Abstracts of KOTAC
Volume 10(2008), 16–17

On backward extensions of operators

Il Bong Jung
Kyungpook National University, Korea
ibjung@knu.ac.kr

There is a description of subnormality for an abstract operator T in terms of weighted shifts; namely T is subnormal if and only if for each $h \neq 0$ in $\mathcal{H}$, the weighted shift with weight sequence $\alpha_h := \{\|T^{n+1}h\| / \|T^n h\|\}$ is subnormal ([Lam],[St]). We define W_h to be the weighted shift on l^2_+ with the weight sequence α_h. Then by C. Berger's theorem, for each $h \neq 0$ there is a Borel probability measure μ_h over $[0, \|W_h\|^2]$ such that, for all $n \in \mathbb{N} \cup \{0\}$, $\|T^n h\|^2 / \|h\|^2 = \int t^n d\mu_h$. A nonzero vector $h \in \mathcal{H}$ is called a *k-step backward extension vector* for T if $1/t^k \in L^1(\mu_h)$. We write $\mathcal{E}_T$ for the set of all k-step backward extension vectors for T. If $\mathcal{E}_T = \mathcal{H}$, then we say that T has *k-step full backward extension.* If this holds for all integers $k \geq 1$, then we say that T has ∞-*step full backward extension.* Suppose that $N \in \mathcal{L}(\mathcal{K})$ is its minimal normal extension. Then we have the following theorem.

Theorem A. *If $k \geq 1$ is integer, then the following assertions are equivalent:*
(i) *T has k-step full backward extension;*
(ii) $\mathcal{H} = \mathcal{H} \cap N^k(\mathcal{K})$;
(iii) $\operatorname{ran} T^k$ *is closed and* $\ker T^{*k}$ *is contained in* $\operatorname{ran} N^k$;
(iv) *for every $h \in \mathcal{H}$, there exists a positive real constant c_h such that*

$$\left| \left(\sum_{i=0}^{n} T^{*i} h_i, h \right) \right|^2 \leq c_h \sum_{i,j=0}^{n} \left(T^{j+k} h_i, T^{i+k} h_j \right), \quad h_0, \cdots, h_n \in \mathcal{H}, \quad n \geq 0.$$

Corollary B. *Suppose T has one-step full backward extension and dense range. Then T has ∞-step full backward extension.*

In particular, if T is cyclic, then T is unitarily equivalent to a multiplication operator S_ν on $H^2(\nu)$, with normal extension N_ν on $L^2(\nu)$, where the subscript ν indicates multiplication by the independent variable z on $H^2(\nu)$ or $L^2(\nu)$. For nonzero h in $H^2(\nu)$, let $d\nu_h = |h|^2 d\nu$. Then we have the following proposition.

Proposition C. *Let T be a cyclic subnormal operator on $\mathcal{H}$ with associated measure ν. Then T has full backward extension if and only if $\int_{\mathbb{D}} |h(z)/z|^2 \, d\nu < \infty$ for all $h \in H^2(\nu)$.*

Corollary D. *Suppose S_ν has full backward extension and dense range. Then $R^2(\nu) = H^2(\nu)$, where $R^2(\nu) := H^2(\nu) \vee \{\frac{1}{z^k} : k \geq 1\}$.*

Proposition E. *Let T be a quasinormal operator on $\mathcal{H}$. Then T has ∞-step backward extension.*

Remark F. The above works on backward extensions also can be extended to completely hyperexpansive operators in $\mathcal{L}(\mathcal{H})$ (cf. [JJS]).

This talk is based on [HJL], [JJS], and [JLS]. In addition, we will discuss further research on this area.

References

[HJL] T. Hoover, I. Jung, A. Lambert, *Moment sequences and backward extensions of subnormal weighted shifts*, J. Aust. Math. Soc. **73**(2002), 27-36.

[JJS] Z. Jablonski, I. Jung, J. Stochel, *Backward extensions of hyperexpansive operators*, Studia Math. **173**(2006), 233-257.

[JLS] I. Jung, A. Lambert, J. Stochel, *Backward extensions of subnormal operators*, Proc. Amer. Math. Soc. **132**(2004), 2291-2302.

[Lam] A. Lambert, *Subnormality and weighted shifts*, J. London Math. Soc. **14**(1976), 476-480.

[St] J. Stochel, *Seminormality of operators from their tensor product*, Proc. Amer. Math. Soc. 124(1996), 135-140.