ON M-HYPONORMAL WEIGHTED SHIFTS

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Abstract. Let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence and let W_α denote the associated unilateral weighted shift on $\ell^2(\mathbb{Z}_+)$. In this paper we prove that if α is eventually increasing, then W_α is M-hyponormal and that if α has exactly two subsequential limits such that the larger one is different from the spectral radius of W_α then W_α is not M-hyponormal.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called normal if $T^*T = TT^*$ and hyponormal if $T^*T \geq TT^*$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *M-hyponormal* if there exists $M > 0$ such that

$$||(T-\lambda)^*x|| \leq M\,||(T-\lambda)x|| \quad \text{for all } \lambda \in \mathbb{C} \text{ and for all } x \in \mathcal{H}.$$

If $M \leq 1$ then M-hyponormality implies hyponormality. The notion of an M-hyponormal operator is due to J. Stampfli (unpublished) (see [11]). The class of M-hyponormal operators has been studied by many authors (cf.[1–3],[5–12]). However examples of M-hyponormal non-hyponormal operators seem to be scarce from the literature. The aim of the present article is to give abundant examples of M-hyponormal non-hyponormal operators. Our strategy involves the unilateral weighted shifts.

2. Results

Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$, i.e., α is monononically increasing. B. Wadha [11] gave an example of an M-hyponormal non-hyponormal weighted shift W_α of the form:

$$W_\alpha = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 2 & 0 & & & \\ & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}.$$

On the other hand, M. Radjabalipour [8] showed that the only quasinilpotent M-hyponormal operator is 0. Thus if W_α is a weighted shift with weight sequence $\{\alpha_n\}$ converging to 0 then W_α is not M-hyponormal. In this paper we consider the question: *Which weighted shifts are M-hyponormal?* Our main theorem now follows:

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Theorem 1. *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$. If α is eventually increasing then T is M-hyponormal.*

Proof. For any $x = \sum_{n=0}^\infty x_n e_n \in \ell^2$ and for any $\lambda \in \mathbb{C}$,

$$(T-\lambda)x = -\lambda x_0 e_0 + \sum_{i=0}^\infty (\alpha_i x_i - \lambda x_{i+1}) e_{i+1}$$

and

$$(T^* - \bar{\lambda})x = \sum_{i=0}^\infty (\alpha_i x_{i+1} - \bar{\lambda} x_i) e_i.$$

Thus a straightforward calculation shows that

$$\begin{aligned}||(T-\lambda)x||^2 - ||(T^*-\bar{\lambda})x||^2 &= |\lambda x_0|^2 + \sum_{i=0}^\infty |\alpha_i x_i - \lambda x_{i+1}|^2 - \sum_{i=0}^\infty |\alpha_i x_{i+1} - \bar{\lambda} x_i|^2 \\ &= |\alpha_0 x_0|^2 + \sum_{i=1}^\infty (\alpha_i^2 - \alpha_{i-1}^2)\, |x_i|^2.\end{aligned}$$

Suppose that $\alpha = \{\alpha_n\}_{n=0}^\infty$ is monotonically increasing for $n \geq k$. We now claim that if $c := \min\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$, then

$$||(T-\lambda)x|| \;\geq\; (c - |\lambda|)\, ||x|| \quad \text{for any } \lambda \in \mathbb{C}:$$

indeed we have that

$$\begin{aligned}||(T-\lambda)x|| \geq ||Tx|| - |\lambda|\, ||x|| &= \left(\sum_{i=0}^\infty \alpha_i^2\, |x_i|^2\right)^{\frac{1}{2}} - |\lambda|\, ||x|| \\ &\geq \left(\sum_{i=0}^\infty c^2\, |x_i|^2\right)^{\frac{1}{2}} - |\lambda|\, ||x|| = (c - |\lambda|)\, ||x||.\end{aligned}$$

Thus we have that

$$||(T-\lambda)x|| \;\geq\; \frac{c}{2}\, ||x|| \quad \text{for } |\lambda| < \frac{c}{2}. \tag{1.1}$$

