum $\sigma(T)$.

**Theorem 2.11.** *The restriction of $\partial\sigma$ to the set of all Toeplitz operator with pseudo-piecewise continuous symbol is lower-semicontinuous at each Toeplitz operator with Douglas symbol; that is, if $\varphi_n \in PPC$, $\varphi \in D(\mathbb{T})$ and $||T_{\varphi_n} - T_\varphi|| \longrightarrow 0$ then $\partial\sigma(T_\varphi) \subseteq \liminf \partial\sigma(T_{\varphi_n})$.*

*Proof.* Observe that $\liminf \partial\sigma(T_{\varphi_n}) = \partial\big(\liminf \sigma(T_{\varphi_n})\big)$. Since $\liminf \sigma(T_{\varphi_n}) \subseteq \sigma(T_\varphi)$ and hence $\operatorname{int}\big(\liminf \sigma(T_{\varphi_n})\big) \subseteq \operatorname{int}\sigma(T_\varphi)$, it suffices to show that $\partial\sigma(T_\varphi) \subseteq \liminf \sigma(T_{\varphi_n})$. Assume $\lambda \notin \liminf \sigma(T_{\varphi_n})$. Then there exists a neighborhood $\mathcal{N}_1(\lambda)$ of $\lambda$ such that does not intersect infinitely many $\sigma(T_{\varphi_n})$. Thus we can choose a subsequence $\{\varphi_{n_i}\}$ of $\{\varphi_n\}$ such that $T_{\varphi_{n_i}} - \mu$ is invertible for each $\mu \in \mathcal{N}_1(\lambda)$, which says that $\varphi_{n_i}(\mathbb{T}) \cap \mathcal{N}_1(\lambda) = \emptyset$ for each $n_i$. Since $||\varphi_n - \varphi||_\infty = ||T_{\varphi_n} - T_\varphi|| \to 0$, there exists a neighborhood $\mathcal{N}_2(\lambda)$ of $\lambda$ such that $\varphi(\mathbb{T}) \cap \mathcal{N}_2(\lambda) = \emptyset$ and $\mathcal{N}_2(\lambda) \subseteq \mathcal{N}_1(\lambda)$. There are two cases to consider.

(Case i) Suppose $\varphi(\mathbb{T})$ winds around $\mathcal{N}_2(\lambda)$. Theorem 7.42 of [8] states that if $\varphi \in L^\infty(\mathbb{T})$ and $C$ is a rectifiable simple closed curve lying in $\mathbb{C} \setminus \sigma_e(T_\varphi)$, then $\operatorname{conv}\varphi(\mathbb{T})$ lies either entirely inside or entirely outside of $C$. But since by (2.7.2), $\mathcal{N}_2(\lambda) \subseteq \mathbb{C} \setminus \sigma_e(T_\varphi)$, it follows that either $\mathcal{N}_2(\lambda) \subseteq \sigma_e(T_\varphi)$ or $\mathcal{N}_2(\lambda) \cap \sigma_e(T_\varphi) = \emptyset$. Therefore $\lambda \notin \partial\sigma(T_\varphi)$.

(Case ii) Suppose $\varphi(\mathbb{T})$ does not wind around $\mathcal{N}_2(\lambda)$. We now claim that

$$\lambda \notin \bigcup_{\lambda_0 \in \mathbb{T}} \operatorname{conv} V_{\lambda_0}(\varphi).$$

On the contrary, we assume that $\lambda \in \operatorname{conv} V_{\lambda_0}(\varphi)$ for some $\lambda_0 \in \mathbb{T}$. Since $\varphi(\mathbb{T}) \cap \mathcal{N}_2(\lambda) = \emptyset$, and $\varphi \in D(\mathbb{T})$, $\lambda$ must lie in some line segment $L_{\lambda_0}(\varphi)$ such that $L_{\lambda_0}(\varphi) \cap \varphi(\mathbb{T}) \neq \emptyset$. Since $||\varphi_{n_i} - \varphi|| \to 0$, we have $V_{\lambda_0}(\varphi_{n_i}) \to V_{\lambda_0}(\varphi)$ and hence $\partial\operatorname{conv} V_{\lambda_0}(\varphi_{n_i}) \to \partial \operatorname{conv} V_{\lambda_0}(\varphi)$. But since $\partial \operatorname{conv} V_{\lambda_0}(\varphi)$ is contained in a line segment and, by (2.9.1), $\partial \operatorname{conv} V_{\lambda_0}(\varphi_{n_i}) \subseteq$

