ON GENERALIZED RIESZ POINTS

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"Weyl's theorem" for an operator is a statement about the complement in its spectrum of the "Weyl spectrum", which we shall call "generalized Riesz points". In this note we observe how Weyl's theorem and its more relaxed relative "Browder's theorem" do not generally survive under small perturbations.

Recall [7],[8] that a bounded linear operator $T \in BL(X,X)$ on a complex Banach space X is *Fredholm* if $T(X)$ is closed and both $T^{-1}(0)$ and $X/T(X)$ are finite dimensional. If $T \in BL(X,X)$ is Fredholm we can define the *index* of T by $\operatorname{index}(T) = \dim T^{-1}(0) - \dim X/T(X)$. An operator $T \in BL(X,X)$ is called *Weyl* if it is Fredholm of index zero. The essential spectrum $\sigma_{\text{ess}}(T)$ and the Weyl spectrum $\omega_{\text{ess}}(T)$ of $T \in BL(X,X)$ are defined by

$$\sigma_{\text{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\} \tag{0.1}$$

and

$$\omega_{\text{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}. \tag{0.2}$$

If $T \in BL(X,X)$ we shall write

$$\pi^{\text{left}}(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)^{-1}(0) \neq \{0\}\} \tag{0.3}$$

for the *eigenvalues* of T,

$$\pi_0^{\text{left}}(T) = \{\lambda \in \operatorname{iso}\,\sigma(T) : 0 < \dim(T - \lambda I)^{-1}(0) < \infty\} \tag{0.4}$$

for the isolated eigenvalues of finite multiplicity and

$$\pi_{00}(T) = \operatorname{iso}\sigma(T) \setminus \sigma_{\text{ess}}(T) \tag{0.5}$$

for the *Riesz points* of T. From the continuity of the index we have

$$\pi_{00}(T) = \operatorname{iso}\sigma(T) \setminus \omega_{\text{ess}}(T). \tag{0.6}$$

1. Definition. The *generalized Riesz points* of $T \in BL(X,X)$ are the complement of the Weyl spectrum in the spectrum of T:

$$\pi_0(T) = \sigma(T) \setminus \omega_{\text{ess}}(T). \tag{1.1}$$

Thus a necessary and sufficient condition for $0 \in \pi_0(T)$ is

$$0 < \dim T^{-1}(0) = \dim X/T(X) < \infty; \tag{1.2}$$

in particular (1.2) guarantees that T has closed range $T(X) = \operatorname{cl}(TX)$. We recall [2],[9] that "Weyl's theorem holds for T" iff

$$\pi_0(T) = \pi_0^{\text{left}}(T), \tag{1.3}$$

and ([9] Definition 1) "Browder's theorem holds for T" iff

$$\pi_0(T) = \pi_{00}(T). \tag{1.4}$$

The difference between Riesz and generalized Riesz points is environmental:

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2. Theorem. *Each of the following is equivalent to Browder's theorem for $T \in BL(X,X)$:*

$$\pi_0(T) \subseteq \pi_{00}(T); \tag{2.1}$$

$$\pi_0(T) \subseteq \mathit{iso}\ \sigma(T); \tag{2.2}$$

$$\pi_0(T) \subseteq \partial\sigma(T); \tag{2.3}$$

$$\mathit{int}\ \pi_0(T) = \emptyset. \tag{2.4}$$

Proof. Since we always have $\pi_{00}(T) \subseteq \pi_0(T)$ it follows (2.1) $\Longleftrightarrow$ (1.4). Inclusion $\pi_{00}(T) \subseteq \text{iso}\ \sigma(T) \subseteq \partial\sigma(T)$ gives implication (2.1) $\Longrightarrow$ (2.2) $\Longrightarrow$ (2.3). Since int $\partial\sigma(T) = \emptyset$ we have implication (2.3) $\Longrightarrow$ (2.4). For implication (2.4) $\Longrightarrow$ (2.3) $\Longrightarrow$ (2.2), we argue

$$\text{int}\big(\sigma(T) \setminus \omega_{\text{ess}}(T)\big) = \emptyset \Longrightarrow \sigma(T) \subseteq \omega_{\text{ess}}(T) \cup \partial\sigma(T) \subseteq \omega_{\text{ ess}}(T) \cup \text{iso}\ \sigma(T), \tag{2.5}$$

since by the punctured neighbourhood theorem ([8] Theorem 9.8.4) the boundary is contained in the union of the Weyl spectrum and the isolated points. Finally since $\pi_{00}(T) = \pi_0(T) \cap \text{iso}\ \sigma(T)$ we have (2.2) $\Longrightarrow$ (2.1). $\square$

Recall that ([8] Theorem 9.8.4)

$$\sigma_{\text{ess}}(T) \subseteq \omega_{\text{ess}}(T) \subseteq \eta\sigma_{\text{ess}}(T), \tag{2.6}$$

where ηK is the "connected hull" of the compact set $K \subseteq \mathbb{C}$ in the sense ([8] Definition 7.10.1) of the complement in $\mathbb{C}$ of the unbounded component of the complement of K.

