Fourier algebras on locally compact groups

Hun Hee Lee,
Seoul National University

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Locally compact groups and Banach algebras

- (**Def**) A **locally compact (LC) group** is a topological group whose topology is locally compact and Hausdorff.
- Examples of LC groups: (1) Lie groups such as $\mathbb{T}$, $\mathbb{R}$, $SU(2)$, Heisenberg groups, $\cdots$) (2) discrete group, i.e. any groups with discrete topology.
- (**Def**) A **Banach algebra** is a Banach space $\mathcal{A}$ with a vector multiplication s.t.

$$||x \cdot y|| \leq ||x|| \cdot ||y||, \; x, y \in \mathcal{A}.$$

A Banach algebra from a LC group

- G: Locally compact group
  $\Rightarrow \exists \mu$: A (left) Haar measure - (left) translation invariant measure (a generalization of the Lebesgue measure)
- (**Convolution**)
  $L^1(G) = L^1(G, \mu)$,
  $f * g(x) = \int_G f(y)g(x^{-1}y)d\mu(y),\ f, g \in L^1(G).$
- The space $(L^1(G), *)$ is a Banach algebra, i.e. we have

$$||f * g||_1 \leq ||f||_1 \cdot ||g||_1,\ f, g \in L^1(G).$$

  This algebra is called the **convolution algebra on** G.
- (**Wendel '52**, $L^1(G)$ "determines" G)
  $L^1(G) \cong L^1(H)$ isometrically isomorphic
  $\Leftrightarrow G \cong H$ as topological groups.
- For non-abelian G the algebra $(L^1(G), *)$ is non-commutative!

Another Banach algebra from a LC group

- $G = \mathbb{R}$
  $\mathcal{F}^{\mathbb{R}} : L^1(\mathbb{R}) \to C_0(\mathbb{R}),\ f \mapsto \hat{f},\ \widehat{f * g} = \hat{f} \cdot \hat{g},\ \widehat{f \cdot g} = \hat{f} * \hat{g}$
  $A(\mathbb{R}) \overset{\text{def}}{=} \mathcal{F}^{\mathbb{R}}(L^1(\mathbb{R})),\ ||\hat{f}||_{A(\mathbb{R})} \overset{\text{def}}{=} ||f||_1$: a subalgebra of $C_0(\mathbb{R})$
- Note that $f \in L^1$ can be written $f = f_1 \cdot f_2$ with $f_1, f_2 \in L^2$ with $||f||_1 = ||f_1||_2 \cdot ||f_2||_2$, then by Plancherel's thm

$$A(\mathbb{R}) = \{f * g : f, g \in L^2(\mathbb{R})\}$$

- For a general locally compact group we have a limited understanding of the **group Fourier transform**, which requires a heavy group representation theory, so that we take the last approach to define $A(G)$.

Fourier algebra [Kreĭn '49, Stinespring '59, Eymard '64]

- We defien the Fourier algebra $A(G)$ by

$$A(G) \stackrel{\text{def}}{=} \{f = g * \check{h} : g, h \in L^2(G)\},$$

  where $\check{h}(x) = h(x^{-1})$.
- $A(G)$ is equipped with the norm

$$||f||_{A(G)} = \inf_{f = g * \check{h}} ||g||_2 ||h||_2,$$

  and is known to be a subalgebra of $C_0(G)$, so a commutative Banach algebra with respect to the pointwise multiplication.
- But still $A(G)$ "in my mind" is the collection of functions whose "group Fourier transform" is in (non-commutative) L^1.
- For abelian G we do have $A(G) \cong L^1(\widehat{G})$ via group Fourier transform, where $\widehat{G}$ is the "dual group" of G. In this sense we regard $A(G)$ as the "dual object" of $L^1(G)$.

Fourier algebra: continued

- (**Walters '72**, $A(G)$ "determines" G)
  $A(G) \cong A(H)$ isometrically isomorphic $\Leftrightarrow G \cong H$.
- The algebra $A(G)$ has the advantage of being commutative.
- (**Def**) $\mathcal{A}$: a commutative Banach algebra
  $\text{Spec}\mathcal{A} = \{\varphi : \mathcal{A} \to \mathbb{C}, \text{ non-zero, multiplicative, linear}\}$
- (**Eymard '64**, Gelfand spectrum "detects" G)
  $\text{Spec}A(G) \cong G$ as topological spaces.
- The difficulty lies in the calculation of the Fourier algebra norm.