$\sigma_e(T_{\varphi_{n_i}})$, if follows that for each neighborhood $\mathcal{N}(\lambda)$, there exists a $\mu \in \mathcal{N}(\lambda)$ such that $T_{\varphi_{n_i}} - \mu$ is not Fredholm, which gives a contradiction. Therefore $\lambda \notin \bigcup_{\lambda_0 \in \mathbb{T}} \operatorname{conv} V_{\lambda_0}(\varphi)$. Thus by (2.8.2), $T_\varphi - \lambda$ is Fredholm. Now because for every $T \in L(H)$, $\partial\sigma(T) \setminus \sigma_e(T)$ consists of isolated points of $\sigma(T)$, we can conclude $\lambda \notin \partial\sigma(T_\varphi)$ because $\sigma(T_\varphi)$ is connected. This completes the proof. □

We now have our extension of Corollary 2.6 with the following result.

**Theorem 2.12.** *The restriction of $\sigma$ to the set of all Toeplitz operators with pseudo-piecewise continuous symbols is continuous at each Toeplitz operator with Douglas symbol.*

*Proof.* Suppose $\varphi_n \in PPC$, $\varphi \in D(\mathbb{T})$ and $||T_{\varphi_n} - T_\varphi|| \to 0$. By Theorem 2.11

$$\sigma(T_\varphi)^\wedge = \big(\liminf \sigma(T_{\varphi_n})\big)^\wedge,$$

where $K^\wedge$ denotes the polynomial-convex hull of $K$. Consequently, the passage from $\liminf \sigma(T_{\varphi_n})$ to $\sigma(T_\varphi)$ is the filling of some holes of $\liminf \sigma(T_{\varphi_n})$. Thus if $\sigma(T_\varphi)$ has no holes, then evidently $\sigma(T_\varphi) = \liminf\ \sigma(T_{\varphi_n})$. If $\sigma(T_\varphi)$ has a hole $\Omega$, then $\partial\Omega$ can be regarded as a "local closed curve "(see [9]) determined by $\operatorname{conv} V_\lambda(\varphi)$. As $\partial\Omega \subseteq \bigcup_{\lambda_0 \in \mathbb{T}} \operatorname{conv} V_{\lambda_0}(\varphi) = \bigcup_{\lambda_0 \in \mathbb{T}} \partial\operatorname{conv} V_{\lambda_0}(\varphi)$, we have

$$\partial\Omega = \bigcup_{\lambda_0 \in \mathbb{S}} \partial\operatorname{conv} V_{\lambda_0}(\varphi) \qquad \text{for some subset } \mathbb{S} \text{ of } \mathbb{T}.$$

Because also $\bigcup_{\lambda_0 \in \mathbb{T}} \partial\operatorname{conv} V_{\lambda_0}(\varphi_{n_i}) \to \bigcup_{\lambda_0 \in \mathbb{T}} \partial\operatorname{conv} V_{\lambda_0}(\varphi)$, we conclude that for sufficiently large $n_i$, $\varphi_{n_i}$ behaves like a Douglas function locally on $\mathbb{S}$. Thus the index theory for continuous symbols can be applied for this local closed curve ([9]). But $||\varphi_n - \varphi||_\infty \to 0$ and so for sufficiently large $n$,

$$-\operatorname{ind}(T_\varphi - \lambda) = wn\,(\varphi - \lambda) = wn(\varphi_n - \lambda) = -\operatorname{ind}(T_{\varphi_n} - \lambda) \quad \text{for each } \lambda \in \Omega\,.$$

Hence $\sigma(T_\varphi) \setminus \liminf \sigma(T_{\varphi_n})$ has no hole with non-zero winding number, and consequently $\sigma(T_\varphi) = \liminf \sigma(T_{\varphi_n})$. □

We were unable to decide whether or not, in Theorem 2.12, $D(\mathbb{T})$ can be replaced by $PPC$. (If we could have equality in (2.8.1), then the answer would be yes.) More interesting still is the the following open problem.

