The spectrum is continuous on the set of p–hyponormal operators

In Sung Hwang and Woo Young Lee[1]

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
(e-mail:wylee@yurim.skku.ac.kr)

Mathematics Subject Classification (1991): 47A10, 47B20

Abstract. In this paper it is shown that the spectrum σ, a set valued function, is continuous when the function is restricted to the set of all p-hyponormal operators on a Hilbert space.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if

$$(T^*T)^p - (TT^*)^p \geq 0.$$

If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal. It is well-known that q-hyponormal operators are p-hyponormal for $p \leq q$. Throughout this paper we assume $0 < p < 1$. The notion of semi-hyponormal operators was introduced by D. Xia [12] and the notion of p-hyponormal operators was introduced by A. Aluthge [1]. The p-hyponormal operators have been studied by many authors (cf. [1],[3],[4],[5]). Let $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ be a polar decomposition, where U is a partial isometry with initial and final spaces $\text{cl}(\text{ran}\, T^*)$ and $\text{cl}(\text{ran}\, T)$, respectively. Note that if $T \in \mathcal{L}(\mathcal{H})$ then $\ker T = \ker |T|^\alpha$ for every $\alpha > 0$. Thus if $T = U|T|$ is a p-hyponormal operator then $\ker(|T|^{2p}) \subseteq \ker(|T^*|^{2p})$, so that $\ker T \subseteq \ker T^*$, which implies $\text{cl}(\text{ran}\, T) \subseteq \text{cl}(\text{ran}\, T^*)$. Thus, in the polar decomposition $T = U|T|$, the operator U can be extended to an isometry from $\mathcal{H}$ to $\mathcal{H}$. Thus for a p-hyponormal operator $T = U|T|$, we may assume that the operator U is an isometry. The following are the basic properties of a p-hyponormal operator T:

(i) ([3]) T is *normaloid*, i.e., $r(T) = ||T||$, where $r(T)$ denotes the spectral radius of T.
(ii) ([1]) If T is invertible, then T^{-1} is also p-hyponormal.

[1]Supported in part by the KOSEF through the GARC at Seoul National University and the KOSEF 971-0102-010-2.

Let $\mathfrak{S}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. If $\mathfrak{A}$ is a unital Banach algebra then the spectrum can be viewed as a function $\sigma : \mathfrak{A} \to \mathfrak{S}$, mapping each $T \in \mathfrak{A}$ to its spectrum $\sigma(T)$. It is well-known that the function σ is upper semicontinuous and that in noncommutative algebras, σ does have points of discontinuity. The work of J. Newburgh [11] contains the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [7] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^*-algebra of all operators acting on a complex separable Hilbert space. Of interest is the identification of points of spectral continuity, as in [7], and of classes $\mathfrak{C}$ of operators for which σ becomes continuous when restricted to $\mathfrak{C}$. Recently in [8] and [10], the continuity of the spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators. The set of normal operators is perhaps the most immediate in the latter direction: σ is continuous on the set of normal operators. As noted in Solution 104 of [9], Newburgh's argument uses the fact that the inverses of normal resolvents are normaloid. This argument can be easily extended to the set of hyponormal operators because the inverses of hyponormal resolvents are also hyponormal and hence normaloid. Although p-hyponormal operators are normaloid, p-hyponormality is not translation-invariant and hence their inverses of p-hyponormal resolvents need not, in general, be normaloid. Thus the arguments of Newburgh cannot apply to show that σ is continuous when restricted to the set of p-hyponormal operators. The purpose of the present paper is to show that σ is continuous when restricted to the set of p-hyponormal operators.

Theorem. *The spectrum σ is continuous on the set of all p-hyponormal operators.*

If $\{T_n\}$ is a sequence of elements in a unital Banach algebra $\mathfrak{A}$, then $\liminf_n \sigma(T_n)$ is the set of all limit points of convergent sequences of the form $\{\lambda_n\}$, where $\lambda_n \in \sigma(T_n)$ for each n. Because the set of invertible elements in $\mathfrak{A}$ is open, we conclude that $\liminf_n \sigma(T_n) \subseteq \sigma(T)$ whenever the sequence of elements T_n converges to T in $\mathfrak{A}$. Therefore proving the above theorem is to show equality in this relation.

