HYPONORMALITY OF TOEPLITZ OPERATORS WITH POLYNOMIAL SYMBOLS

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1 Introduction

A bounded linear operator A on a Hilbert space $\mathfrak{H}$ with inner product $(\cdot,\cdot)$ is said to be hyponormal if its selfcommutator $[A^*,A]=A^*A-AA^*$ induces a positive semidefinite quadratic form on $\mathfrak{H}$ via $\xi\mapsto([A^*,A]\xi,\xi)$, for $\xi\in\mathfrak{H}$. Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T}=\partial\mathbb{D}$ in the complex plane. Recall that given $\varphi\in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on $H^2(\mathbb{T})$ defined by $T_\varphi f=P(\varphi\cdot f)$, where $f\in H^2(\mathbb{T})$ and P denotes the projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The hyponormality of Toeplitz operators has been studied by C. Cowen [**1**],[**2**], P. Fan [**4**], C. Gu [**8**], T. Ito and T. Wong [**9**], T. Nakazi and K. Takahashi [**11**], D. Yu [**13**], K. Zhu [**14**], R. Curto, D. Farenick, the second and the third named authors [**3**],[**5**],[**6**],[**10**] and others. An elegant theorem of C. Cowen [**2**] characterizes the hyponormality of a Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi\in L^\infty(\mathbb{T})$. K. Zhu [**14**] reformulated Cowen's criterion and then showed that the hyponormality of T_φ with polynomial symbols φ can be decided by a method based on the classical interpolation theorem of I. Schur [**12**]. Also Farenick and the third named author [**5**] characterized the hyponormality of T_φ in terms of the Fourier coefficients of the trigonometric polynomial φ in the cases that the outer coefficients of φ have the same modulus. But the case of arbitrary trigonometric polynomials φ, though solved in principle by Cowen's theorem or Zhu's theorem, is in practice very complicated. On the other hand, Nakazi and Takahashi [**11**, Corollary 5] showed that *if $\varphi(z)=\sum_{n=-m}^N a_nz^n$ is a trigonometric polynomial with $m\le N$ and if for every zero ζ of $z^m\varphi$ such that $|\zeta|>1$, the number $1/\overline{\zeta}$ is a zero of $z^m\varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ, then T_φ is hyponormal.* But the converse is not true in general. To see this consider the following trigonometric polynomial: $\varphi(z)=z^{-2}(z-2)(z-1)(z-\frac{1}{5})(z-\frac{1}{3})$. Then $\varphi(z)=\frac{2}{15}z^{-2}-\frac{19}{15}z^{-1}+\frac{55}{15}-\frac{53}{15}z+z^2$. Using an argument of P. Fan [**4**, Theorem 1] – for every trigonometric polynomial φ of the form $\varphi(z)=\sum_{n=-2}^2 a_nz^n$,

$$(0.1)\qquad T_\varphi \text{ is hyponormal } \iff \left|\det\begin{pmatrix} a_{-1} & a_{-2}\\ a_1 & a_2\end{pmatrix}\right|\le|a_2|^2-|a_{-2}|^2,$$

a straightforward calculation shows that T_φ is hyponormal. In this paper we consider how the converse of the above result due to Nakazi and Takahashi survives for arbitrary

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trigonometric polynomials. The main results are as follows. Suppose $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$ with $m\le N$ and write

$$\mathfrak{F}:=\{\zeta,\,1/\overline{\zeta}:\ \text{the complex numbers } \zeta \text{ and } 1/\overline{\zeta} \text{ are zeros of } z^m\varphi\}.$$

If $\mathfrak{F}$ contains at least $(N+1)$ elements then the following statements are equivalent.

(i) T_φ is a hyponormal operator.

(ii) For every zero ζ of $z^m\varphi$ such that $|\zeta|>1$, the number $1/\overline{\zeta}$ is a zero of $z^m\varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.

Moreover, in the cases where T_φ is a hyponormal operator, the rank of the selfcommutator of T_φ is computed from the formula $\text{rank}\,[T_\varphi^*,T_\varphi]=N-m+Z_{\mathbb{D}}-Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$, where $Z_{\mathbb{D}}$ and $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ are the number of zeros of $z^m\varphi$ in $\mathbb{D}$ and in $\mathbb{C}\setminus\overline{\mathbb{D}}$ counting multiplicity. In addition, a new necessary condition for hyponormality of T_φ with polynomial symbols φ is presented: if $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$ is such that T_φ is hyponormal and if $z^m\,\varphi=a_N\prod_{j=1}^{m+N}(z-\zeta_j)$, then $\left|\sum_{j=1}^{m+N}\left(\zeta_j-1/\overline{\zeta_j}\right)\right|\ \le\ \frac{1}{\prod_{j=1}^{m+N}|\zeta_j|}-\prod_{j=1}^{m+N}|\zeta_j|$.

2 Main results

We shall use a variant of Cowen's theorem [**1**] that was first proposed by Nakazi and Takahashi [**11**].

Cowen's Theorem. *Suppose $\varphi\in L^\infty(\mathbb{T})$ is arbitrary and write*

$$\mathcal{E}(\varphi)=\{k\in H^\infty(\mathbb{T}):||k||_\infty\le 1 \text{ and } \varphi-k\overline{\varphi}\in H^\infty(\mathbb{T})\}\,.$$

Then T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty.