**Problem A.** Is the restriction of $\sigma$ to the set of all Toeplitz operators continuous?

## 3. Weyl's theorem for analytic functions of Toeplitz operators

We follow [4] in saying that *Weyl's theorem holds for* $T$ if

$$w(T) = \sigma(T) \setminus \pi_{00}(T)\,,$$

where $\pi_{00}(T)$ is the set of isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. The set of operators for which Weyl's theorem holds includes all seminormal operators and all Toeplitz operators [1],[4],[21]. The following old question of K. Oberai [22] led to the work in this section: *if* $T_\varphi$ *is a Toeplitz operator, then does Weyl's theorem hold for* $T_\varphi^2$*?*

To answer the Oberai's question, we begin with a spectral property of Toeplitz operators with continuous symbols.

**Lemma 3.1.** *Suppose that* $\varphi$ *is continuous and that* $f$ *is an analytic function defined on some open set containing* $\sigma(T_\varphi)$*. Then*

$$\sigma(T_{f\circ\varphi}) \subseteq f(\sigma(T_\varphi))\,, \tag{3.1.1}$$

*and equality occurs if and only if Weyl's theorem holds for* $f(T_\varphi)$*.*

*Proof.* By Corollary 2.2, $\sigma(T_{f\circ\varphi}) = w(f(T_\varphi)) \subseteq \sigma(f(T_\varphi)) = f(\sigma(T_\varphi))$. Because $\sigma(T_\varphi)$ is connected, so is $f(\sigma(T_\varphi)) = \sigma(f(T_\varphi))$; therefore the set $\pi_{00}(f(T_\varphi))$ is empty. Again by Corollary 2.2, $w(f(T_\varphi)) = \sigma(T_{f\circ\varphi})$ and so $w(f(T_\varphi)) = \sigma(f(T_\varphi)) \setminus \pi_{00}(f(T_\varphi))$ if and only if $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$. $\square$

**Remark 3.2.** If $\varphi$ is not continuous, it is possible for Weyl's theorem to hold for some $f(T_\varphi)$ without $\sigma(T_{f\circ\varphi})$ being equal to $f(\sigma(T_\varphi))$. One example is as follows. Let $\varphi(e^{i\,\theta}) = e^{\frac{i\,\theta}{3}}$ $(0 \le \theta < 2\pi)$, a piecewise continuous function. The operator $T_\varphi$ is invertible but $T_{\varphi^2}$ is not; hence $0 \in \sigma(T_{\varphi^2}) \setminus \{\sigma(T_\varphi)\}^2$. However $w(T_\varphi^2) = \sigma(T_\varphi^2)$, and $\pi_{00}(T_\varphi^2)$ is empty (see Figure 2); therefore Weyl's theorem holds for $T_\varphi^2$.

Figure 2

Figure 3

We can now answer Oberai's question: the answer is no.

**Example 3.3.** *There exists a continuous function $\varphi \in C(\mathbb{T})$ such that $\sigma(T_{\varphi^2}) \neq \{\sigma(T_\varphi)\}^2$.*

*Proof.* Let $\varphi$ be defined by

$$\varphi(e^{i\,\theta}) = \begin{cases} -e^{2i\,\theta}+1 & (0 \le \theta \le \pi) \\ e^{-2i\,\theta}-1 & (\pi \le \theta \le 2\pi)\,. \end{cases}$$