2. Lemmas

If $T \in \mathcal{L}(\mathcal{H})$, write $\rho(T)$ for the resolvent of T. Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be *bounded below* if there exists $k > 0$ for which $||x|| \leq k\, ||Tx||$ for each $x \in \mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, write

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \ T - \lambda \text{ is not bounded below}\}.$$

It is well-known that $\partial\, \sigma(T) \subseteq \sigma_{ap}(T)$, where $\partial(\cdot)$ denotes the topological boundary. To prove the theorem we need several lemmas.

Lemma 1. *Suppose that $T, T_n \in \mathcal{L}(\mathcal{H})$, for $n \in \mathbb{Z}^+$, are positive operators. If T_n converges to T then T_n^p converges to T^p.*

Proof. Straightforward from the fact ([2] Corollary 2) that for any $A, B \geq 0$,

$$||A^p - B^p|| \leq ||\, |A - B|\, ||^p \quad \text{for } 0 < p \leq 1.$$

□

Lemma 2. *Let $\lambda \in \mathbb{C}$ and suppose that $T, T_n \in \mathcal{L}(\mathcal{H})$, for $n \in \mathbb{Z}^+$, are such that $(T_n - \lambda)^{-1}$ is normaloid whenever $\lambda \in \rho(T_n)$. If T_n converges strongly to T then*

$$\lambda \in \sigma_{ap}(T) \implies \lambda \in \liminf \sigma(T_n). \tag{2.1}$$

Furthermore if T_n converges to T then (2.1) implies

$$\lambda \in \sigma(T) \implies \lambda \in \liminf \sigma(T_n). \tag{2.2}$$

Proof. For the implication (2.1) assume that $\lambda \notin \liminf \sigma(T_n)$. Then there exists $\epsilon > 0$ such that $\text{dist}\,(\lambda,\ \sigma(T_n)) \geq \epsilon$ for infinitely many values of n. Thus we can find a subsequence $\{T_{n_k}\}_k$ of $\{T_n\}$ such that $\text{dist}\,(\lambda,\ \sigma(T_{n_k})) \geq \epsilon$ for all $k \in \mathbb{Z}^+$. There is no loss in simplifying the notation and assuming that $\text{dist}\,(\lambda,\ \sigma(T_n)) \geq \epsilon$ for all $n \in \mathbb{Z}^+$. If $S \in \mathcal{L}(\mathcal{H})$, write $\gamma(S)$ for the *reduced minimum modulus* of S: i.e.,

$$\gamma(S) = \inf_{x \in \mathcal{H}} \frac{||Sx||}{\text{dist}\,(x,\ \ker S)},$$

where $\frac{0}{0}$ is defined to be ∞. Since by assumption $(T_n - \lambda)^{-1}$ is normaloid we have

$$\gamma(T_n - \lambda) = \frac{1}{||(T_n - \lambda)^{-1}||} = \frac{1}{\max_{\mu \in \sigma(T_n - \lambda)^{-1}} |\mu|} = \min_{\mu \in \sigma(T_n - \lambda)} |\mu| = \text{dist}\,(\lambda,\ \sigma(T_n)) \geq \epsilon. \tag{2.3}$$

Assume to the contrary that $T - \lambda$ is not bounded below. Then there exists a sequence $\{x_j\}_j$ of unit vectors in $\mathcal{H}$ such that $||(T - \lambda)x_j|| \to 0$. Since $\gamma(T_n - \lambda) = \inf_{||x||=1} ||(T_n - \lambda)x||$, it follows from (2.3) that $||(T_n - \lambda)x_j|| \geq \epsilon$ for all n, j. Since by assumption, T_n converges strongly to T it follows

$$||(T_n - \lambda)x_j|| \leq ||(T_n - T)x_j|| + ||(T - \lambda)x_j|| \xrightarrow{n,j \to \infty} 0,$$

giving a contradiction. This proves the implication (2.1).

