On Generalized criss-cross, near commutativity and common spectral properties

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Many authors studied its applications and consequences in local and global spectral theory (B. Barnes, P. Aiena, M. Cho, R. Curto, R. Harte, T. Huruya, I.H. Jeon, I.B. Jung, E. Ko, K. Tanahashi, S. Li, C. Lin, Y. Ruan, Z. Yan, E. Zerouali and C.B., ...)

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We loose “somehow” 0 by passing from local to global spectra.

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Actually, there are two known possiblities:

A and **B** are called criss-cross (R. Harte) commuting if

$$\left\{ \begin{array}{l} A_iB_jA_k = A_kB_jA_i \quad \text{for every } 1 \le i,j,k \le n \\ \end{array} \right.$$

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Here $\sigma_{\mathbf{T}}$ stands for Taylor spectrum for commuting n-tuples introduced by J. L. Taylor.

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$$\sigma_{\mathbf{T}}(\mathbf{AB}) \setminus [0] = \sigma_{\mathbf{T}}(\mathbf{BA}) \setminus [0], \tag{3.3}$$

where

$$[0] := \{(z_1, \cdots, z_n) \in \mathbb{C}^n \ : \prod_{i=1}^{n} z_i = 0\}.$$

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provided that $ABA = ACA$.

Of course the last condition is obviously true when $B = C$ and in this case we obtain (1.1). Thus, (4.1) could be considered as an extension of Jacobson's lemma.

Our goal is to give the n-tuple version of 4.1. In fact, we give two versions as it was done for Jacobson's lemma.

Definition (Generalized criss-cross)

$$\begin{cases} A_iB_jA_k = A_kB_jA_i \text{ and } B_iA_jB_k = B_kA_jB_i \quad \forall \quad 1 \le i, j, k \le n \\ \end{cases} \qquad (i')$$

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Remark:

When $\mathbf{C} = \mathbf{B}$, conditions (ii') and (iii') are empty and we end with the criss-cross definition.

Theorem

et $\mathbf{A} = (A_1, \cdots, A_n) \in \mathcal{L}(X)^n$, $\mathbf{B} = (B_1, \cdots, B_n) \in \mathcal{L}(X)^n$ *and* $\mathbf{C} = (C_1, \cdots, C_n) \in \mathcal{L}(X)^n$ *be commuting n-tuples that are generalized criss-cross. We have*

$$\Sigma(\mathbf{AB}) \setminus \{0\} = \Sigma(\mathbf{CA}) \setminus \{0\}$$

for any $\Sigma \in \{\sigma, \sigma_e, \sigma^{\pi,k}, \sigma_e^{\pi,k}, \sigma^{\delta,k}, \sigma_e^{\delta,k}\}$ *and* $0 \leq k \leq n$.

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$$\begin{cases} A_iB_j = B_jA_i \qquad \text{for every} \quad 1 \leq i \neq j \leq n \quad (i) \\ \end{cases}$$

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Remark:

Notice that if $\mathbf{C} = \mathbf{B}$, conditions (ii) and (iii) are empty. Thus, we overhaul near commutativity.

Let $\mathcal{I}$ be a subset of $\{1, \cdots, n\}$ and denote

$$(\mathbf{CA})_{\mathcal{I}} = ((C_1 A_1)_{\mathcal{I}}, \cdots, (C_n A_n)_{\mathcal{I}}),$$

with $(C_i A_i)_{\mathcal{I}} = A_i B_i$ if $i \in \mathcal{I}$ and $(C_i A_i)_{\mathcal{I}} = C_i A_i$ otherwise.
Clearly $(\mathbf{CA})_{\emptyset} = \mathbf{CA}$ and $(\mathbf{CA})_{\{1,\cdots,n\}} = \mathbf{AB}$.
We also write

$$[0]^{\mathcal{I}} = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n : \prod_{i \in \mathcal{I}} \lambda_i = 0\}.$$

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- *For* $\mathcal{I}, \mathcal{J} \subset \{1, \cdots, n\}$, *We have*

$$\Sigma((\mathbf{CA})_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \Sigma((\mathbf{CA})_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}.$$

Theorem

Let $\mathbf{A} = (A_1, \cdots, A_n) \in \mathcal{L}(X)^n$, $\mathbf{B} = (B_1, \cdots, B_n) \in \mathcal{L}(X)^n$ *and* $\mathbf{C} = (C_1, \cdots, C_n) \in \mathcal{L}(X)^n$ *be commuting n-tuples that are generalized near-commuting*

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- *In particular*

$$\Sigma(\mathbf{CA}) \setminus [0] = \Sigma(\mathbf{AB}) \setminus [0]$$

for $\Sigma \in \{\sigma, \sigma_e, \sigma^{\pi,k}, \sigma_e^{\pi,k}, \sigma^{\delta,k}, \sigma_e^{\delta,k}\}$.