In this note we are interested to see whether, for "small" perturbations $T + K$ of $T \in BL(X,X)$, "small" $\pi_0(T)$ gives rise to "small" $\pi_0(T+K)$. For example if $K \in BL(X,X)$ is compact then the Weyl spectrum of $T+K$ is the same as that of T; since however the spectrum of $T+K$ can be very different than that of T, we need not expect the sets of "generalized Riesz points" to agree. But if $K \in BL(X,X)$ is quasinilpotent and commutes with T then (*cf.* [10] Lemma 2)

$$\sigma(T+K) = \sigma(T) \quad \text{and} \quad \omega_{\text{ess}}(T+K) = \omega_{\text{ess}}(T), \tag{2.7}$$

so that also $\pi_0(T+K) = \pi_0(T)$. For quasinilpotents which do not commute it is not so clear what happens:

3. Problem. *For which operators $T \in BL(X,X)$ is there implication, for quasinilpotent $K \in BL(X,X)$,*

$$\pi_0(T) = \emptyset \Longrightarrow \pi_0(T+K) = \emptyset, \tag{3.1}$$

or implication

$$\mathit{int}\ \pi_0(T) = \emptyset \Longrightarrow \mathit{int}\ \pi_0(T+K) = \emptyset? \tag{3.2}$$

We can also ask analogous questions for polynomials and for direct sums: is there implication, for polynomials p,

$$\pi_0(T) = \emptyset \Longrightarrow \text{int}\ \pi_0 p(T) = \emptyset, \tag{3.3}$$

or implication

$$\pi_0(T) = \emptyset = \pi_0(S) \Longrightarrow \text{int } \pi_0(S \oplus T) = \emptyset? \tag{3.4}$$

As we are about to see, the condition (3.1) fails for compact and for quasinilpotent operators T, while the condition (3.2) fails for quasinilpotents but holds for compact operators. Both (3.1) and (3.2) can fail for self adjoint and for unitary operators. Many of our counterexamples are generated by taking 2×2 operator matrices built from just three operators on the sequence spaces ℓ_p or c_0: the *forward shift*

$$u : (x_1, x_2, x_3, \cdots) \mapsto (0, x_1, x_2, \cdots), \tag{3.5}$$

the *backward shift*

$$v : (x_1, x_2, x_3, \cdots) \mapsto (x_2, x_3, x_4 \cdots) \tag{3.6}$$

and the *standard weight*

$$w : (x_1, x_2, x_3, \cdots) \mapsto (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \cdots). \tag{3.7}$$

We recall that

$$vu = 1 \neq uv \tag{3.8}$$

is Fredholm; also w is compact but not of finite rank, and commutes with the projection uv. For operator matrices we observe that for most of the familiar kinds of spectrum ϖ there is equality, for $T \in BL(X, X)$ and $S \in BL(Y, Y)$,

$$\varpi \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} = \varpi(T) \cup \varpi(S), \tag{3.9}$$

also, for $V \in BL(X, Y)$ and $U \in BL(Y, X)$,

$$\varpi \begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix} = \sqrt{\varpi(UV) \cup \varpi(VU)}, \tag{3.10}$$

the set of those $\lambda \in \mathbb{C}$ for which λ^2 is in the spectrum of one of the products. When the entries in an operator matrix commute then ([6] Solution 70; [8] Theorem 11.7.7) spectra can be calculated by determinants: if $\{S, T, U, V\} \subseteq BL(X, X)$ is commutative then

$$\varpi \begin{pmatrix} T & U \\ V & S \end{pmatrix} = \{\lambda \in \mathbb{C} : 0 \in \varpi\big((S - \lambda I)(T - \lambda I) - UV\big)\}. \tag{3.11}$$