For the implication (2.2), if $\lambda \notin \liminf \sigma(T_n)$ then there exists a sequence $\{T_{n_k}\}_k$ such that $T_{n_k} - \lambda$ is invertible for every $k \in \mathbb{Z}^+$ and $T_{n_k} - \lambda$ converges to $T - \lambda$ as $k \to \infty$. Since by (2.1), $T - \lambda$ is bounded below and hence semi-Fredholm, it follows from the continuity of the (semi-Fredholm) index that $\text{ind}\,(T - \lambda) = 0$, which forces $T - \lambda$ to be invertible. This proves the implication (2.2). □

Corollary 3. *If $T, T_n \in \mathcal{L}(\mathcal{H})$, for $n \in \mathbb{Z}^+$, are hyponormal operators and if T_n converges strongly to T then*

$$\sigma_{ap}(T) \subseteq \liminf \sigma(T_n).$$

Proof. This follows from Lemma 2 together with the fact that if T_n is hyponormal and $\lambda \in \rho(T_n)$ then $(T_n - \lambda)^{-1}$ is also hyponormal and hence normaloid. □

Corollary 4. *If $T, T_n \in \mathcal{L}(\mathcal{H})$, for $n \in \mathbb{Z}^+$, are p-hyponormal operators and if T_n converges to T then*

$$0 \in \sigma(T) \implies 0 \in \liminf \sigma(T_n).$$

Proof. Apply Lemma 2 with $\lambda = 0$. □

Let τ be the set of all strictly monotone increasing continuous nonnegative functions on $[0, \infty)$. Let $\tau_0 = \{\psi \in \tau : \psi(0) = 0\}$. Let $T = U|T|$ and U be unitary. For $\psi \in \tau_0$, the mapping $\tilde{\psi}$ is defined by

$$\tilde{\psi}(re^{i\theta}) = e^{i\theta}\psi(r) \quad \text{and} \quad \tilde{\psi}(T) = U\psi(|T|).$$

Then $\sigma(T)$ and $\sigma_{ap}(T)$ obey the spectral mapping theorem for $\tilde{\psi}$:

Lemma 5 ([5] Theorem 3). *Let $T = U|T|$ be p-hyponormal and U be unitary. If $\psi \in \tau_0$ and $\tilde{\psi}(T)$ is p-hyponormal then*

$$\tilde{\psi}(\omega(T)) = \omega(\tilde{\psi}(T)), \quad \textit{where } \omega = \sigma, \sigma_{ap}.$$

Lemma 6 ([12] Lemma II.3.5; [5] Theorem B). *Let $T = U|T|$ be p-hyponormal. Then there exists a Hilbert space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$ such that U extends to a unitary operator $\widetilde{U}$ in $\widetilde{\mathcal{H}}$ and $|T|$ extends to a positive operator $|\widetilde{T}|$ on $\widetilde{\mathcal{H}}$. Also T extends to a p-hyponormal operator $\widetilde{T} = \widetilde{U}|\widetilde{T}|$ satisfying*

$$\sigma(T) \subseteq \sigma(\widetilde{T}) \subseteq \sigma(T) \cup \{0\}.$$

It is instructive to write $\widetilde{U}, \widetilde{T}$ in the matrix form with respect to the decomposition $\widetilde{\mathcal{H}} = \mathcal{H} \oplus (\widetilde{\mathcal{H}} \ominus \mathcal{H})$:

$$\widetilde{U} = \begin{pmatrix} U & * \\ 0 & * \end{pmatrix} \quad \text{and} \quad \widetilde{T} = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}.$$

In fact the *von Neumann-Wold decomposition of an isometry* shows that if U is any isometry on $\mathcal{H}$ then U extends to a unitary operator $\widetilde{U}$ on $\mathcal{H} \oplus \mathcal{H}$ (cf. [6] p.14): more concretely, if U is an isometry on $\mathcal{H}$, then we can write

$$U = \begin{pmatrix} V & 0 \\ 0 & S_\alpha \end{pmatrix} : \mathcal{H} \longrightarrow \mathcal{H},$$

where V is a unitary operator acting on $\bigcap_{n=1}^{\infty} U^n\mathcal{H}$ and S_α is a unilateral shift with multiplicity α, and thus one can get

$$\widetilde{U} = \begin{pmatrix} V & 0 & 0 & 0 \\ 0 & S_\alpha & 1 - S_\alpha S_\alpha^* & 0 \\ 0 & 0 & S_\alpha^* & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H},$$

where 1 in the lower-right corner acts on $\bigcap_{n=1}^{\infty} U^n\mathcal{H}$.