Extracting information of G from $L^1(G)$: Amenability

- (**Def, von Neumann**) G is called **amenable** if there is a mean φ on $L^\infty(G)$, i.e. $\varphi \in (L^\infty(G))^*$ satisfying (1) (positive) $\varphi(f) \geq 0$ for $f \geq 0$, (2) (unital) $\varphi(1) = 1$, and (3) (translation invariant) $\varphi({}_xf) = f$, $x \in G$, where ${}_xf(y) = f(x^{-1}y)$.
- Compact groups, abelian groups are amenable and $\mathbb{F}_n$, $n \geq 2$ are non-amenable.
- (**Def, Johnson '72**) A Banach algebra $\mathcal{A}$ is called **amenable** if any bounded derivation $D : \mathcal{A} \to X^*$ for a $\mathcal{A}$-bimodule X is **inner**, i.e. $\exists \psi \in X^*$ s.t. $D(a) = a \cdot \psi - \psi \cdot a$.
- (**Johnson '72**) $L^1(G)$ is amenable $\Leftrightarrow$ G is amenable.

Extracting information of G from $A(G)$: Amenability

- (**Question**) $A(G)$ amenable $\Leftrightarrow$ G amenable?
- (**Johnson '94**) $A(SU(2))$ is not amenable.
- (**Ruan, '95**) $A(G)$ amenable in the **operator space** category $\Leftrightarrow$ G amenable.
- Canonical operator space structure on $A(G)$ is quite informative.

  $$A(G \times H) \cong A(G) \widehat{\otimes} A(H) \ncong A(G) \otimes_\pi A(H),$$

  where $\widehat{\otimes}$ and $\otimes_\pi$ are projective tensor products of operator spaces and Banach spaces, resp.
- Note that the canonical operator space structure on $L^1(G)$ is not so new.

Spectrum of weighted Fourier algebras and complexification of Lie groups

- (**Question**) Recall Spec$A(G) \cong G$. Now we make $A(G)$ weighted and investigate its spectrum. What can we say about the underlying group G??
- (**Answer**) We can say something about the complexification of G!
- (1) Weighted $A(G)$? (2) complexification of G?

Simplest case $G = \mathbb{R}$

- Recall $A(\mathbb{R}) \cong L^1(\widehat{\mathbb{R}})$. We call $w : \widehat{\mathbb{R}} \cong \mathbb{R} \to (0, \infty)$ a **weight** if it is sub-multiplicative. Then, $A(\mathbb{R}, w) := L^1(\widehat{\mathbb{R}}, w)$ is still a Banach algebra w.r.t. convolution (we call this a weighted convolution algebra or a Beurling algebra).
- Examples of weights:
  (1) $w_\alpha(t) = (1 + |t|)^\alpha$, $t \in \mathbb{R}$, $\alpha \geq 0$
  (2) $w_\beta(t) = \beta^{|t|}$, $t \in \mathbb{R}$, $\beta \geq 1$.
- $\mathrm{Spec} A(\mathbb{R}, w_\beta) \cong \{c \in \mathbb{C} : |\mathrm{Im} c| \leq \log \beta\}$
- The proof uses $C_c(\widehat{\mathbb{R}}) \subseteq A(\mathbb{R}, w_\beta)$ and solve a Cauchy functional equation to get the conclusion.

Another simple case $G = \mathbb{T}$

- Recall $A(\mathbb{T}) \cong L^1(\mathbb{Z})$. We call $w : \mathbb{Z} \cong \mathbb{R} \to (0, \infty)$ a **weight** if it is sub-multiplicative. Then, $A(\mathbb{T}, w) := L^1(\mathbb{Z}, w)$ is still a Banach algebra w.r.t. convolution.
- Examples of weights:
  (1) $w_\alpha(n) = (1 + |n|)^\alpha$, $n \in \mathbb{Z}$, $\alpha \geq 0$
  (2) $w_\beta(n) = \beta^{|n|}$, $n \in \mathbb{Z}$, $\beta \geq 1$.
- $\mathrm{Spec} A(\mathbb{T}, w_\beta) \cong \{c \in \mathbb{C} : \frac{1}{\log \beta} \leq |c| \leq \log \beta\}$
- The proof uses $\mathrm{Trig}(\mathbb{T}) \subseteq A(\mathbb{T}, w_\beta)$ and note that $\mathrm{Trig}(\mathbb{T})$ is singly generated to get the conclusion.