On the other hand Nakazi and Takahashi [**11**] showed that if T_φ is a hyponormal operator such that $\text{rank}\,[T_\varphi^*,T_\varphi]<\infty$, then there exists a finite Blaschke product $k\in\mathcal{E}(\varphi)$ of the form

$$k(z)=e^{i\theta}\prod_{j=1}^{n}\frac{z-\beta_j}{1-\overline{\beta_j}z}\qquad(|\beta_j|<1 \text{ for } j=1,\cdots,n).$$

It is rephrased explicitly as follows.

Lemma 1 (Nakazi-Takahashi Theorem). *A Toeplitz operator T_φ is hyponormal and the rank of the selfcommutator $[T_\varphi^*,T_\varphi]$ is finite (e.g., φ is a trigonometric polynomial) if and only if there exists a finite Blaschke product $k\in\mathcal{E}(\varphi)$ such that deg$(k)=$ rank$[T_\varphi^*,T_\varphi]$, where deg(k) denotes the degree of k – meaning the number of zeros of k in the open unit disk $\mathbb{D}$ ([**7**, Page 6]).*

We record here results on the hyponormality of Toeplitz operators with polynomial symbols, which have been recently developed in [**3**],[**5**],[**6**],[**9**],[**10**],[**11**], and [**14**].

Lemma 2. *Suppose that φ is a trigonometric polynomial of the form $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$, where a_{-m} and a_N are nonzero.*

(i) *If T_φ is a hyponormal operator then $m\le N$ and $|a_{-m}|\le|a_N|$.*

(ii) *If T_φ is a hyponormal operator then $N-m\le$ rank$[T_\varphi^*,T_\varphi]\le N$.*

(iii) *The hyponormality of T_φ is independent of the particular values of the Fourier coefficients $a_0, a_1, \cdots, a_{N-m}$ of φ. Moreover the rank of the selfcommutator $[T_\varphi^*, T_\varphi]$ is also independent of those coefficients.*

(iv) *If $|a_{-m}| = |a_N| \neq 0$, then T_φ is hyponormal if and only if the following equation holds:*

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m}\begin{pmatrix} \overline{a_{N-m+1}} \\ \overline{a_{N-m+2}} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix}. \tag{2.1}$$

In this case, the rank of $[T_\varphi^*, T_\varphi]$ is $N-m$.

(v) *T_φ is normal if and only if $m = N$, $|a_{-m}| = |a_N|$, and (2.1) holds with $m = N$.*

We begin with:

Lemma 3. *Suppose that B is a finite Blaschke product of degree n. If*

$$\prod_{j=1}^{r}(z-\zeta_j) - B\prod_{j=1}^{r}(1-\overline{\zeta_j}z) \in z^m H^\infty \qquad (n+r<2m;\ \zeta_j \in \mathbb{C} \text{ for } j=1,\cdots,r), \tag{3.1}$$

then B is of the form

$$B(z) = \prod_{j=1}^{r}\frac{z-\zeta_j}{1-\overline{\zeta_j}z}. \tag{3.2}$$

Thus, in particular, if B_1 and B_2 are finite Blaschke products such that deg$(B_1) \le$ deg$(B_2) < N$, and if $B_1 - B_2 \in z^N H^\infty$, then $B_1 = B_2$.

Proof. Suppose that B is of the form

$$B(z) = e^{i\omega}\prod_{j=1}^{n}\frac{z-\beta_j}{1-\overline{\beta_j}z} \quad (|\beta_j|<1 \text{ for } j=1,\cdots,n).$$

Without loss of generality we assume that $m \le n+r$ (if $m > n+r$ then evidently (3.2) holds). Multiplying $\prod_{j=1}^{n}(1-\overline{\beta_j}z)$ on both sides of (3.1) gives

$$\prod_{j=1}^{r}(z-\zeta_j)\prod_{j=1}^{n}(1-\overline{\beta_j}z) - e^{i\omega}\prod_{j=1}^{n}(z-\beta_j)\prod_{j=1}^{r}(1-\overline{\zeta_j}z) := \sum_{j=m}^{n+r} d_j z^j$$

for some d_j $(j=m,\cdots,n+r)$. Put $f(z) := \sum_{j=m}^{n+r} d_j z^j$. Now it suffices to show that $f=0$. Observe that for $z \neq 0$,

$$\begin{aligned} f\left(\frac{1}{z}\right) &= \sum_{j=m}^{n+r} d_j\left(\frac{1}{z}\right)^j = \prod_{j=1}^{r}\left(\frac{1}{z}-\zeta_j\right)\prod_{j=1}^{n}\left(1-\frac{\overline{\beta_j}}{z}\right) - e^{i\omega}\prod_{j=1}^{n}\left(\frac{1}{z}-\beta_j\right)\prod_{j=1}^{r}\left(1-\frac{\overline{\zeta_j}}{z}\right) \\ &= \left(\frac{1}{z}\right)^{n+r}\left[\prod_{j=1}^{r}(1-\zeta_j\,\overline{z})\prod_{j=1}^{n}(\overline{z}-\overline{\beta_j}) - e^{i\omega}\prod_{j=1}^{n}(1-\beta_j\,\overline{z})\prod_{j=1}^{r}(\overline{z}-\overline{\zeta_j})\right]. \end{aligned}$$