The orientation of the graph of $\varphi$ is shown in Figure 3. Evidently, $\varphi$ is continuous and, in Figure 3, $\varphi$ has winding number $+1$ with respect to the hole of $C_1$; the hole of $C_2$ has winding number $-1$. Thus we have $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$ and $\sigma(T_\varphi) = \operatorname{conv}\varphi(\mathbb{T})$. On the other hand, a straightforward calculation shows that $\varphi^2(\mathbb{T})$ is the Cardioid $r = 2(1+\cos\theta)$. In particular, $\varphi^2(\mathbb{T})$ traverses the Cardioid once in a counterclockwise direction and then traverses the Cardioid once in a clockwise direction. Thus $wn(\varphi^2 - \lambda) = 0$ for each $\lambda$ in the hole of $\varphi^2(\mathbb{T})$. Hence $T_{\varphi^2-\lambda}$ is a Weyl operator and is, therefore, invertible for each $\lambda$ in the hole of $\varphi^2(\mathbb{T})$. This implies that $\sigma(T_{\varphi^2})$ is the Cardioid $r = 2(1+\cos\theta)$. But because $\{\sigma(T_\varphi)\}^2 = \{\operatorname{conv}\varphi(\mathbb{T})\}^2 = \{(r,\theta) : r \le 2(1+\cos\theta)\}$, it follows that $\sigma(T_{\varphi^2}) \neq \{\sigma(T_\varphi)\}^2$. $\square$

**Remark 3.4.** It is instructive to observe that Lemma 3.1 gives a necessary condition for $T_\varphi$ to be hyponormal. We recall [17] that if $T \in L(H)$ is hyponormal, then Weyl's theorem holds for every $f(T)$. In conjunction with Lemma 3.1, this is to say that if $T_\varphi$ is hyponormal, then $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$. But this necessary condition is not sufficient, for a slight extension of Theorem 1 in [17] shows that Weyl's theorem holds for $f(T_\varphi)$, where $T_\varphi$ is the cohyponormal Toeplitz operator with symbol $\varphi(e^{i\,\theta}) = e^{-i\,\theta}$; hence $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$.

We conclude our work by studying continuous symbols $\varphi$ that have the property that Weyl's theorem holds for $f(T_\varphi)$, for every analytic function $f$ on a neigbourhood of $\sigma(T_\varphi)$.

**Theorem 3.5.** *If $\varphi \in C(\mathbb{T})$ is such that $\sigma(T_\varphi)$ has planar Lebesgue measure zero, then $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$ for every analytic function $f$ defined on an open set containing $\sigma(T_\varphi)$.*

*proof.* As $\varphi$ is continuous, so is $f\circ\varphi$ and thus $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$ and $\sigma_e(f(T_\varphi)) = \sigma_e(T_{f\circ\varphi}) = f\circ\varphi(\mathbb{T})$. The planar measure of $\sigma(T_\varphi)$ is zero; because $\sigma(T_\varphi)$ is a compact connected set consisting of $\varphi(\mathbb{T})$ and some of its holes, it follows that $\partial\sigma(T_\varphi) = \sigma_e(T_\varphi) = \sigma(T_\varphi)$, which is just a continuous curve. Furthermore, as analytic functions map open connected sets onto open connected open sets, we have that $\partial\sigma(f(T_\varphi)) = \sigma_e(f(T_\varphi)) = \sigma(f(T_\varphi))$. Thus $\sigma(f(T_\varphi)) \subseteq \sigma(T_{f\circ\varphi})$, which together with (3.1.1) gives the result. $\square$

**Remark 3.6.** We note here that Toeplitz operators whose symbol satisfies the hypothesis of Theorem 3.5 are essentially normal of the type "normal + compact." To see this, let $D$ be a diagonal operator whose spectrum is $\varphi(\mathbb{T})$. Because $T_\varphi$ and $D$ are both essentially normal and $\mathcal{SP}(T_\varphi) = \mathcal{SP}(D)$, it follows from the Brown-Douglas-Fillmore theorem that $T_\varphi$ and $D$ are compalent; that is, $T_\varphi = N + K$ for some normal operator $N$ on $H^2$ and some $K \in K(H^2)$. This observation is of interest because if $\sigma(T_\varphi)$ has planar Lebesgue measure zero and, further, if $T_\varphi$ is hyponormal, then by Putnam's inequality $T_\varphi$ is normal and $\varphi(\mathbb{T})$ must be a line segment.