Indeed (3.9) is easily checked for the ordinary spectrum, the left and the right spectrum, the essential spectrum, the eigenvalues and the approximate eigenvalues while, for the same spectra ϖ, (3.10) follows from (3.9) together with the spectral mapping theorem for the polynomial z^2: simply observe that

$$\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}^2 = \begin{pmatrix} UV & 0 \\ 0 & VU \end{pmatrix}. \tag{3.12}$$

For commutative matrices (3.11) just write down the "classical adjoint":

$$\begin{pmatrix} S & -U \\ -V & T \end{pmatrix} \begin{pmatrix} T & U \\ V & S \end{pmatrix} = \begin{pmatrix} ST - UV & 0 \\ 0 & ST - UV \end{pmatrix} = \begin{pmatrix} T & U \\ V & S \end{pmatrix} \begin{pmatrix} S & -U \\ -V & T \end{pmatrix}. \tag{3.13}$$

For the Weyl spectrum $\varpi = \omega_{\text{ess}}$ it is more delicate:

4. Theorem. *If $T \in BL(X,X)$ and $S \in BL(Y,Y)$ there is inclusion*

$$\omega_{\text{ess}} \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \subseteq \omega_{\text{ess}}(T) \cup \omega_{\text{ess}}(S), \tag{4.1}$$

with equality if

$$\text{either } \ \omega_{\text{ess}}(T) = \sigma_{\text{ess}}(T) \ \text{ or } \ \omega_{\text{ess}}(S) = \sigma_{\text{ess}}(S). \tag{4.2}$$

If $V \in BL(X,Y)$ and $U \in BL(Y,X)$ then with no restriction there is equality

$$\omega_{\text{ess}} \begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix} = \sqrt{\omega_{\text{ess}}(UV) \cup \omega_{\text{ess}}(VU)}. \tag{4.3}$$

If $\{S,T,U,V\}$ commute there is inclusion

$$\{\lambda \in \mathbb{C} : 0 \in \omega_{\text{ess}}\big((T-\lambda I)(S-\lambda I) - UV\big)\} \subseteq \omega_{\text{ess}} \begin{pmatrix} T & U \\ V & S \end{pmatrix}. \tag{4.4}$$

Proof. For (4.1) recall that the index of a direct sum is the sum of the indexes. The inclusion (4.1) and Theorem 5 in [9] imply (4.2). For (4.3) notice that if $0 \neq \lambda \in \mathbb{C}$

$$\begin{pmatrix} -\lambda I & U \\ V & -\lambda I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda I \\ V \end{pmatrix} (UV - \lambda^2 I)^{-1}(0) = \begin{pmatrix} U \\ \lambda I \end{pmatrix} (VU - \lambda^2 I)^{-1}(0) \tag{4.5}$$

and

$$\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V^{-1}(0) \\ U^{-1}(0) \end{pmatrix}. \tag{4.6}$$

Taking adjoints and combining shows that, whether or not $\lambda = 0 \in \mathbb{C}$,

$$\text{index} \begin{pmatrix} -\lambda I & U \\ V & -\lambda I \end{pmatrix} = \text{index}\,(UV - \lambda^2 I) = \text{index}\,(VU - \lambda^2 I) \tag{4.7}$$

if U and V are Fredholm. For (4.4) remember that the product of Weyl operators is Weyl. □

In contrast both (3.9), (3.10) and (3.11) are all valid without restriction if $\varpi = \omega_{\text{ess}}^{\text{comm}}$ is the "Browder spectrum".

The failure of both (3.1) and (3.2) for quasinilpotent operators T is easy:

5. Example. *If*

$$S = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \tag{5.1}$$

with the forward and backward shifts u and v on ℓ_p or c_0, put

$$T = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix}: \tag{5.2}$$

then T and K are both nilpotents and

$$\pi_0(T) = \emptyset \neq \textit{int}\ \pi_0(T+K) = \{|z| < 1\}. \tag{5.3}$$

Proof. It is clear T and K are both nilpotents; thus $\pi_0(T) = \emptyset$. To compute $\pi_0(T+K)$ use (3.10) and (4.3): by (3.10) we have

$$\sigma(T+K) = \sqrt{\sigma(S^2)} = \sqrt{\sigma(u^2) \cup \sigma(v^2)} = \mathbb{D},$$

the closed unit disc, while again by (3.10)

$$\sigma_{\text{ess}}(T+K) = \sqrt{\sigma_{\text{ess}}(S^2)} = \sqrt{\sigma_{\text{ess}}(u^2) \cup \sigma_{\text{ess}}(v^2)} = \mathbb{S},$$

the unit circle. Also since $\text{index}(S^2 - \lambda^2 I) = \text{index}(u^2 - \lambda^2) + \text{index}(v^2 - \lambda^2) = 0$ whenever $|\lambda| < 1$, (4.3) gives

$$\omega_{\text{ess}}(T+K) = \sqrt{\omega_{\text{ess}}(S^2)} = \mathbb{S}.$$

$\square$

The implication (3.2) is satisfied by compact operators T:

6. Theorem. *If $T \in BL(X,X)$ is a compact operator then*

$$\textit{int}\ \pi_0(T+K) = \emptyset \quad \textit{for every quasinilpotent } K \in BL(X,X). \tag{6.1}$$

Proof. If T is compact then

$$\sigma_{\text{ess}}(T+K) = \sigma_{\text{ess}}(K) = \{0\} \quad \text{for every quasinilpotent } K \in BL(X,X), \tag{6.2}$$

so that $T+K$ is a "Riesz operator". By the punctured neighborhood theorem $\sigma(T+K)$ is at most countable and hence $\text{int}\ \pi_0(T+K) = \emptyset$. $\square$

Towards the failure of (3.1) for compact operators, notice that if $\pi_0(T) = \emptyset$ for a compact operator T then also $\sigma(T) = \{0\}$, so that T is quasinilpotent:

7. Example. *If*

$$T = \begin{pmatrix} 0 & uw \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad K = \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \tag{7.1}$$

on ℓ_p or c_0 then T is compact, K is nilpotent and

$$\pi_0(T) = \emptyset \neq \pi_0(T+K) = \{\pm\frac{1}{\sqrt{n}} : n \in \mathbb{N}\}. \tag{7.2}$$

Proof. It is clear that T is a compact nilpotent and K is nilpotent. Since T is nilpotent, $\pi_0(T) = \emptyset$. By (3.10) the spectrum of $T+K$ is given by

$$\sigma(T+K) = \sqrt{\sigma(w) \cup \sigma(uwv)} = \{\pm\frac{1}{\sqrt{n}} : n \in \mathbb{N}\} \cup \{0\},$$

while (3.10) and (2.6) together give

$$\omega_{\text{ess}}(T+K) = \sigma_{\text{ess}}(T+K) = \sqrt{\sigma_{\text{ess}}(w) \cup \sigma_{\text{ess}}(uwv)} = \{0\}.$$

$\square$

Both (3.1) and (3.2) fail for unitary and for self adjoint operators. In the unitary case we can do both at once:

8. Example. *If on ℓ_2*

$$T = \begin{pmatrix} u & 1-uv \\ 0 & v \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 1-uv \\ 0 & 0 \end{pmatrix} \tag{8.1}$$

then T is unitary, K is a finite rank nilpotent and

$$\pi_0(T) = \emptyset \neq \text{int } \pi_0(T-K) = \{|z| < 1\}. \tag{8.2}$$

Proof. This calculation is done in Example 6 of [9]. □

Example 8 shows we cannot guarantee that if $T \in BL(X, X)$ then

$$\pi_0(T) = \emptyset \implies \operatorname{int} \pi_0(T+K) = \emptyset \text{ for every compact operator } K \in BL(X, X). \tag{8.3}$$

If however there is no hole of $\sigma_{\text{ess}}(T)$ associated with index zero, in particular if T is self-adjoint, then the right hand side of (8.3) is always satisfied. Indeed if

$$\omega_{\text{ess}}(T) = \eta\sigma_{\text{ess}}(T) \tag{8.4}$$

then Browder's theorem holds for T, since ([8] Theorem 9.8.4) the spectrum is included in the connected hull of the essential spectrum union with the Riesz points. Now notice that if T satisfies (8.4) then so do all its compact perturbations $T + K$.