Thus we have

$$\sum_{j=m}^{n+r} d_j\, \overline{z}^{n+r-j} = \prod_{j=1}^{r}\bigl(1-\zeta_j\, \overline{z}\bigr)\prod_{j=1}^{n}\bigl(\overline{z}-\overline{\beta_j}\bigr) - e^{i\omega}\prod_{j=1}^{n}\bigl(1-\beta_j\, \overline{z}\bigr)\prod_{j=1}^{r}\bigl(\overline{z}-\overline{\zeta_j}\bigr),$$

so that

$$\sum_{j=m}^{n+r} \overline{d_j}\, z^{n+r-j} = \prod_{j=1}^{r}\bigl(1-\overline{\zeta_j}\, z\bigr)\prod_{j=1}^{n}\bigl(z-\beta_j\bigr) - e^{-i\omega}\prod_{j=1}^{n}\bigl(1-\overline{\beta_j}\, z\bigr)\prod_{j=1}^{r}\bigl(z-\zeta_j\bigr) = -e^{-i\omega} f(z),$$

which implies

$$\sum_{j=m}^{n+r} d_j z^j = -e^{i\omega}\sum_{j=m}^{n+r}\overline{d_j} z^{n+r-j}. \tag{3.3}$$

Note that the equality (3.3) holds also for $z=0$ and hence for all $z \in \mathbb{C}$. But since $n+r-m<m$, it follows from (3.3) that $d_j = 0$ for every $j = m, \cdots, n+r$, and therefore $f = 0$. This completes the proof. □

We need not expect that in the second assertion of Lemma 3, the condition "$\deg(B_1) \le \deg(B_2) < N$" is relaxed to the condition "$\deg(B_1) \le \deg(B_2) \le N$". For example if

$$B_1(z) := -\frac{z-\frac{1}{2}}{1-\frac{1}{2}z} \quad \text{and} \quad B_2(z) := \frac{z+\frac{1}{2}}{1+\frac{1}{2}z},$$

then $B_1 - B_2 \in z\,H^\infty$ and $\deg(B_1) = \deg(B_2) = 1$, while $B_1 \ne B_2$.

The following corollary shows that if φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ such that the rank of the selfcommutator $[T_\varphi^*, T_\varphi]$ is less than N, then the finite Blaschke product of degree less than N in Lemma 1 is uniquely determined.

Corollary 4. *Suppose that $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is such that T_φ is hyponormal and rank$[T_\varphi^*, T_\varphi] < N$. If we put*

$$\mathfrak{B}(\varphi) := \{b \in \mathcal{E}(\varphi):\ b \text{ is a finite Blaschke product of degree less than } N\},$$

then $\mathfrak{B}(\varphi)$ contains exactly one element.

Proof. In view of Lemma 1, there exists a finite Blaschke product $b_1 \in \mathfrak{B}(\varphi)$ such that $\deg(b_1) = \operatorname{rank}[T_\varphi^*, T_\varphi] < N$. For the uniqueness we assume that there exists a finite Blaschke product $b_2 \in \mathfrak{B}(\varphi)$ of degree less than N. Then $\varphi - b_i\overline{\varphi} \in H^\infty$ for $i = 1, 2$, so that $(b_1 - b_2)\overline{\varphi} \in H^\infty$. But since $\overline{\varphi} = z^{-N}\bigl(\overline{a_N} + \overline{a_{N-1}}z + \cdots + \overline{a_{-m}}z^{m+N}\bigr)$, it follows that $b_1 - b_2 \in z^N H^\infty$. Now applying Lemma 3 gives that $b_1 = b_2$. □

We don't however guarantee that if φ is a trigonometric polynomial such that T_φ is hyponormal then there exists a unique finite Blaschke product in $\mathcal{E}(\varphi)$. For example if $\varphi(z) = z^{-1} + 3z$ then

$$b_1 = \frac{z+\frac{1}{3}}{1+\frac{1}{3}z} \quad \text{and} \quad b_2 = \frac{(z+\frac{2}{3})(z+\frac{1}{2})}{(1+\frac{2}{3}z)(1+\frac{1}{2}z)}$$

are both finite Blaschke products in $\mathcal{E}(\varphi)$.

We now have:

Theorem 5. *Suppose* $\varphi(z)=\sum_{n=-m}^{N} a_n z^n$ *with* $m \le N$ *and write*

$$\mathfrak{F} := \{\zeta,\, 1/\overline{\zeta} :\ \text{the complex numbers } \zeta \text{ and } 1/\overline{\zeta} \text{ are zeros of } z^m\varphi\}.$$

If $\mathfrak{F}$ *contains at least* $(N+1)$ *elements then the following statements are equivalent.*

(i) T_φ *is a hyponormal operator.*

(ii) *For every zero* ζ *of* $z^m\varphi$ *such that* $|\zeta|>1$, *the number* $1/\overline{\zeta}$ *is a zero of* $z^m\varphi$ *in* $\mathbb{D}$ *of multiplicity greater than or equal to the multiplicity of* ζ.

