Fourier algebras on locally compact groups

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KOTAC2015, June 18th, 2015
Locally compact groups and Banach algebras

- **Definition** A **locally compact (LC) group** is a topological group whose topology is locally compact and Hausdorff.

- Examples of LC groups: (1) Lie groups such as $\mathbb{T}$, $\mathbb{R}$, $SU(2)$, Heisenberg groups, · · · ) (2) discrete group, i.e. any groups with discrete topology.

- **Definition** A **Banach algebra** is a Banach space $\mathcal{A}$ with a vector multiplication s.t.

\[ ||x \cdot y|| \leq ||x|| \cdot ||y||, \ x, y \in \mathcal{A}. \]
A Banach algebra from a LC group

- $G$: Locally compact group
  $\Rightarrow \exists \mu$: A (left) Haar measure - (left) translation invariant measure (a generalization of the Lebesgue measure)

- (Convolution)
  $L^1(G) = L^1(G, \mu)$,
  $$f \ast g(x) = \int_G f(y)g(x^{-1}y)d\mu(y), \quad f, g \in L^1(G).$$

- The space $(L^1(G), \ast)$ is a Banach algebra, i.e. we have
  $$\|f \ast g\|_1 \leq \|f\|_1 \cdot \|g\|_1, \quad f, g \in L^1(G).$$

  This algebra is called the convolution algebra on $G$.

- (Wendel ’52, $L^1(G)$ “determines” $G$)
  $L^1(G) \cong L^1(H)$ isometrically isomorphic
  $\Leftrightarrow G \cong H$ as topological groups.

- For non-abelian $G$ the algebra $(L^1(G), \ast)$ is non-commutative!
Another Banach algebra from a LC group

- $G = \mathbb{R}$
- $\mathcal{F}^\mathbb{R} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$, $f \mapsto \hat{f}$, $\hat{f} \ast g = \hat{f} \cdot \hat{g}$, $\hat{f} \cdot g = \hat{f} \ast \hat{g}$
- $A(\mathbb{R}) \overset{\text{def}}{=} \mathcal{F}^\mathbb{R}(L^1(\mathbb{R}))$, $\|\hat{f}\|_{A(\mathbb{R})} \overset{\text{def}}{=} \|f\|_1$: a subalgebra of $C_0(\mathbb{R})$

- Note that $f \in L^1$ can be written $f = f_1 \cdot f_2$ with $f_1, f_2 \in L^2$ with $\|f\|_1 = \|f_1\|_2 \cdot \|f_2\|_2$, then by Plancherel’s thm

$$A(\mathbb{R}) = \{ f \ast g : f, g \in L^2(\mathbb{R}) \}$$

- For a general locally compact group we have a limited understanding of the group Fourier transform, which requires a heavy group representation theory, so that we take the last approach to define $A(G)$. 
Fourier algebras on locally compact groups

Fourier algebra [Kreĭn ’49, Stinespring ’59, Eymard ’64]

We define the Fourier algebra $A(G)$ by

$$A(G) \overset{\text{def}}{=} \{ f = g \ast \check{h} : g, h \in L^2(G) \},$$

where $\check{h}(x) = h(x^{-1})$.

$A(G)$ is equipped with the norm

$$\|f\|_{A(G)} = \inf_{f = g \ast \check{h}} \|g\|_2 \|h\|_2,$$

and is known to be a subalgebra of $C_0(G)$, so a commutative Banach algebra with respect to the pointwise multiplication.

But still $A(G)$ “in my mind” is the collection of functions whose “group Fourier transform” is in (non-commutative) $L^1$.

For abelian $G$ we do have $A(G) \cong L^1(\hat{G})$ via group Fourier transform, where $\hat{G}$ is the “dual group” of $G$. In this sense we regard $A(G)$ as the “dual object” of $L^1(G)$. 
Fourier algebra: continued

- (Walters ‘72, $A(G)$ “determines” $G$)
  $A(G) \cong A(H)$ isometrically isomorphic $\iff G \cong H$.
- The algebra $A(G)$ has the advantage of being commutative.
- (Def) $\mathcal{A}$: a commutative Banach algebra
  $\text{Spec} \mathcal{A} = \{\varphi : \mathcal{A} \to \mathbb{C}, \text{ non-zero, multiplicative, linear}\}$
- (Eymard ‘64, Gelfand spectrum “detects” $G$)
  $\text{Spec} A(G) \cong G$ as topological spaces.
- The difficulty lies in the calculation of the Fourier algebra norm.
Extracting information of $G$ from $L^1(G)$: Amenability

- (Def, von Neumann) $G$ is called **amenable** if there is a mean $\varphi$ on $L^\infty(G)$, i.e. $\varphi \in (L^\infty(G))^*$ satisfying (1) (positive) $\varphi(f) \geq 0$ for $f \geq 0$, (2) (unital) $\varphi(1) = 1$, and (3) (translation invariant) $\varphi(xf) = f$, $x \in G$, where $xf(y) = f(x^{-1}y)$.