Write $J := \{j \geq 1 : \; \alpha_j < \alpha_{j-1}\}$. If $J = \emptyset$ then T must be hyponormal. Evidently, $J \subseteq \{1, \cdots, k\}$. Write $m := \sharp(J)$. We argue that

$$||(T-\lambda)x||^2 \geq \left[\sum_{l=0}^{j-1} \left(\prod_{i=l}^{j-1} \alpha_i^2 |\lambda|^{2l}\right) + |\lambda|^{2j}\right]^{-1} |\lambda|^{2(j+1)} |x_j|^2 \quad (j \geq 1). \tag{1.2}$$

Towards (1.2) observe that

$$||(T-\lambda)x||^2 = |\lambda x_0|^2 + \sum_{i=0}^\infty |\alpha_i x_i - \lambda x_{i+1}|^2,$$

and so it suffices to show that

$$|\lambda x_0|^2 + \sum_{i=0}^{j-1} |\alpha_i x_i - \lambda x_{i+1}|^2 \;\geq\; K_j^{-1} |\lambda|^{2(j+1)} |x_j|^2 \quad (j \geq 1), \tag{1.3}$$

where

$$K_j := \sum_{l=0}^{j-1} \left(\prod_{i=l}^{j-1} \alpha_i^2 |\lambda|^{2l} \right) + |\lambda|^{2j}$$

for each $j = 1, 2, \cdots$. We use an induction on j. First, observe that

$$\begin{aligned}
|\lambda x_0|^2 + |\alpha_0 x_0 - \lambda x_1|^2 &= |\lambda|^2 |x_0|^2 + \alpha_0^2 |x_0|^2 - \bar{\lambda} \bar{x}_1 \alpha_0 x_0 - \lambda x_1 \alpha_0 \bar{x}_0 + |\lambda|^2 |x_1|^2 \\
&= \left| \sqrt{|\lambda|^2 + \alpha_0^2}\, x_0 - \frac{\alpha_0 \lambda\, x_1}{\sqrt{|\lambda|^2 + \alpha_0^2}} \right|^2 - \frac{\alpha_0^2 |\lambda|^2}{|\lambda|^2 + \alpha_0^2} |x_1|^2 + |\lambda|^2 |x_1|^2 \\
&\geq \frac{|\lambda|^4}{|\lambda|^2 + \alpha_0^2}\, |x_1|^2 \\
&= \frac{|\lambda|^4}{K_1}\, |x_1|^2,
\end{aligned}$$

which shows that (1.3) holds for $j = 1$. We now suppose that (1.3) holds for $j = n$. Then a straightforward calculation shows that

$$\begin{aligned}
&|\lambda x_0|^2 + \sum_{i=0}^{n} |\alpha_i x_i - \lambda x_{i+1}|^2 \\
\geq& K_n^{-1} |\lambda|^{2(n+1)} |x_n|^2 + |\alpha_n x_n - \lambda x_{n+1}|^2 \\
=& \left(K_n^{-1} |\lambda|^{2(n+1)} + \alpha_n^2 \right) |x_n|^2 - \alpha_n x_n \bar{\lambda} \bar{x}_{n+1} - \alpha_n \bar{x}_n \lambda x_{n+1} + |\lambda|^2 \, |x_{n+1}|^2 \\
=& \left| \sqrt{K_n^{-1} |\lambda|^{2(n+1)} + \alpha_n^2}\; x_n - \frac{\alpha_n \lambda\, x_{n+1}}{\sqrt{K_n^{-1} |\lambda|^{2(n+1)} + \alpha_n^2}} \right|^2 - \frac{\alpha_n^2 |\lambda|^2}{K_n^{-1} |\lambda|^{2(n+1)} + \alpha_n^2}\; |x_{n+1}|^2 + |\lambda|^2 |x_{n+1}|^2 \\
\geq& \frac{K_n^{-1} |\lambda|^{2(n+1)} |\lambda|^2}{K_n^{-1} |\lambda|^{2(n+1)} + \alpha_n^2}\; |x_{n+1}|^2 \\
=& \frac{|\lambda|^{2(n+2)}}{\sum_{l=0}^{n} \left(\prod_{i=l}^{n} \alpha_i^2 |\lambda|^{2l} \right) + |\lambda|^{2(n+1)}}\; |x_{n+1}|^2 \\
&= K_{n+1}^{-1} |\lambda|^{2(n+2)} |x_{n+1}|^2,
\end{aligned}$$

which shows that (1.3) holds for $j = n + 1$. This proves (1.3). On the other hand, for $|\lambda| \geq \frac{c}{2}$ and for each $j \in J$ we can find a constant $\gamma_j > 0$ for which

$$\left[\sum_{l=0}^{j-1} \left(\prod_{i=l}^{j-1} \alpha_i^2 |\lambda|^{2l} \right) + |\lambda|^{2j} \right]^{-1} |\lambda|^{2(j+1)} \;\geq\; \gamma_j^2, \tag{1.4}$$

where γ_j is independent of λ with $|\lambda| \geq \frac{c}{2}$.