Moreover, in the cases where T_φ *is a hyponormal operator, the rank of the selfcommutator of* T_φ *is computed from the formula*

$$\text{rank}\,[T_\varphi^*, T_\varphi] = N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}},$$

where $Z_{\mathbb{D}}$ *and* $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ *are the number of zeros of* $z^m\varphi$ *in* $\mathbb{D}$ *and in* $\mathbb{C}\setminus\overline{\mathbb{D}}$ *counting multiplicity.*

Proof. By assumption we can write

$$z^m\varphi = a_N \prod_{j=1}^{r}(z-\gamma_j)(z-\frac{1}{\overline{\gamma_j}})\prod_{j=1}^{t}(z-\alpha_j)\prod_{j=1}^{m+N-2r-t}(z-\zeta_j),$$

where $|\gamma_j|>1$ $(1\le j\le r)$, $|\alpha_j|=1$ $(1\le j\le t)$, $0<|\zeta_j|\ne 1$ $(1\le j\le m+N-2r-t)$, and $2r+t\ge N+1$. So we have

$$\begin{aligned} z^N\overline{\varphi} &= \overline{a_N}\prod_{j=1}^{r}(1-\overline{\gamma_j}z)(1-\frac{1}{\gamma_j}z)\prod_{j=1}^{t}(1-\overline{\alpha_j}z)\prod_{j=1}^{m+N-2r-t}(1-\overline{\zeta_j}z)\\ &= \overline{a_N}\prod_{j=1}^{r}\left(\frac{\overline{\gamma_j}}{\gamma_j}\right)\prod_{j=1}^{t}(-\overline{\alpha_j})\prod_{j=1}^{r}(z-\gamma_j)(z-\frac{1}{\overline{\gamma_j}})\prod_{j=1}^{t}(z-\alpha_j)\prod_{j=1}^{m+N-2r-t}(1-\overline{\zeta_j}z). \end{aligned}$$

Combining Cowen's theorem, Lemma 1, and Lemma 2(ii), we can see that T_φ is hyponormal if and only if there exists a finite Blaschke product $k\in\mathcal{E}(\varphi)$ with $N-m\le \deg(k)\le N$, say

$$k(z)=e^{i\theta}\prod_{j=1}^{s}\frac{z-\beta_j}{1-\overline{\beta_j}z}\quad (N-m\le s\le N).$$

Observe

$$\varphi - k\,\overline{\varphi} = \frac{a_N}{z^N}\prod_{j=1}^{r}(z-\gamma_j)(z-\frac{1}{\overline{\gamma_j}})\prod_{j=1}^{t}(z-\alpha_j)\,g(z),$$

where

$$g(z) = z^{N-m}\prod_{j=1}^{m+N-2r-t}(z-\zeta_j) - e^{i\phi}\prod_{j=1}^{s}\frac{z-\beta_j}{1-\overline{\beta_j}z}\prod_{j=1}^{m+N-2r-t}(1-\overline{\zeta_j}z) \tag{5.1}$$

with

$$e^{i\phi} = e^{i\theta}\frac{\overline{a_N}}{a_N}\prod_{j=1}^{r}\left(\frac{\overline{\gamma_j}}{\gamma_j}\right)\prod_{j=1}^{t}(-\overline{\alpha_j}).$$

Thus $\varphi - k\overline{\varphi} \in H^\infty$ if and only if g should be of the form $g(z) = \sum_{j=N}^\infty c_j z^j$. If $m < N$ then substituting $z = 0$ into (5.1) gives that $\prod_{j=1}^s(-\beta_j) = 0$ and hence $\beta_j = 0$ for some j $(1 \le j \le s)$. Repeating this process we can see that $(N-m)$'s β_j should be zero, say $\beta_{s+m-N+1} = \cdots = \beta_s = 0$. Therefore from (5.1) we can write

$$\prod_{j=1}^{m+N-2r-t}(z-\zeta_j) - e^{i\phi}\prod_{j=1}^{s+m-N}\frac{z-\beta_j}{1-\overline{\beta_j}z}\prod_{j=1}^{m+N-2r-t}(1-\overline{\zeta_j}z) = \sum_{j=m}^\infty c_{N-m+j}z^j. \tag{5.2}$$

If instead $m = N$ then evidently (5.2) holds. Note that $(m+N-2r-t)+(s+m-N) < 2m$. Therefore applying Lemma 3 gives that the equality (5.2) holds if and only if $\sum_{j=m}^\infty c_{N-m+j}z^j = 0$ and hence $g = 0$. Thus $\varphi - k\overline{\varphi} \in H^\infty$ if and only if $g = 0$. Therefore a Toeplitz operator T_φ is hyponormal if and only if $\varphi = k\overline{\varphi}$ for some Blaschke product $k \in H^\infty$. Observe

$$k = \frac{\varphi}{\overline{\varphi}} = \frac{a_N}{\overline{a_N}}\prod_{j=1}^r\left(\frac{\gamma_j}{\overline{\gamma_j}}\right)\prod_{j=1}^t(-\alpha_j)\, z^{N-m}\prod_{j=1}^{m+N-2r-t}\frac{z-\zeta_j}{1-\overline{\zeta_j}z}. \tag{5.3}$$

Therefore T_φ is hyponormal if and only if k is analytic on $\mathbb{D}$ if and only if for every zero ζ of $z^m\varphi$ such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of $z^m\varphi$ in $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.

