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Limits of iterations of the Aluthge transformation in 2 by 2 matix algebras

Jun Tomiyama
Emeritus Professor of Tokyo Metropolitan University,Japan
juntomi@med.email.ne.jp

Let T be an n by n matrix and $T = U|T|$ be its polar decomposition. The Aluthge transform for T, $\Delta(T)$, is defined as $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Write its (n times) iteration by $\Delta^n(T)$. Since $\Delta(T)$ is norm decreasing,the map;$T \to \Delta(T)$ gives an interesting and important dynamical system in the unit ball of the matrix algebra M_n , and the system has been studied by many authors. As of now however we know the convergence of iterations only for 2 by 2 matrices by Ando and Yamazaki [1] and the general concrete form of their limits has not been known yet.

This talk concerns with the introduction of recent work [3] by T.Takasaki about such concrete form of limits for 2 by 2 matrices. In fact, in that article , though under the restriction of the spectrum of a matrix Takasaki has shown the precise form of limits of iterations as well as provides an alternative proof of general convergence.

Since $\Delta(\zeta T) = \zeta\Delta(T)$, in the following we often assume that $\det(T) = 1$, hence $\det \Delta^n(T) = 1$.

His first contribution is the following exact form of Aluthge transform. Write the given matrix as $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and put

$$\alpha = |a|^2 + |b|^2 + |c|^2 + |d|^2 + 2|ad - bc| = tr(T^\star T) + 2\det(T).$$

Proposition 1 *Assume that* $\det(T) = 1$. *Then we have*

$$\Delta(T) = \frac{(\sqrt{\alpha}+1)tr(T) + tr(T^\star)}{(\sqrt{\alpha}+2)\alpha}T^\star T + \frac{1}{\sqrt{\alpha}}(T - T^\star) + \frac{(\sqrt{\alpha}+1)tr(T^\star) + tr(T)}{(\sqrt{\alpha}+2)\alpha}I$$

Corollary 1 *If* $tr(T) = -tr(T^\star)$, *in particular when* $tr(T) = 0$, $\Delta(T)$ *becomes normal, hence*

$$\lim_{n\to\infty} \Delta^n(T) = \Delta(T).$$

Let λ and μ be eigenvalues of the matrix T, then Takasaki's main result is the following

Theorem 1 *The iterations* $\{\Delta^n(T)\}$ *converge for any 2 by 2 matrix* T. *Moreover, when* $|\lambda| = |\mu| = 1$ *and* $\det(T) = 1$ *we have*

$$\lim_{n\to\infty} \Delta^n(T) = \frac{tr(T)}{2}I + \frac{\sqrt{4 - tr(T)^2}}{2\sqrt{\alpha - tr(T)^2}}(T - T^\star)$$

.

Two key lemmas are used to show the converence. Write the number α_n for $\Delta^n(T)$ corresponding to α for T, and let $r(T)$ be the spectral radius of T. Assume that $\det(T) = 1$.

Lemma 1 *The sequences* $\{tr(\Delta^n(T)^\star \Delta^n(T))\}$ *and* $\{\alpha_n\}$ *are monotone decreasing. Their limits are*

$$\lim_{n\to\infty} tr(\Delta^n(T)^\star \Delta^n(T)) = r(T)^2 + \frac{1}{r(T)^2} \qquad \textit{and}$$

$$\sqrt{\alpha_\infty} = \lim_{n\to\infty} \sqrt{\alpha_n} = r(T) + \frac{1}{r(T)}.$$

Here when $|\lambda| = |\mu| = 1$ we see that $\alpha_\infty = 4$, which is used to determine the concrete limit form in this case.

Next Lemma is the estimation, which is used to show the convergence.

Lemma 2

$$\|\Delta(T) - \Delta^2(T)\| \leq \frac{2}{\sqrt{\alpha_\infty}} \|T - \Delta(T)\|.$$

Mail address of T.Takasaki: takasaki.toshiyuki@nein.ed.jp

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