It thus follows from (1.2) and (1.4) that if $|\lambda| \geq \frac{c}{2}$ then

$$||(T - \lambda)x||^2 \;\geq\; \gamma_j^2\, |x_j|^2 \quad \text{for each } j \in J.$$

Thus if $|\lambda| \geq \frac{c}{2}$ then

$$||(T - \lambda)x||^2 \geq \frac{\gamma^2}{m} \sum_{j \in J} |x_j|^2, \quad \text{where } \gamma = \min_{j \in J} \gamma_j. \tag{1.5}$$

Write $d := \max_{j\in J}\left\{\alpha_{j-1}^2-\alpha_j^2\right\}$ and put

$$M^2 := \max\left\{\frac{md}{\gamma^2},\ \frac{4d}{c^2}\right\}+1.$$

Then we claim that

$$(M^2-1)\,||(T-\lambda)x||^2 \ \geq\ d\sum_{j\in J}|x_j|^2 \quad \text{for all } \lambda\in\mathbb{C}:$$

indeed if $|\lambda|\geq\frac{c}{2}$ then by (1.5),

$$(M^2-1)\,||(T-\lambda)x||^2 \geq \frac{md}{\gamma^2}\,||(T-\lambda)x||^2 \geq \frac{md}{\gamma^2}\cdot\frac{\gamma^2}{m}\sum_{j\in J}|x_j|^2 = d\sum_{j\in J}|x_j|^2$$

and if instead $|\lambda|<\frac{c}{2}$ then by (1.1),

$$(M^2-1)\,||(T-\lambda)x||^2 \geq \frac{4d}{c^2}\,||(T-\lambda)x||^2 \geq d\,||x||^2 \geq d\sum_{j\in J}|x_j|^2.$$

Therefore we have that

$$\begin{aligned}
&M^2\,||(T-\lambda)x||^2-||(T^*-\bar{\lambda})x||^2\\
=&(M^2-1)\,||(T-\lambda)x||^2+||(T-\lambda)x||^2-||(T^*-\bar{\lambda})x||^2\\
\geq& d\sum_{j\in J}|x_j|^2+|\alpha_0x_0|^2+\sum_{i=1}^{\infty}\left(\alpha_i^2-\alpha_{i-1}^2\right)\,|x_i|^2\\
\geq& |\alpha_0x_0|^2+\sum_{i\in\mathbb{N}\setminus J}\left(\alpha_i^2-\alpha_{i-1}^2\right)\,|x_i|^2\geq 0.
\end{aligned}$$

This completes the proof. □

We were unable to decide whether or not the converse of Theorem 1 is true. However we conjecture that it is:

Conjecture 2. Let W_α be a weighted shift with weight sequence $\alpha=\{\alpha_n\}_{n=0}^{\infty}$. Then W_α is M-hyponormal if and only if α is eventually increasing.

We now provide evidence for the validity of the conjecture.

Theorem 3. *Let $T\equiv W_\alpha$ be a weighted shift with weight sequence $\alpha=\{\alpha_n\}_{n=0}^{\infty}$. If α has exactly two subsequential limits such that the larger one is different from the spectral radius $r(T)$ of T, then T is not M-hyponormal.*

Proof. Suppose that there are infinite sets B and C such that $\mathbb{N}=B\cup C$, where (i) B and C are disjoint; (ii) $\beta_n:=\alpha_n$ if $n\in B$ and $\gamma_n:=\alpha_n$ if $n\in C$; (iii) $\beta_n\to\beta$ and $\gamma_n\to\gamma$; and (iv) $\beta<\gamma$.