3. Proof of the theorem

Suppose that $T = U|T|$ and $T_n = U_n|T_n|$, for $n \in \mathbb{Z}^+$, are p-hyponormal operators such that T_n converges to T. It will suffice to show that $\sigma(T) \subseteq \liminf \sigma(T_n)$. We first claim that

(3.1) $\quad U|T|^p$ and $U_n|T_n|^p$, for $n \in \mathbb{Z}^+$, are hyponormal;

(3.2) $\quad U_n|T_n|^p$ converges strongly to $U|T|^p$.

For (3.1) observe

$$(U|T|^p)^*(U|T|^p) = |T|^{2p} = (T^*T)^p \geq (TT^*)^p = U|T|^{2p}U^* = (U|T|^p)(U|T|^p)^*,$$

which implies that $U|T|^p$ is hyponormal and similarly so is $U_n|T_n|^p$. This proves (3.1). For (3.2) observe

$$\ker|T| = \ker|T|^\alpha \quad \text{for every } \alpha > 0.$$

Thus we can decompose $\mathcal{H}$ as follows:

$$\mathcal{H} = \ker|T|^p \bigoplus \operatorname{cl}(\operatorname{ran}|T|^{1-p}).$$

If $y \in \ker|T|^p$ then it follows from Lemma 1 that

$$||(U_n|T_n|^p - U|T|^p)y|| = ||U_n|T_n|^p y|| \leq ||\,|T_n|^p y|| \xrightarrow{n\to\infty} 0. \tag{3.3}$$

If $w \in \operatorname{ran}|T|^{1-p}$, say $w = |T|^{1-p}v$, then

$$||(U_n|T_n|^p - U|T|^p)w|| = ||(U_n|T_n|^p - U|T|^p)|T|^{1-p}v|| = ||(U_n|T_n|^p|T|^{1-p} - U|T|)v||.$$

But since, by Lemma 1,

$$\begin{aligned} ||\,U_n|T_n|^p|T|^{1-p} - U|T|\,|| &\leq ||\,U_n|T_n|^p|T|^{1-p} - U_n|T_n| + U_n|T_n| - U|T|\,|| \\ &\leq ||\,|T_n|^p\,||\,||\,|T|^{1-p} - |T_n|^{1-p}|| + ||T_n - T|| \xrightarrow{n\to\infty} 0, \end{aligned}$$

it follows that

$$||(U_n|T_n|^p - U|T|^p)w|| \xrightarrow{n\to\infty} 0. \tag{3.4}$$

By (3.3) and (3.4) we have that $U_n|T_n|^p$ converges strongly to $U|T|^p$ on $\big(\ker|T|^p \oplus \operatorname{ran}|T|^{1-p}\big)$. But since $\big(\ker|T|^p \oplus \operatorname{ran}|T|^{1-p}\big)$ is dense in $\mathcal{H}$, it follows that $U_n|T_n|^p$ converges strongly to $U|T|^p$. This proves (3.2). By the remark below Lemma 6, we can find a common Hilbert space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$ such that U (U_n, resp.) extends to a unitary operator $\widetilde{U}$ ($\widetilde{U}_n$, resp.) in $\widetilde{\mathcal{H}}$ and $|T|$ ($|T_n|$, resp.) extends to a positive operator $|\widetilde{T}|$ ($|\widetilde{T}_n|$, resp.) on $\widetilde{\mathcal{H}}$ satisfying

$$\sigma(T) \subseteq \sigma(\widetilde{T}) \subseteq \sigma(T) \cup \{0\} \quad \text{and} \quad \sigma(T_n) \subseteq \sigma(\widetilde{T}_n) \subseteq \sigma(T_n) \cup \{0\} \quad \text{for all } n \in \mathbb{Z}^+.$$