For the failure of (3.1) for self-adjoint operators we have

9. Example. *If on ℓ_2*

$$T = \begin{pmatrix} uv & uv \\ uv & uv \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & uv \\ 0 & 0 \end{pmatrix} \tag{9.1}$$

then

$$\pi_0(T) = \emptyset \neq \pi_0(T-K) = \{0\}. \tag{9.2}$$

Proof. By (3.11) we have $\sigma(T) = \sigma_{\text{ess}}(T) = \{0, 2\}$ while $\sigma(T-K) = \{0, 1\}$ and $\sigma_{\text{ess}}(T-K) = \{1\}$; now use (2.6). □

For the failure of (3.2) we have:

10. Example. *If* $S = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ *put*

$$Q = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix}, \tag{10.1}$$

and then

$$T = \begin{pmatrix} 0 & Q^* \\ Q & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & R-Q^* \\ 0 & 0 \end{pmatrix}, \tag{10.2}$$

now T is self adjoint, K is nilpotent and

$$\pi_0(T) = \{0\} \neq \text{int } \pi_0(T+K) = \{|z| < 1\}. \tag{10.3}$$

Proof. Evidently T is self-adjoint and K is nilpotent. By (3.9), (3.10) and (2.6) we have $\sigma(T) = \{0, 1\}$ and $\pi_0(T) = \{0\}$, while $\sigma(T+K)$ is the closed unit disc and $\sigma_{\text{ess}}(T+K)$ is the circle. By (4.3) the Weyl spectrum of $T+K$ agrees with the essential spectrum, so that $\pi_0(T+K)$ is the open disc. □

For an alternative to Example 10, take

$$T = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{10.4}$$

Problem 3 is liable to have a negative solution for Toeplitz operators:

11. Theorem. *If $T \in BL(X,X)$ satisfies the conditions*

$$\sigma(T) = \omega_{\text{ess}}(T) \neq \eta\sigma(T) \tag{11.1}$$

then (3.1) fails.

Proof. It is clear that $\pi_0(T)$ is empty. Choose a point $\lambda \in \mathbb{C}$ for which

$$\lambda \in \eta\sigma(T) \setminus \sigma(T). \tag{11.2}$$

Since T cannot be a scalar there must exist $x \in X$ for which

$$(x, Tx - \lambda x) \in X^2 \text{ is linearly independent,} \tag{11.2}$$

and hence a bounded linear functional $f \in X^\dagger$ for which

$$f(x) - 1 = 0 = f(Tx - \lambda x) : \tag{11.3}$$

now put

$$K = f \odot (T - \lambda I)x : y \mapsto f(y)(Tx - \lambda x). \tag{11.4}$$

Evidently $K^2 = 0$, so that K is nilpotent; also since K is finite rank and hence compact the Weyl spectrum of $T - K$ is the same as that of T:

$$\omega_{\text{ess}}(T - K) = \sigma(T). \tag{11.5}$$

On the other hand, since evidently $Kx = Tx - \lambda x$,

$$\lambda \in \pi^{\text{left}}(T - K) \subseteq \sigma(T - K). \tag{11.6}$$

This puts the point $\lambda \in \pi_0(T - K)$. □

The condition (11.1) can easily be satisfied by a "Toeplitz operator": recall ([2]; [4] Definition 7.2) that T_φ, induced by a function $\varphi \in L^\infty(\mathbb{S})$, is the operator on the Hardy space $H^2(\mathbb{S})$ given by setting

$$\mathrm{T}_\varphi(f) = \mathbf{P}(\varphi f) \quad \text{for each } f \in H^2(\mathbb{S}), \tag{11.7}$$

where $\mathbf{P}$ is the orthogonal projection from $L^2(\mathbb{S})$ onto $H^2(\mathbb{S})$. It is familiar ([4] Corollary 7.46) that the spectrum of a Toeplitz operator is always connected, and that the spectrum and the Weyl spectrum coincide ([4] Corollary 7.25; [3] Theorem 4.1). The essential spectrum of the Toeplitz operator induced by a continuous symbol coincides with the range of the function ([4] Theorem 7.26):

$$\sigma_{\text{ess}}(\mathrm{T}_\varphi) = \sigma(\varphi) = \varphi(\mathbb{S}). \tag{11.8}$$

The spectrum and the Weyl spectrum both coincide ([4] Corollary 7.25) with the *exponential spectrum* ([8] Definition 9.3.1) of the symbol:

$$\sigma(\mathrm{T}_\varphi) = \omega_{\text{ess}}(\mathrm{T}_\varphi) = \varepsilon(\varphi) \tag{11.9}$$

is the set of $\lambda \in \mathbb{C}$ for which either $\varphi - \lambda$ vanishes somewhere on the circle $\mathbb{S}$, or if not then $\varphi - \lambda$ winds non-trivially around the origin $0 \in \mathbb{C}$. Thus for (11.1) we want a continuous function from $\mathbb{S}$ to $\mathbb{S}$ whose range is the whole of $\mathbb{S}$, but which does not wind round the origin. For a specific example we may take

$$\varphi(e^{i\theta}) = \begin{cases} e^{2i\theta} & (0 \le \theta \le \pi) \\ e^{-2i\theta} & (\pi \le \theta \le 2\pi). \end{cases} \tag{11.10}$$