Assume to the contrary that T is M-hyponormal. Then there exists $M\geq 1$ such that

$$||(T-\lambda)^*x||\leq M\,||(T-\lambda)x|| \quad \text{for all } \lambda\in\mathbb{C} \text{ and for all } x\in\ell^2.$$

Suppose $\beta=0$. Since $\gamma>0$ we can choose δ such that $\gamma_n\geq\delta>0$ for all $n\in C$. Since $\beta_n\to 0$, we can find an $N\in B$ such that $\beta_N<\frac{\delta}{M}$. Since C is infinite there exists $N_0>N$ such that $N_0\in B$ and $N_0-1\in C$. Thus if we take $x=e_{N_0}$, then

$$M^2\,||Tx||^2-||T^*x||^2=M^2\alpha_{N_0}^2-\alpha_{N_0-1}^2<M^2\left(\frac{\delta}{M}\right)^2-\delta^2=0,$$

which shows that T is not M-hyponormal. We now suppose $\beta > 0$. Note that span$\{e_k : k \geq N\}$ is an invariant subspace for T and the restriction of T to such a subspace still yields a weighted shift. But since the restriction of an M-hyponormal operator to an invariant subspace is also M-hyponormal we may assume, without loss of generality, that for sufficiently small $\epsilon > 0$,

$$\alpha_0 = \beta_0, \quad \beta_m < \gamma_k, \quad |\beta_m^2 - \beta^2| < \epsilon \quad \text{and} \quad |\gamma_k^2 - \gamma^2| < \epsilon \quad \text{for each } m \in B \text{ and } k \in C.$$

If $\{\gamma_n\}$ occurs infinitely often, in arbitrary long blocks, then γ must be in the approximate point spectrum of T: indeed if $\{\alpha_n\}$ has the consecutive terms such as $\gamma_{m+1}, \gamma_{m+2}, \cdots, \gamma_{m+k}$ and if f_k is a unit vector such as $f_k = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} e_{m+j}$ then

$$||(T-\gamma)f_k|| = \frac{1}{\sqrt{k}}\left(\gamma^2 + (\gamma_{m+1}-\gamma)^2 + \cdots + (\gamma_{m+k-1}-\gamma)^2 + \gamma_{m+k}^2\right)^{\frac{1}{2}} \leq \sqrt{\epsilon} + \sqrt{\frac{2}{k}}\,\gamma \longrightarrow 0 \text{ as } k \to \infty.$$

But since (cf. [4, Solution 91])

$$r(T) = \lim_k \sup_n \left|\prod_{i=0}^{k-1} \alpha_{n+i}\right|^{\frac{1}{k}} \leq \gamma,$$

it follows that $r(T) = \gamma$, which contradicts to our assumption. Thus $\{\gamma_n\}$ occurs infinitely often, in at most finite length of blocks. Suppose h is the largest number of such the lengths. Let $\lambda > \beta$ be a positive number and choose a sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 = 1 \quad \text{and} \quad x_n = \frac{1}{\lambda^n} \prod_{j=0}^{n-1} \alpha_j \quad (n = 1, 2, \cdots).$$

Consider $\sum c_n z^n$, where $c_0 := 1$ and $c_n := \prod_{j=0}^{n-1} \alpha_j$ $(n = 1, 2, \cdots)$. If ρ is the radius of convergence of this power series, then $\sum_{n=0}^{\infty} x_n^2$ will converge whenever $\frac{1}{\lambda} < \rho$, or $\lambda > \frac{1}{\rho} =: R$. Thus $x = \sum_{n=0}^{\infty} x_n e_n \in \ell_2$ for $\lambda > R$. Observe that if $x = \sum_{n=0}^{\infty} x_n e_n \in \ell^2$ then