The case of a general Lie group G

- (**Weighted** $A(G)$**?**) There is a natural way of constructing "sub-multiplicative" weights on the "dual of G" extending the ones from the commutative Lie subgroups of G (or from the Lie derivative directions).
- (**Complexification of** G**?**) G has its Lie algebra $\mathfrak{g}$, which can be easily complexified $\mathfrak{g}_{\mathbb{C}}$. If there is a "canonical" Lie group whose Lie algebra is $\mathfrak{g}_{\mathbb{C}}$, then we call it the complexification of G with the notation $G_{\mathbb{C}}$. For example, $\mathbb{R}_{\mathbb{C}} = \mathbb{C}$, $\mathbb{T}_{\mathbb{C}} = \mathbb{C}\backslash\{0\}$, $SU(2)_{\mathbb{C}} = SL(2, \mathbb{C})$.
- (**Conjecture**) $\mathrm{Spec}A(G, w) \subseteq G_{\mathbb{C}}$ and it reflects the structure of $G_{\mathbb{C}}$ in a various way (e.g. translation, automorphism).
- The conjecture is somehow true for compact (Lie) groups and for some non-compact Lie groups (Heisenberg group, Euclidean motion group).

The case of Heisenberg group H_1

- Heisenberg group $H_1 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \cong \mathbb{R}^3$.
- (**The complexification of** H_1)
  $(H_1)_{\mathbb{C}} = \left\{ (x, y, z) = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x, y, z \in \mathbb{C} \right\} \cong \mathbb{C}^3$.
- (**L./Spronk, preprint**) Let w_β be the exponential weight extended from the subgroup $Y = \{(0, y, 0) : y \in \mathbb{R}\}$, then

$$\text{Spec}A(H_1, w_\beta) \cong \{(x, y, z) \in (H_1)|_{\mathbb{C}} \cong \mathbb{C}^3 \\ : \text{Im}x = \text{Im}z = 0, |\text{Im}y| \leq \log \beta\}.$$

Other subgroups have similar results.

The case of Heisenberg group H_1: ingredients

- $\widehat{H_1}$: irreducible unitary representations on H_1.
- Very careful choice of a subalgebra (or a subspace) of $A(H_1, w_\beta)$ replacing $C_c(H_1)$!
- Solving Cauchy functional equation for distributions!

Group Fourier transform of H_1

- For any $r \in \mathbb{R}\backslash\{0\}$ we have the Schrödinger representation

$$\pi^r(x,y,z)\xi(w) = e^{2\pi i r(-wy+z)}\xi(-x+w),\ \xi \in L^2(\mathbb{R}).$$

- For $f \in L^1(H_1)$ and $r \in \mathbb{R}\backslash\{0\}$ we define $\widehat{f}^{H_1}$ by

$$\widehat{f}^{H_1}(r) := \int_{H_1} f(x,y,z)\pi^r(x,y,z)d\mu(x,y,z) \in B(L^2(\mathbb{R})).$$

- We have

$$A(H_1) \cong L^1(\mathbb{R}\backslash\{0\}, |r|dr; S^1(L^2(\mathbb{R}))),$$

where $S^1(\mathcal{H})$ is the trace class on $\mathcal{H}$.

The case of $E(2)$, the Euclidean motion group

- The group $E(2)$ and its complexification $E(2)_{\mathbb{C}}$ are

$$
\begin{aligned}
E(2) &= \left\{ \begin{bmatrix} e^{i\alpha} & z \\ 0 & 1 \end{bmatrix} : \alpha \in [0, 2\pi], z \in \mathbb{C} \right\} \\
&= \{(\alpha, z_1, z_2) : \alpha \in [0, 2\pi], z_1, z_2 \in \mathbb{R}\} \cong \mathbb{T} \times \mathbb{R}^2. \\
E(2)_{\mathbb{C}} &= \{(\alpha, z_1, z_2) : \alpha, z_1, z_2 \in \mathbb{C}\} \cong \mathbb{C}^3.
\end{aligned}
$$

- For $a > 0$ we consider the representation $\pi^a(\alpha, z)F(\theta) = e^{ia(z_1 \cos\theta + z_2 \sin\theta)} F(\theta - \alpha),\ F \in L^2[0, 2\pi]$.
- **(Group Fourier transform on $E(2)$)** For $f \in L^1(E(2))$ and $a > 0$ we define

$$\widehat{f}^{E(2)}(a) = \int_{E(2)} f(\alpha, z)\pi^a(\alpha, z)d\mu(\alpha, z).$$

- $A(E(2)) \cong L^1(\mathbb{R}^+, ada; S^1(L^2[0, 2\pi]))$.

The case of $E(2)$: Results

- **(L./Spronk, preprint)** Let w_β is the exponential weight coming from the subgroup $\{(\alpha, 0) : \alpha \in [0, 2\pi]\}$, then we have

$$\mathrm{Spec}A(G; w_\beta) \cong$$
$$\{(\alpha, z_1, z_2) \in G_{\mathbb{C}} : \mathrm{Im} z_1 = \mathrm{Im} z_2 = 0, |\mathrm{Im}\alpha| \le \log\beta\}.$$

Thank you for your attention!