As we did in the first part of the proof, if there exists a finite Blaschke product $k \in \mathcal{E}(\varphi)$, then k should be of the form $k = \frac{\varphi}{\overline{\varphi}}$; thus in this case the Blaschke product k is determined uniquely. On the other hand if $Z_\mathbb{D}$ and $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ denote the number of zeros of $z^m\varphi$ in $\mathbb{D}$ and in $\mathbb{C}\setminus\overline{\mathbb{D}}$ counting multiplicity, then since

$$z^m\varphi = a_N\prod_{j=1}^r(z-\gamma_j)(z-\frac{1}{\overline{\gamma_j}})\prod_{j=1}^t(z-\alpha_j)\prod_{j=1}^{m+N-2r-t}(z-\zeta_j),$$

it follows that $Z_\mathbb{D} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ equals to the value

$$\#\left(\text{zeros of} \prod_{j=1}^{m+N-2r-t}(z-\zeta_j) \text{ in } \mathbb{D}\right) - \#\left(\text{zeros of} \prod_{j=1}^{m+N-2r-t}(z-\zeta_j) \text{ in } \mathbb{C}\setminus\overline{\mathbb{D}}\right).$$

Thus the expression (5.3) shows that the degree of k is $N - m + Z_\mathbb{D} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$. It therefore follows from Lemma 1 that rank $[T_\varphi^*, T_\varphi] = N - m + Z_\mathbb{D} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$. □

Corollary 6. *Let $\varphi(z) = \sum_{n=-m}^N a_n z^n$ with $m \le N$. If $z^m\varphi$ has at least $(N+1)$'s zeros on the unit circle, then the following statements are equivalent.*

(i) *T_φ is a hyponormal operator.*

(ii) *For every zero ζ of $z^m\varphi$ such that $|\zeta| > 1$, the number $1/\overline{\zeta}$ is a zero of $z^m\varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.*

Proof. If $z^m\varphi$ has at least $(N+1)$'s zeros on the unit circle, then this satisfies the assumption of Theorem 5. Thus the result immediately follows from Theorem 5. □

Corollary 7([5, Theorem 2]). *Suppose $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$, with $m\le N$, is a circulant polynomial with argument ω, i.e., $a_{-k}=e^{i\omega}a_{N-k+1}$ for every $1\le k\le m$ and for some fixed $\omega\in[0,2\pi)$. If $f(z)=a_{N-m+1}+a_{N-m+2}z+\cdots+a_Nz^{m-1}$, then the following statements are equivalent.*

(i) *T_φ is a hyponormal operator.*

(ii) *For every zero ζ of f such that $|\zeta|>1$, the number $1/\overline{\zeta}$ is a zero of f in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.*

Proof. In view of Lemma 2(iii), we assume that $a_0=a_1=\cdots=a_{N-m}=0$. Note that

$$z^m\,\varphi=\left(e^{i\omega}+z^{N+1}\right)f(z).$$

Thus $z^m\varphi$ has at least $(N+1)$'s zeros on the unit circle, and the set of zeros of $z^m\varphi$ not on the unit circle is the set of zeros of f not on the unit circle. Therefore the result immediately follows from Corollary 6. □

Theorem 8. *Suppose $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$, where $m\le N$ and $|a_N|=|a_{-m}|\neq 0$, and let $\psi:=\varphi-\sum_{n=0}^{N-m}a_nz^n$. Then the following statements are equivalent.*

(i) *T_φ is a hyponormal operator.*

(ii) *For every zero ζ of $z^m\psi$, the number $1/\overline{\zeta}$ is a zero of $z^m\psi$ of multiplicity equal to the multiplicity of ζ.*

Proof. In view of Lemma 2(iii), T_φ is hyponormal if and only if T_ψ is. By Lemma 2(iv), T_ψ is hyponormal if and only if the Fourier coefficients of ψ satisfy the following equation:

$$\overline{a_N}\begin{pmatrix}a_{-1}\\ a_{-2}\\ \vdots\\ \vdots\\ a_{-m}\end{pmatrix}=a_{-m}\begin{pmatrix}\overline{a_{N-m+1}}\\ \overline{a_{N-m+2}}\\ \vdots\\ \vdots\\ \overline{a_N}\end{pmatrix}. \tag{8.1}$$

Since $|a_N|=|a_{-m}|$, there exists $\theta\in[0,2\pi)$ such that $a_{-m}=\overline{a_N}e^{i\theta}$. Then by (8.1) we have that $a_{-j}=\overline{a_{N-m+j}}e^{i\theta}$ for every $j=1,\cdots,m$. Thus we may rewrite ψ as

$$\psi(z)=e^{i\theta}\left(\overline{a_N}z^{-m}+\cdots+\overline{a_{N-m+1}}z^{-1}\right)+a_{N-m+1}z^{N-m+1}+\cdots+a_Nz^N.$$