**Theorem 3.7.** *If the winding number of $\varphi \in C(\mathbb{T})$ with respect to each hole of $\varphi(\mathbb{T})$ is nonnegative (or is nonpositive), then $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$ for every analytic function $f$ defined an an open set containing $\sigma(T_\varphi)$.*

*Proof.* Suppose that the holes of $\varphi(\mathbb{T})$ have only nonnegative winding numbers. Since $\varphi$ is continuous, it follows that $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$ and

$$\sigma_e(T_{f\circ\varphi}) = \sigma_e(f(T_\varphi)) = f(\sigma_e(T_\varphi))\,. \tag{3.7.1}$$

If $\varphi(\mathbb{T})$ has no holes or has holes of winding number zero only, then $\sigma(T_\varphi) = \sigma_e(T_\varphi)$; thus

$$f(\sigma(T_\varphi)) = f(\sigma_e(T_\varphi)) = \sigma_e(f(T_\varphi)) = \sigma_e(T_{f\circ\varphi}) \subseteq \sigma(T_{f\circ\varphi})\,,$$

which together with (3.1.1) gives $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$. Now assume that there exists at least a hole $\Omega$ of $\varphi(\mathbb{T})$ such that $wn(\varphi - \lambda) \neq 0$ for all $\lambda \in \Omega$. Namely, $wn(\varphi - \lambda) = w > 0$ for all $\lambda \in \Omega$. In view of (3.7.1), it suffices to show that $f(\sigma(T_\varphi)) \setminus f(\sigma_e(T_\varphi)) \subseteq \sigma(T_{f\circ\varphi}) \setminus \sigma_e(T_{f\circ\varphi})$. Thus the proof is completed by showing that if $\lambda \in \Omega$, then $f(\lambda) \in \sigma(T_{f\circ\varphi})$. Suppose that $\lambda \in \Omega$; thus $T_\varphi - \lambda$ is Fredholm with $\operatorname{ind}(T_\varphi - \lambda) = -wn(\varphi - \lambda) = -w < 0$. Write

$$f(z) - f(\lambda) = (z-\lambda)(z-\mu_1)^{\alpha_1}\cdots(z-\mu_n)^{\alpha_n}F(z)\,,$$

where $\alpha_i \in \mathbb{Z}^+,\ \mu_i \in \sigma(T_\varphi)$ $(1 \le i \le n)$ and $F(z)$ is analytic and has no zeros in $\sigma(T_\varphi)$. We have

$$f\circ\varphi - f(\lambda) = (\varphi-\lambda)(\varphi-\mu_1)^{\alpha_1}\cdots(\varphi-\mu_n)^{\alpha_n}F\circ\varphi\,.$$

From (3.7.1), $T_{f\circ\varphi - f(\lambda)}$ is Fredholm and hence $f\circ\varphi - f(\lambda)$ is invertible on $\mathbb{T}$. So each $\varphi - \mu_i$ $(1 \le i \le n)$ and $F\circ\varphi$ vanish nowhere on $\mathbb{T}$. Therefore $T_{\varphi-\mu_i}$ and $T_{F\circ\varphi}$ are all Fredholm. By assumption, $wn(\varphi - \mu_i) \ge 0$, and because $F\circ\varphi$ has no zeros in $\sigma(T_\varphi)$, $wn(F\circ\varphi) = 0$. Thus

$$\begin{aligned}\operatorname{ind}(T_{f\circ\varphi-f(\lambda)}) &= -wn\{(\varphi-\lambda)(\varphi-\mu_1)^{\alpha_1}\cdots(\varphi-\mu_n)^{\alpha_n}F(\varphi)\}\\ &= -wn(\varphi-\lambda) - \sum_{i=1}^n \alpha_i\, wn(\varphi-\mu_i) < 0\,,\end{aligned}$$

which shows that $T_{f\circ\varphi-f(\lambda)}$ is not a Weyl operator and hence is not invertible. We conclude that $f(\lambda) \in \sigma(T_{f\circ\varphi})$. The proof of the case of nonpositive winding numbers is similar. $\square$

