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Positivity of operator-matrices of Hua-type

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Let A_j $(j = 1, 2, \ldots, n)$ be strirct contractions on a Hilbert soace, i.e., $\|A_j\| < 1$ $(j = 1, 2, \ldots, n)$. Our concern is the $n \times n$ operator-matrix

$$\mathbf{H}_n(A_1, A_2, \ldots, A_n) = \Big[(I - A_j^* A_i)^{-1}\Big]_{i,j=1}^{n}.$$

Our interest is in positivity, i.e., positive semi-definiteness of $\mathbf{H}_n$.

To make the situation more visible, when $n = 2$ write $A = A_1$ and $B = A_2$. Then

$$\mathbf{H}_2(A, B) = \begin{bmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{bmatrix}.$$

In the case of matrices, L.K. Hua observed the block-matrix $\mathbf{H}_2(A, B)$ for the first time and proved its positivity. When $n \geq 3$ the operator-matrix $\mathbf{H}_3(A_1, A_2, A_3)$ is not always positive. Our aim is to find a map $\Theta(\cdot)$ of the open unit disc $\mathcal{D}$ of strict contractions, which preserves the positivity of $\mathbf{H}_n$ in the sense

$$\mathbf{H}_n(A_1, A_2, \ldots, A_n) \geq 0 \quad \Longrightarrow \quad \mathbf{H}_n\Big(\Theta(A_1), \Theta(A_2), \ldots, \Theta(A_n)\Big) \geq 0.$$

We will show that, given $B \in \mathcal{D}$, the operator Möbius map $\Theta_B(\cdot)$ at B, defined as

$$\Theta_B(Z) = D_{B^*}^{-1}(B - Z)(I - B^*Z)^{-1} D_B \qquad (Z \in \mathcal{D})$$

meets the requirment. Here D_B is the defect operator $D_B = (I - B^*B)^{1/2}$.

References

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