Therefore T_ψ is hyponormal if and only if $\psi=k\overline{\psi}$ with $k=\frac{a_{-m}}{\overline{a_N}}z^{N-m}$. Observe

$$\zeta \text{ is a zero of } z^m\psi \Longleftrightarrow 1/\overline{\zeta} \text{ is a zero of } z^N\overline{\psi}.$$

But since

$$k=z^{N-m}\,\frac{z^m\psi}{z^N\overline{\psi}}=\frac{a_{-m}}{\overline{a_N}}\,z^{N-m}\prod_{j=1}^{N+m}\frac{z-\zeta_j}{z-1/\overline{\zeta_j}}$$

(note that $\zeta_j\neq 0$ for every $1\le j\le N+m$ because $a_{-m}\neq 0$), it follows that T_ψ is hyponormal if and only if for every zero ζ of $z^m\psi$, the number $1/\overline{\zeta}$ is a zero of $z^m\psi$ of multiplicity equal to the multiplicity of ζ. This completes the proof. □

Corollary 9. *If $\varphi(z)=\sum_{n=-N}^{N} a_n z^n$ then the following statements are equivalent.*

(i) *T_φ is a normal operator.*

(ii) *For every zero ζ of $z^N(\varphi - a_0)$, the number $1/\overline{\zeta}$ is a zero of $z^N(\varphi - a_0)$ of multiplicity equal to the multiplicity of ζ.*

Thus in particular if $\varphi(z)=\sum_{n=-N}^{N} a_n z^n$, where $a_{-N}=\overline{a_N}$ and a_0 is real (e.g., the Fourier coefficients of φ are real) then T_φ is a normal operator if and only if for any zero ζ of $z^N\varphi$, the number $1/\overline{\zeta}$ is a zero of $z^N\varphi$ of multiplicity equal to the multiplicity of ζ.

Proof. From Lemma 2(v), we have that T_φ is normal if and only if $|a_{-N}|=|a_N|$ and the Fourier coefficients of φ satisfy the symmetry condition (8.1) with $m=N$. Note that the condition (ii) implies the condition "$|a_{-N}|=|a_N|$". Therefore the first assertion immediately follows from Theorem 8. For the second assertion observe that if $a_{-N}=\overline{a_N}$ then we can see from the proof of Theorem 8 that the Blaschke product k is in $\mathcal{E}(\varphi)$ if and only if $k=1$. Therefore if $k\in\mathcal{E}(\varphi)$ and a_0 is real then $0=(\varphi-a_0)-k\,(\overline{\varphi}-\overline{a_0})=\varphi-\overline{\varphi}$, i.e., $\varphi=\overline{\varphi}$ and therefore T_φ is normal if and only if for every zero ζ of $z^N\varphi$, the number $1/\overline{\zeta}$ is a zero of $z^N\varphi$ of multiplicity equal to the multiplicity of ζ. □

Example 10. Consider the following trigonometric polynomials:

$$\varphi_1(z)=z^{-4}(z-1)^5(z-2)(z-\frac{1}{2})^2 \quad\text{and}\quad \varphi_2(z)=z^{-4}(z-1)^5(z-2)(z-\frac{1}{10})^2.$$

In [**5**, Remark 1.2], it was shown that if $\varphi(z)=\sum_{n=-m}^{N} a_n z^n$ and if $|a_N|$ is sufficiently large in comparison with other coefficients, then T_φ is hyponormal. Thus intuition suggests that φ_1 is less likely than φ_2 to induce a hyponormal Toeplitz operator, as the modulus of the "co-analytic" outer coefficient of φ_1 is greater than that of φ_2. However the opposite is true: Theorem 5 shows that T_{φ_1} is hyponormal whereas T_{φ_2} is not.

Example 11. Consider the following trigonometric polynomial:

$$\varphi(z)=z^{-3}(z-2)^2(z-\frac{1}{2})^2(z-\alpha)(z-\beta).$$

Theorem 5 shows that T_φ is hyponormal if and only if $\overline{\alpha}\beta=1$. Also Corollary 9 shows that if $\alpha,\beta\in\mathbb{R}$ then hyponormality and normality coincide for T_φ.

In view of the preceding results, one might guess that if $\varphi(z)=\sum_{n=-m}^{N} a_n z^n$ is such that T_φ is hyponormal, then the number of zeros of $z^m\varphi$ in the open unit disk $\mathbb{D}$ is greater than or equal to the number of zeros of $z^m\varphi$ outside $\mathbb{D}$. In the sequel we provide an example which shows that this guess is wrong (see Example 16 below). For this we give a necessary condition for hyponormality of T_φ with polynomial symbol φ of the form $\varphi(z)=\sum_{n=-m}^{N} a_n z^n$, in terms of zeros of the analytic polynomial $z^m\varphi$; in fact, with arbitrary trigonometric polynomials, the known necessary conditions for practical use are only the statements (i) and (ii) in Lemma 2. To do this we review here Schur's algorithm, due to K. Zhu [**14**], determining hyponormality for Toeplitz operators with polynomial symbols.