**Example 3.8.** *If $\varphi$ is of the form of $p(\frac{a}{z}+bz)$, where $a, b \in \mathbb{R}$ and $p$ is any polynomial, then $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$.*

*Proof.* If $a = b$, then $T_\varphi$ is hermitian and the desired conclusion is evident. If $a \neq b$, set $\psi = \frac{a}{z} + bz$. Then

$$\psi(\mathbb{T}) = \{(u, v) \in \mathbb{C} : (\frac{u}{b+a})^2 + (\frac{v}{b-a})^2 = 1\},$$

which is a circle or an ellipse. Thus $\varphi(\mathbb{T}) = (p \circ \psi)(\mathbb{T}) = p(\psi)(\mathbb{T})$, which has no holes or has exactly one hole (because polynomials map continuous curves onto continuous curves and open sets onto open sets). The conclusion nows follows from Theorem 3.7. □

**Remark 3.9.** Lemma 3.1 and Theorems 3.5, 3.7 hold for quasicontinuous symbol $\varphi$. In this case, if $T_\varphi$ is Fredholm, then the index of $T_\varphi$ is the negative of the winding number with respect to the origin of the curve $\hat{\varphi}(re^{i\theta})$ for $1-\delta < r < 1$, and

$$\sigma_e(T_\varphi) = \bigcap_{0<\delta<1} \mathrm{cl}\,\{\hat{\varphi}(re^{i\theta}) : 1-\delta < r < 1\},$$

where $\hat{\varphi}$ is the harmonic extension of $\varphi$ to the open unit disk $\mathbb{D}$ (cf. [8]).

**Remark 3.10.** The index of a hyponormal opertor is always nonpositive and therefore, in general, the holes of the essential spectrum of a hyponormal operator cannot have negative winding numbers. This fact may lead one to believe that if $\varphi(\mathbb{T})$ has no hole with negative winding number (in particular, in case that $\varphi$ is a trigonometric polynomial), then $T_\varphi$ is hyponormal. But such is not the case. For example, if

$$\varphi_1(e^{i\theta}) = e^{-2i\theta} + e^{i\theta} + e^{2i\theta}, \quad \text{and} \quad \varphi_2(e^{i\theta}) = e^{-2i\theta} - e^{-i\theta} + e^{i\theta} + e^{2i\theta},$$

then $\varphi_1(\mathbb{T})$ has just one essential hole whose winding number is $+1$, and $\varphi_2(\mathbb{T})$ has no hole, as shown in Figure 4. But by Theorem 1.4, $T_{\varphi_1}$ and $T_{\varphi_2}$ both fail to be hyponormal.

Figure 4

**Remark 3.11.** Recall [23] that an operator $T \in L(H)$ is quasitriangular if there exists an increasing sequence $\{P_n\}$ of projections of finite rank in $L(H)$ that converges strongly to the identity and satisfies $||P_nTP_n - TP_n|| \to 0$. By the work of Apostol, Foias, and Voiculescu, it is known that $T$ is quasitriangular if and only if $\mathcal{SP}(T)$ contains no hole or pseudohole with negative winding number. Rewrite Theorem 3.7 as follows: if $T_\varphi$ is a quasitriangular (or $T_\varphi^*$ is a quasitriangular) Toeplitz operator with continuous symbol $\varphi$, then $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$. In Remark 3.10 we showed that even if $T_\varphi^*$ is a quasitriangular Toeplitz operator (with trigonometric polynomial symbol $\varphi$), $T_\varphi$ may fail to be hyponormal. In spite of this, it would be interesting to have a method by which one could determine the winding numbers of curves given by trigonometric polynomials with respect to the various holes these polynomials produce. We expect the solution will make extensive use of Theorem 3.7. The following open problem is of particular interest in operator theory.

**Problem B.** If $\varphi$ is a trigonometric polynomial, find necessary and sufficient conditions, in terms of the coefficients of $\varphi$, in order for the Toeplitz operator $T_\varphi^*$ to be quasitriangular.

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