Now define $\psi(r) := r^p$ on $[0, \infty)$. Since by (3.1), $\widetilde{U}|\widetilde{T}|^p$ and $\widetilde{U}_n|\widetilde{T}_n|^p$ are hyponormal it follows from Lemma 5 that

$$\sigma_{ap}(\widetilde{U}|\widetilde{T}|^p) = \tilde{\psi}(\sigma_{ap}(\widetilde{T})) \quad \text{and} \quad \sigma(\widetilde{U}_n|\widetilde{T}_n|^p) = \tilde{\psi}(\sigma(\widetilde{T}_n)).$$

Since by (3.2), $\widetilde{U}_n|\widetilde{T}_n|^p$ converges strongly to $\widetilde{U}|\widetilde{T}|^p$, it follows from Corollary 3 that

$$\sigma_{ap}(\widetilde{U}|\widetilde{T}|^p) \subseteq \liminf \sigma(\widetilde{U}_n|\widetilde{T}_n|^p),$$

so that

$$\tilde{\psi}(\sigma_{ap}(\widetilde{T})) \subseteq \liminf \tilde{\psi}(\sigma(\widetilde{T}_n)),$$

which implies

$$\sigma_{ap}(\widetilde{T}) \subseteq \liminf \sigma(\widetilde{T}_n).$$

Since $\widetilde{T}_n$ converges to $\widetilde{T}$, it follows from (2.2) that

$$\sigma(\widetilde{T}) \subseteq \liminf \sigma(\widetilde{T}_n).$$

We thus have

$$\sigma(T) \subseteq \sigma(\widetilde{T}) \subseteq \liminf \sigma(\widetilde{T}_n) \subseteq \liminf \sigma(T_n) \cup \{0\}.$$

If $0 \notin \sigma(T)$ then evidently $\sigma(T) \subseteq \liminf \sigma(T_n)$. If instead $0 \in \sigma(T)$ then by Corollary 4, $0 \in \liminf \sigma(T_n)$, so that $\sigma(T) \subseteq \liminf \sigma(T_n)$. This completes the proof. □

Acknowledgements. The authors wish to thank M. Chō for discussions concerning this work. They are also grateful to the referee for helpful comments concerning this paper.

References

1. A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Int. Eq. Op. Th. **13** (1990), 307–315
2. T. Ando, Comparison of norms $|||f(A) - f(B)|||$ and $|||f(|A - B|)|||$, Math. Z. **197** (1988), 403–409
3. M. Chō, T. Huruya, p-hyponormal operators $(0 < p < \frac{1}{2})$, Commentationes Math. **33** (1993), 23–29
4. M. Chō, Spectral properties of p-hyponormal operators, Glasgow Math. J. **36** (1994), 117–122
5. M. Chō, M. Itoh, Putnam's inequality for p-hyponormal operators, Proc. Amer. Math. Soc. **123** (1995), 2435–2440
6. J.B. Conway, The Theory of Subnormal Operators, Math. Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991
7. J.B. Conway, B.B. Morrel, Operators that are points of spectral continuity, Int. Eq. Op. Th. **2** (1979) 174–198
8. D.R. Farenick, W.Y. Lee, Hyponormality and spectra of Toeplitz operators, Trans. Amer. Math. Soc. **348** (1996), 4153–4174
9. P.R. Halmos, A Hilbert Space Problem Book, Springer-Verlag, New York, 1982
10. I.S. Hwang, W.Y. Lee, On the continuity of spectra of Toeplitz operators, Arch. Math. **70** (1998), 66-73
11. J.D. Newburgh, The variation of spectra, Duke Math. J. **18** (1951), 165–176
12. D. Xia, Spectral Theory of Hyponormal Operators, Birkhäuser Verlag, Boston, 1983