Suppose that $k(z)=\sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$. If $k_0=k$, define by induction a sequence $\{k_n\}$ of functions in the closed unit ball of $H^\infty(\mathbb{T})$ as follows:

$$k_{n+1}(z)=\frac{k_n(z)-k_n(0)}{z\left(1-\overline{k_n(0)}\,k_n(z)\right)},\quad |z|<1,\ n=0,1,2,\cdots.$$

We write

$$k_n(0) = \Phi_n(c_0, \cdots, c_n), \quad n = 0, 1, 2, \cdots,$$

where Φ_n is a function of $n+1$ complex variables. We call the Φ_n's *Schur's functions*. Then Zhu's theorem can be written as follows: if $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if

$$\text{(11.1)} \qquad \begin{pmatrix} \overline{c_0} \\ \overline{c_1} \\ \vdots \\ \overline{c_{N-1}} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \dots & a_{N-1} & a_N \\ a_2 & a_3 & \dots & a_N & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_N & 0 & \dots & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \overline{a_{-1}} \\ \overline{a_{-2}} \\ \vdots \\ \overline{a_{-N}} \end{pmatrix},$$

then T_φ is hyponormal if and only if $|\Phi_n(c_0, \cdots, c_n)| \leq 1$ for every $n = 0, 1, \cdots, N-1$.

If $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is a function in H^∞ such that $\varphi - k\overline{\varphi} \in H^\infty$, then $c_0, \cdots, c_{N-1}$ are just the values given in (11.1). Thus Zhu's theorem shows that if $k(z) = \sum_{j=0}^{\infty} c_j z^j$ satisfies $\varphi - k\overline{\varphi} \in H^\infty$, then the hyponormality of T_φ is determined by the values of c_j's for $0 \leq j \leq N-1$. On the other hand, Zhu's theorem can be reformulated as follows:

Lemma 12 (Zhu's Theorem). *If $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where $m \leq N$ and $a_N \neq 0$, then T_φ is hyponormal if and only if $|\Phi_n(c_0, \cdots, c_n)| \leq 1$ for every $n = 0, 1, \cdots, N-1$, where the c_n are given by the following recurrence relation:*

$$\text{(12.1)} \qquad \begin{cases} c_0 = c_1 = \cdots = c_{N-m-1} = 0 \\ c_{N-m} = \frac{a_{-m}}{\overline{a_N}} \\ c_n = (\overline{a_N})^{-1}\big(a_{-N+n} - \sum_{j=N-m}^{n-1} c_j \overline{a_{N-n+j}}\big) \quad \textit{for } n = N-m+1, \cdots, N-1. \end{cases}$$

Proof. See [**10**, Proposition 1]. □

We also recall:

Lemma 13 ([10, Proposition 3]). *Suppose that $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$ and that $\{\Phi_n\}$ is a sequence of Schur's functions associated with $\{c_n\}$. If $c_1 = \cdots = c_{n-1} = 0$ and $c_n \neq 0$, then we have that $\Phi_0 = c_0$, $\Phi_1 = \cdots = \Phi_{n-1} = 0$,*

$$\Phi_n = \frac{c_n}{1 - |c_0|^2} \quad \textit{and} \quad \Phi_{n+1} = \frac{c_{n+1}}{\big(1 - |c_0|^2\big)\big(1 - |\Phi_n|^2\big)}.$$

We now get a condition that φ must necessarily satisfy in order for T_φ to be a hyponormal operator.

Theorem 14 (A Necessary Condition for Hyponormality). *Suppose that $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero. If $z^m \varphi = a_N \prod_{j=1}^{m+N} (z - \zeta_j)$ then*

$$\text{(14.1)} \qquad T_\varphi \textit{ is hyponormal} \implies \left| \sum_{j=1}^{m+N} \big(\zeta_j - 1/\overline{\zeta_j}\big) \right| \leq \frac{1}{\prod_{j=1}^{m+N} |\zeta_j|} - \prod_{j=1}^{m+N} |\zeta_j|.$$

Proof. Observe that $\zeta_j \neq 0$ for every $1 \le j \le m+N$, and that

$$\frac{a_{N-1}}{a_N} = -\sum_{j=1}^{m+N} \zeta_j;$$
$$\frac{a_{-m}}{a_N} = (-1)^{m+N} \prod_{j=1}^{m+N} \zeta_j;$$
$$\frac{a_{-m+1}}{a_N} = (-1)^{m+N-1} \prod_{j=1}^{m+N} \zeta_j \cdot \sum_{j=1}^{m+N} \frac{1}{\zeta_j}.$$