- Compact groups, abelian groups are amenable and $\mathbb{F}_n$, $n \geq 2$ are non-amenable.

- (Def, Johnson ’72) A Banach algebra $\mathcal{A}$ is called **amenable** if any bounded derivation $D : \mathcal{A} \to X^*$ for a $\mathcal{A}$-bimodule $X$ is **inner**, i.e. $\exists \psi \in X^*$ s.t. $D(a) = a \cdot \psi - \psi \cdot a$.

- (Johnson ’72) $L^1(G)$ is amenable $\iff$ $G$ is amenable.
Extracting information of $G$ from $A(G)$: Amenability

- **(Question)** $A(G)$ amenable $\iff$ $G$ amenable?
- **(Johnson ’94)** $A(SU(2))$ is not amenable.
- **(Ruan, ’95)** $A(G)$ amenable in the operator space category $\iff G$ amenable.
- Canonical operator space structure on $A(G)$ is quite informative.

$$A(G \times H) \cong A(G) \hat{\otimes} A(H) \not\cong A(G) \otimes_\pi A(H),$$

where $\hat{\otimes}$ and $\otimes_\pi$ are projective tensor products of operator spaces and Banach spaces, resp.
- Note that the canonical operator space structure on $L^1(G)$ is not so new.
Spectrum of weighted Fourier algebras and complexification of Lie groups

- *(Question)* Recall $\text{Spec} A(G) \cong G$. Now we make $A(G)$ weighted and investigate its spectrum. What can we say about the underlying group $G$?

- *(Answer)* We can say something about the complexification of $G$!

- (1) Weighted $A(G)$? (2) complexification of $G$?
Simplest case $G = \mathbb{R}$

- Recall $A(\mathbb{R}) \cong L^1(\hat{\mathbb{R}})$. We call $w : \hat{\mathbb{R}} \cong \mathbb{R} \to (0, \infty)$ a weight if it is sub-multiplicative. Then, $A(\mathbb{R}, w) := L^1(\hat{\mathbb{R}}, w)$ is still a Banach algebra w.r.t. convolution (we call this a weighted convolution algebra or a Beurling algebra).

- Examples of weights:
  1. $w_\alpha(t) = (1 + |t|)^\alpha, \quad t \in \mathbb{R}, \quad \alpha \geq 0$
  2. $w_\beta(t) = \beta|t|, \quad t \in \mathbb{R}, \quad \beta \geq 1$.

- $\text{Spec}A(\mathbb{R}, w_\beta) \cong \{ c \in \mathbb{C} : |\text{Im}c| \leq \log \beta \}$

- The proof uses $C_c(\hat{\mathbb{R}}) \subseteq A(\mathbb{R}, w_\beta)$ and solve a Cauchy functional equation to get the conclusion.
Another simple case $G = \mathbb{T}$

- Recall $A(\mathbb{T}) \cong L^1(\mathbb{Z})$. We call $w : \mathbb{Z} \cong \mathbb{R} \to (0, \infty)$ a **weight** if it is sub-multiplicative. Then, $A(\mathbb{T}, w) := L^1(\mathbb{Z}, w)$ is still a Banach algebra w.r.t. convolution.

- Examples of weights:
  1. $w_\alpha(n) = (1 + |n|)^\alpha, \ n \in \mathbb{Z}, \ \alpha \geq 0$
  2. $w_\beta(n) = \beta^{|n|}, \ n \in \mathbb{Z}, \ \beta \geq 1$.

- $\text{Spec} A(\mathbb{T}, w_\beta) \cong \{c \in \mathbb{C} : \frac{1}{\log \beta} \leq |c| \leq \log \beta\}$

- The proof uses $\text{Trig}(\mathbb{T}) \subseteq A(\mathbb{T}, w_\beta)$ and note that $\text{Trig}(\mathbb{T})$ is singly generated to get the conclusion.
The case of a general Lie group $G$

- **(Weighted $A(G)$?)** There is a natural way of constructing “sub-multiplicative” weights on the “dual of $G$” extending the ones from the commutative Lie subgroups of $G$ (or from the Lie derivative directions).