$$\begin{aligned}
&M^2\,||(T-\lambda)x||^2 - ||(T-\lambda)^* x||^2 \\
=&M^2 \left(|\lambda x_0|^2 + \sum_{n=0}^{\infty} |\alpha_n x_n - \lambda x_{n+1}|^2\right) - \sum_{n=0}^{\infty} |\alpha_n x_{n+1} - \bar{\lambda} x_n|^2 \\
=&(M^2-1)\sum_{n=0}^{\infty} |\alpha_n x_n - \lambda x_{n+1}|^2 + \alpha_0^2 |x_0|^2 + (M^2-1)|\lambda x_0|^2 + \sum_{n=1}^{\infty}\left(\alpha_n^2 - \alpha_{n-1}^2\right)|x_n|^2.
\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}
M^2||(T-\lambda)x||^2 - ||(T-\lambda)^*x||^2 &= (M^2-1)\lambda^2 + \beta_0^2 + \sum_{k'=1}^{\infty}(\beta_{m_{k'}}^2 - \beta_{m_{k'}-1}^2)x_{m_{k'}}^2 \\
&+ \sum_{k''=1}^{\infty}(\gamma_{n_{k''}}^2 - \gamma_{n_{k''}-1}^2)x_{n_{k''}}^2 + \sum_{k=1}^{\infty}(\gamma_{p_k}^2 - \beta_{p_k-1}^2)x_{p_k}^2 + \sum_{k=1}^{\infty}(\beta_{q_k}^2 - \gamma_{q_k-1}^2)x_{q_k}^2,
\end{aligned}$$

where $\sum_{k'=1}^{\infty} x_{m_{k'}}^2 + \sum_{k''=1}^{\infty} x_{n_{k''}}^2 + \sum_{k=1}^{\infty}(x_{p_k}^2 + x_{q_k}^2) = \sum_{n=1}^{\infty} x_n^2$ and

$$x_{q_k}^2 = \left(\frac{\gamma_{v_k}\cdots\gamma_{v_k+e_k-1}}{\lambda^{e_k}}\right)^2 x_{p_k}^2 \quad \text{and} \quad x_{p_k+1}^2 = \left(\frac{\beta_{w_k}\cdots\beta_{w_k+f_k-1}}{\lambda^{f_k}}\right)^2 x_{q_k}^2 \tag{3.1}$$

for some v_k, w_k, e_k, f_k. Note that if $\lambda < \gamma + \epsilon$ then

$$x_{m_{k'}}^2 = \left(\frac{\beta_{t_k} \cdots \beta_{t_k+c_k-1}}{\lambda^{c_k}}\right)^2 x_{q_k}^2 \le \left(\frac{\beta+\epsilon}{\lambda}\right)^{2c_k} x_{q_k}^2$$

and

$$x_{n_{k''}}^2 = \left(\frac{\gamma_{u_k} \cdots \gamma_{u_k+d_k-1}}{\lambda^{d_k}}\right)^2 x_{p_k}^2 \le \left(\frac{(\gamma+\epsilon)^h}{\lambda^h}\right)^2 x_{p_k}^2 \quad (\text{since } d_k \le h)$$

for some t_k, u_k, c_k, d_k. Thus if $\beta + \epsilon < \lambda < \gamma + \epsilon$, then

$$\sum_{k'=1}^{\infty} x_{m_{k'}}^2 \le \sum_{k=0}^{\infty} \sum_{j=1}^{c_k} \left(\frac{\beta+\epsilon}{\lambda}\right)^{2j} x_{q_k}^2 \le \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \left(\frac{\beta+\epsilon}{\lambda}\right)^{2j} x_{q_k}^2 \le \frac{(\beta+\epsilon)^2}{\lambda^2 - (\beta+\epsilon)^2} \sum_{k=0}^{\infty} x_{q_k}^2, \tag{3.2}$$

where $x_{q_0} = x_0 = 1$ and

$$\sum_{k''=1}^{\infty} x_{n_{k''}}^2 \le h \left(\frac{(\gamma+\epsilon)^h}{\lambda^h}\right)^2 \sum_{k=1}^{\infty} x_{p_k}^2. \tag{3.3}$$

Also we have that

$$\begin{aligned} & M^2||(T-\lambda)x||^2 - ||(T-\lambda)^*x||^2 \\ & \le (M^2-1)\lambda^2 + \beta_0^2 + 2\epsilon \left(\sum_{k'=1}^{\infty} x_{m_{k'}}^2 + \sum_{k''=1}^{\infty} x_{n_{k''}}^2\right) + \Big((\gamma^2+\epsilon) - (\beta^2-\epsilon)\Big) \sum_{k=1}^{\infty} x_{p_k}^2 \\ & \qquad + \Big((\beta^2+\epsilon) - (\gamma^2-\epsilon)\Big) \sum_{k=1}^{\infty} x_{q_k}^2, \\ & \le (M^2-1)\lambda^2 + \beta_1^2 + 2\epsilon \sum_{n=1}^{\infty} x_n^2 + (\gamma^2-\beta^2)\Big(\sum_{k=1}^{\infty} x_{p_k}^2 - \sum_{k=1}^{\infty} x_{q_k}^2\Big). \end{aligned} \tag{3.4}$$