By the recurrence relation (12.1) we have

$$c_0 = c_1 = \cdots = c_{N-m-1} = 0 \quad \text{and} \quad |c_{N-m}| = \left|\frac{a_{-m}}{\overline{a_N}}\right| = \prod_{j=1}^{m+N} |\zeta_j|, \tag{14.2}$$

and if we write $\overline{a_N} = e^{i\theta} a_N$ for some $\theta \in [0, 2\pi)$ then

$$\begin{aligned}
|c_{N-m+1}| &= \left|\left(\overline{a_N}\right)^{-1}\left(a_{-m+1} - c_{N-m}\overline{a_{N-1}}\right)\right| \\
&= \left|e^{-i\theta}\frac{a_{-m+1}}{a_N} - e^{-i\theta}\frac{a_{-m}}{a_N}\overline{\left(\frac{a_{N-1}}{a_N}\right)}\right| \\
&= \left|(-1)^{m+N-1}\prod_{j=1}^{m+N} \zeta_j \cdot \sum_{j=1}^{m+N}\frac{1}{\zeta_j} + (-1)^{m+N}\prod_{j=1}^{m+N}\zeta_j \cdot \sum_{j=1}^{m+N}\overline{\zeta_j}\right| \qquad (14.3) \\
&= \prod_{j=1}^{m+N} |\zeta_j| \left|\sum_{j=1}^{m+N} \left(\zeta_j - 1/\overline{\zeta_j}\right)\right|.
\end{aligned}$$

By Lemma 13, we also have

$$\Phi_0 = \cdots = \Phi_{N-m-1} = 0, \quad \Phi_{N-m} = c_{N-m}, \quad \text{and} \quad \Phi_{N-m+1} = \frac{c_{N-m+1}}{1 - |\Phi_{N-m}|^2}.$$

Therefore if T_φ is hyponormal then it follows from Lemma 12 that $|c_{N-m+1}| \le 1 - |c_{N-m}|^2$, which together with (14.2) and (14.3) implies

$$\left|\sum_{j=1}^{m+N}\left(\zeta_j - 1/\overline{\zeta_j}\right)\right| \le \frac{1}{\prod_{j=1}^{m+N}|\zeta_j|} - \prod_{j=1}^{m+N}|\zeta_j|.$$

□

If $m = 2$ in Theorem 14 then the implication (14.1) is reversible.

Corollary 15. *If $z^2\varphi = \prod_{j=1}^{N}(z - \zeta_j)$, where $N \ge 4$ and $\zeta_j \neq 0$ for every $1 \le j \le N$, then*

$$T_\varphi \text{ is hyponormal} \iff \left|\sum_{j=1}^{N}\left(\zeta_j - 1/\overline{\zeta_j}\right)\right| \le \frac{1}{\prod_{j=1}^{N}|\zeta_j|} - \prod_{j=1}^{N}|\zeta_j|. \tag{15.1}$$

Proof. Write $\varphi(z)=\sum_{n=-2}^{N-2} a_n z^n$. Then

$$a_{N-2}=1,\quad a_{N-3}=-\sum_{j=1}^{N}\zeta_j,\quad a_{-2}=(-1)^N\prod_{j=1}^{N}\zeta_j,\quad\text{and}\quad a_{-1}=(-1)^{N-1}\prod_{j=1}^{N}\zeta_j\cdot\sum_{j=1}^{N}\frac{1}{\zeta_j}.$$

Then by the recurrence relation (12.1),

$$\begin{aligned} c_0&=c_1=\cdots=c_{N-5}=0;\\ c_{N-4}&=a_{-2}=(-1)^N\prod_{j=1}^{N}\zeta_j;\\ c_{N-3}&=a_{-1}-c_{N-4}\overline{a_{N-3}}=(-1)^{N-1}\prod_{j=1}^{N}\zeta_j\cdot\sum_{j=1}^{N}\big(1/\zeta_j-\overline{\zeta_j}\big). \end{aligned}$$

On the other hand, by Lemma 13,

$$\Phi_0=\cdots=\Phi_{N-5}=0,\quad \Phi_{N-4}=c_{N-4},\quad\text{and}\quad \Phi_{N-3}=\frac{c_{N-3}}{1-|\Phi_{N-4}|^2}.$$

Since by Lemma 12, T_φ is hyponormal if and only if $|\Phi_n|\le 1$ for every $n=0,1,\cdots,N-3$, it follows that

$$T_\varphi \text{ is hyponormal } \iff |c_{N-4}|\le 1\ \text{ and }\ |c_{N-3}|\le 1-|c_{N-4}|^2,$$

which gives (15.1). □

Example 16. Consider the following trigonometric polynomial:

$$\varphi(z)=z^{-2}\left(z-\frac{4}{5}\right)\left(z-\frac{9}{10}\right)\left(z-\frac{101}{100}\right)\left(z-\frac{102}{100}\right)\left(z-\frac{103}{100}\right).$$

Applying Corollary 15 gives that T_φ is hyponormal. Thus this example shows that when $\varphi(z)=\sum_{n=-m}^{N} a_n z^n$ is such that T_φ is hyponormal, the number of zeros of $z^m\varphi$ in $\mathbb{D}$ need not be greater than or equal to the number of zeros of $z^m\varphi$ outside $\mathbb{D}$.

On the other hand we need not expect that the implication (14.1) is reversible for arbitrary trigonometric polynomials. For example if

$$\varphi(z)=z^{-4}(z-1)^5\left(z-\frac{1}{2}\right)\left(z-\frac{4}{5}\right)\left(z-\frac{10}{9}\right),$$

then by Theorem 5, T_φ is not hyponormal, while the inequality in (14.1) is satisfied.

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