- **(Complexification of $G$?)** $G$ has its Lie algebra $\mathfrak{g}$, which can be easily complexified $\mathfrak{g}_\mathbb{C}$. If there is a “canonical” Lie group whose Lie algebra is $\mathfrak{g}_\mathbb{C}$, then we call it the complexification of $G$ with the notation $G_\mathbb{C}$. For example, $\mathbb{R}_\mathbb{C} = \mathbb{C}$, $\mathbb{T}_\mathbb{C} = \mathbb{C}\setminus\{0\}$, $SU(2)_\mathbb{C} = SL(2, \mathbb{C})$.

- **(Conjecture)** $\text{Spec}A(G, w) \subseteq G_\mathbb{C}$ and it reflects the structure of $G_\mathbb{C}$ in a various way (e.g. translation, automorphism).

- The conjecture is somehow true for compact (Lie) groups and for some non-compact Lie groups (Heisenberg group, Euclidean motion group).
The case of Heisenberg group $H_1$

- Heisenberg group $H_1 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \cong \mathbb{R}^3$.

- (The complexification of $H_1$)

  $$(H_1)_\mathbb{C} = \left\{ (x, y, z) = \begin{bmatrix} 1 & x & z \\ 1 & y & 1 \end{bmatrix} : x, y, z \in \mathbb{C} \right\} \cong \mathbb{C}^3.$$

- (L./Spronk, preprint) Let $w_\beta$ be the exponential weight extended from the subgroup $Y = \{(0, y, 0) : y \in \mathbb{R}\}$, then

  $$\text{Spec} A(H_1, w_\beta) \cong \{(x, y, z) \in (H_1)|_{\mathbb{C}} \cong \mathbb{C}^3 : \text{Im} x = \text{Im} z = 0, |\text{Im} y| \leq \log \beta\}.$$
The case of Heisenberg group $H_1$: ingredients

- $\hat{H}_1$: irreducible unitary representations on $H_1$.
- Very careful choice of a subalgebra (or a subspace) of $A(H_1, w_\beta)$ replacing $C_c(H_1)$!
- Solving Cauchy functional equation for distributions!
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Group Fourier transform of $H_1$

- For any $r \in \mathbb{R}\setminus\{0\}$ we have the Schrödinger representation
  \[
  \pi^r(x, y, z)\xi(w) = e^{2\pi ir(-wy+z)}\xi(-x + w), \quad \xi \in L^2(\mathbb{R}).
  \]

- For $f \in L^1(H_1)$ and $r \in \mathbb{R}\setminus\{0\}$ we define $\widehat{f}^{H_1}$ by
  \[
  \widehat{f}^{H_1}(r) := \int_{H_1} f(x, y, z)\pi^r(x, y, z)d\mu(x, y, z) \in B(L^2(\mathbb{R})).
  \]

- We have
  \[
  A(H_1) \cong L^1(\mathbb{R}\setminus\{0\}, |r|dr; S^1(L^2(\mathbb{R}))),
  \]

  where $S^1(\mathcal{H})$ is the trace class on $\mathcal{H}$. 

The case of $E(2)$, the Euclidean motion group

- The group $E(2)$ and its complexification $E(2)_\mathbb{C}$ are

  $$E(2) = \left\{ \begin{bmatrix} e^{i\alpha} & z \\ 0 & 1 \end{bmatrix} : \alpha \in [0, 2\pi], z \in \mathbb{C} \right\}$$

  $$= \{ (\alpha, z_1, z_2) : \alpha \in [0, 2\pi], z_1, z_2 \in \mathbb{R} \} \cong \mathbb{T} \times \mathbb{R}^2.$$ 

  $$E(2)_\mathbb{C} = \{ (\alpha, z_1, z_2) : \alpha, z_1, z_2 \in \mathbb{C} \} \cong \mathbb{C}^3.$$ 

- For $a > 0$ we consider the representation

  $$\pi^a(\alpha, z) F(\theta) = e^{ia(z_1 \cos \theta + z_2 \sin \theta)} F(\theta - \alpha), \quad F \in L^2[0, 2\pi].$$

- (Group Fourier transform on $E(2)$)

  For $f \in L^1(E(2))$ and $a > 0$ we define

  $$\hat{f}^{E(2)}(a) = \int_{E(2)} f(\alpha, z) \pi^a(\alpha, z) d\mu(\alpha, z).$$

- $A(E(2)) \cong L^1(\mathbb{R}^+, ada; S^1(L^2[0, 2\pi])).$
The case of $E(2)$: Results

- (L./Spronk, preprint) Let $w_\beta$ is the exponential weight coming from the subgroup $\{ (\alpha, 0) : \alpha \in [0, 2\pi] \}$, then we have

$$\text{Spec} A(G; w_\beta) \cong \{ (\alpha, z_1, z_2) \in G_C : \text{Im} z_1 = \text{Im} z_2 = 0, |\text{Im} \alpha| \leq \log \beta \}.$$
Thank you for your attention!