If R_1 and R_2 are the radii of convergence of $\sum_{k=1}^{\infty} x_{p_k}^2$ and $\sum_{k=1}^{\infty} x_{q_k}^2$, respectively then by (3.1), (3.2) and (3.3),

$$\beta \le R_1 \le R_2 = R \le \gamma.$$

If $R_1 < R$, take $\lambda \downarrow R$. Then $\sum_{k=1}^{\infty} x_{p_k}^2$ converges and $\sum_{k=1}^{\infty} x_{q_k}^2 \to \infty$. Since ϵ was arbitrary it follows from (3.4) that $M^2||(T-\lambda)x||^2 - ||(T-\lambda)^*x||^2 < 0$ for $\lambda(> R)$ sufficiently close to R, a contradiction. If $R_1 = R$ then there are two cases to consider.

Case 1 ($R < \gamma$): In this case, take λ so that $R < \lambda < \gamma_{v_1}$, and hence $\frac{\gamma_{v_1}}{\lambda} > 1$. Then we have

$$\begin{aligned} (\gamma^2-\beta^2)\left(\sum_{k=1}^{\infty} x_{p_k}^2 - \sum_{k=1}^{\infty} x_{q_k}^2\right) & \le (\gamma^2-\beta^2)\left(\sum_{k=1}^{\infty} x_{p_k}^2 - \sum_{k=1}^{\infty} x_{p_k}^2 \left(\frac{\gamma_{v_1}}{\lambda}\right)\right) \\ & \le (\gamma^2-\beta^2)\left(1 - \frac{\gamma_{v_1}}{\lambda}\right) \sum_{k=1}^{\infty} x_{p_k}^2. \end{aligned}$$

If we take $\lambda \downarrow R$ then $\sum_{k=1}^{\infty} x_{p_k}^2 \to \infty$. Since ϵ was arbitrary it follows that $M^2||(T-\lambda)x||^2 - ||(T-\lambda)^*x||^2 < 0$ for $\lambda(> R)$ sufficiently close to R, a contradiction.

Case 2 ($R=\gamma$): In this case, take λ so that $\lambda > R$ and hence $\frac{\beta_{s_1}}{\lambda} < 1$. Then we have

$$\begin{aligned}(\gamma^2-\beta^2)\left(\sum_{k=1}^{\infty}x_{p_k}^2-\sum_{k=1}^{\infty}x_{q_k}^2\right) &= (\gamma^2-\beta^2)x_{p_1}^2+(\gamma^2-\beta^2)\left(\sum_{k=2}^{\infty}x_{p_k}^2-\sum_{k=1}^{\infty}x_{q_k}^2\right)\\ &\le (\gamma^2-\beta^2)x_{p_1}^2+(\gamma^2-\beta^2)\left(\sum_{k=1}^{\infty}x_{q_k}^2\left(\frac{\beta_{s_1}}{\lambda}\right)-\sum_{k=1}^{\infty}x_{q_k}^2\right)\\ &\le (\gamma^2-\beta^2)x_{p_1}^2+(\gamma^2-\beta^2)\left(\frac{\beta_{s_1}}{\lambda}-1\right)\sum_{k=1}^{\infty}x_{q_k}^2.\end{aligned}$$

If we take $\lambda \downarrow R$ then $\sum_{k=1}^{\infty} x_{q_k}^2 \to \infty$. Since ϵ was arbitrary it follows that $M^2||(T-\lambda)x||^2-||(T-\lambda)^*x||^2<0$ for $\lambda(>R)$ sufficiently close to R, a contradiction. This completes the proof. □

Example 4. *Let*

$$W_\alpha := \begin{pmatrix} 0 & & & & & \\ \beta & 0 & & & & \\ & \gamma & 0 & & & \\ & & \beta & 0 & & \\ & & & \gamma & 0 & \\ & & & & \ddots & \ddots \end{pmatrix} : \ell^2 \to \ell^2.$$

Then W_α is M-hyponormal if and only if $\beta=\gamma$.

Proof. Since $r(W_\alpha)=\sqrt{\beta\gamma}$, this follows at once from Theorem 3. □

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