We have been unable to decide whether or not int $\pi_0(T+K) = \emptyset$ for every Toeplitz operator T and every quasinilpotent K. Since ([5] Corollary 2.2)

$$\omega_{\mathrm{ess}} p(\mathrm{T}_\varphi) = \sigma(\mathrm{T}_{p\circ\varphi}) \tag{11.11}$$

and by connectedness

$$\pi_0^{\mathrm{left}} p(\mathrm{T}_\varphi) = \emptyset, \tag{11.12}$$

it is clear that Browder's theorem for the operator $p(\mathrm{T}_\varphi)$, with continuous φ and polynomial p, will be no easier than Weyl's theorem (*cf.* [5] Lemma 3.1).

The analogue of Problem 3 is liable to have a negative solution for polynomials:

12. Example. *If $p = z^2$ and*

$$T = \begin{pmatrix} u+1 & 0 \\ 0 & v-1 \end{pmatrix} \tag{12.1}$$

then

$$\pi_0(T) = \emptyset \ne \mathit{int}\ \pi_0 p(T) = \{re^{i\theta} : r < 2(1 + \cos\theta)\}. \tag{12.2}$$

Proof. As in Example 7 of [9]. □

From Theorem 5 of [9] we can see that on Hilbert space X there is equality

$$p\pi_0(T) = \pi_0 p(T) \quad \text{for all polynomials } p \tag{12.3}$$

iff T is "semi-quasitriangular", in the sense that either T or T^* is quasitriangular ([12] Definition 4.8). The analogue of Problem 3 for direct sums can also have negative solution:

13. Example. *If $T = u$ and $S = v$ then*

$$\pi_0(S) = \pi_0(T) = \emptyset \ne \mathit{int}\ \pi_0(S \oplus T) = \{|z| < 1\}. \tag{13.1}$$

Proof. This calculation is part of Example 10. □

This failure cannot occur if for example T is "bi-quasitriangular" ([12] Definition 4.26) in the sense that T and T^* are both quasitriangular: for then ([12] Theorem 6.1) $\sigma_{\mathrm{ess}}(T) = \omega_{\mathrm{ess}}(T)$ and hence (4.2) applies.

We conclude with a converse of Problem 3. Halmos ([6] Problem 106) asks whether if $T, K \in BL(X,X)$ satisfy $\sigma(T + nK) = \sigma(T)$ for $n = 0, 1, 2, \cdots$, the operator K has to be quasinilpotent: the answer ([6] Solution 106), is yes if dim $X < \infty$ and no otherwise.

14. Theorem. *If $T, K \in BL(X,X)$ satisfy*

$$\sigma(T+nK)=\sigma(T) \quad \text{for } n=0,1,2,\cdots \quad \text{with } TK-KT \text{ compact} \tag{14.1}$$

then K is quasinilpotent.

Proof. As in Solution 106 of [6], observe that

$$|K+\tfrac{1}{n}T|_\sigma \longrightarrow 0, \tag{14.2}$$

where $|\cdot|_\sigma$ denotes the spectral radius. The compactness of $TK-KT$ says that the cosets of T and K in the Calkin algebra commute, and hence Newburgh's spectral continuity theorem ([11] Theorem 4; [1] Theorem 3.4.1) applies:

$$\lim_n \sigma_{\text{ess}}(K+\tfrac{1}{n}T)=\sigma_{\text{ess}}(K) \tag{14.3}$$

By (14.2) it follows that

$$\sigma_{\text{ess}}(K)=\{0\}, \tag{14.4}$$

which says that K is a Riesz operator, therefore with totally disconnected spectrum $\sigma(K)$. Now by another of Newburgh's theorems ([11] Theorem 3; [1] Corollary 3.4.5) the operator K is a point of continuity for the spectrum σ:

$$\lim_n \sigma(K+\tfrac{1}{n}T)=\sigma(K). \tag{14.5}$$

But now K must also be a continuity point for the spectral radius:

$$|K|_\sigma=\lim_n |K+\tfrac{1}{n}T|_\sigma=0. \tag{14.6}$$

□

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