The products of Hankel operators

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Motivation

Consider two $N \times N$ Hankel matrices

$$B_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2N-2} \\ a_N & \cdots & a_{2N-2} & a_{2N-1} \end{pmatrix}, \; B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{2N-2} \\ b_N & \cdots & b_{2N-2} & b_{2N-1} \end{pmatrix}$$

When $B_1 B_2 = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-N+1} \\ c_1 & c_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{-1} \\ c_{N-1} & \cdots & c_1 & c_0 \end{pmatrix}$: Toeplitz matrix ?

We first discuss this question in the infinite-dimensional case and answer this question.

Definitions

H^2: the Hardy space on $\mathbb{T}$.

P: the orthogonal projection from L^2 onto H^2.

$J : (H^2)^\perp \to H^2$ is given by $J(z^{-n}) = z^{n-1}$, $n \geq 1$.

Definition

Given $\varphi \in L^\infty$, define the *Toeplitz operator* T_φ and the *Hankel operator* H_φ on H^2 by

$$f \in H^2 \quad \Rightarrow \quad T_\varphi f := P(\varphi f), \quad H_\varphi f := J(I-P)(\varphi f).$$

Note that $I - P$ is the orthogonal projection from L^2 onto $(H^2)^\perp$.

Let $\varphi = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty$ and $H^2 = \vee\{1, z, z^2, \cdots\}$. Then

$$T_\varphi = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \\ a_1 & a_0 & a_{-1} & \ddots \\ a_2 & a_1 & \ddots & \ddots \\ & \ddots & \ddots & \end{pmatrix}, H_\varphi = \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & \ddots & \\ a_{-3} & \ddots & \ddots & \\ \vdots & & & \end{pmatrix}$$

It is interesting to ask that

1. $T_\varphi T_\psi = T_\Phi \quad \Longleftrightarrow \quad ?$
2. $H_\varphi H_\psi = T_\Phi \quad \Longleftrightarrow \quad ?$

They are solved as

1. $T_\varphi T_\psi = T_\Phi \quad \Longleftrightarrow$ either ψ or $\overline{\varphi}$ is analytic. In this case $\Phi = \varphi\psi$,
2. $H_\varphi H_\psi \neq T_\Phi$ for any $\varphi, \psi \in L^\infty$.

For an inner function u on $\mathbb{T}$, let

$$K_u^2 := H^2 \ominus uH^2.$$

For example,

$u = z^N \implies uH^2 = \vee\{z^N, z^{N+1}, \cdots\}$ so that

$$K_u^2 = \vee\{1, z, z^2, \cdots, z^{N-1}\}.$$

Let P_u be the orthogonal projection from H^2 onto K_u^2.

Definition

Given $\varphi \in L^\infty$, define the *truncated Toeplitz operator(TTO)* A_φ and the *truncated Hankel operator(THO)* B_φ on K_u^2 by

$$f \in K_u^2 \quad \Rightarrow \quad A_\varphi f := P_u(T_\varphi f), \quad B_\varphi f := P_u(H_\varphi f).$$

Suppose $u = z^N$ and $\varphi = \sum_{n=-\infty}^{\infty} a_n z^n$. Then

$$A_\varphi = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{1-N} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{N-1} & \cdots & a_1 & a_0 \end{pmatrix}, B_\varphi = \begin{pmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \\ a_{-2} & a_{-3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2-2N} \\ a_{-N} & \cdots & a_{2-2N} & a_{1-2N} \end{pmatrix}$$

However, if u is a general inner function (ex: $u = \prod_{i=1}^{N} \frac{z-\lambda_i}{1-\overline{\lambda_i} z}$, $\lambda \in \mathbb{D}$), we cannot guarantee the Toeplitz or Hankel matrix form.

Notations

For $f \in K_u^2$, define $\tilde{f}$ by

$$\tilde{f}(z) := u(z)\overline{zf(z)}.$$

Then $\sim$ is a conjugate linear isometry of K_u^2 onto itself.

For $\lambda \in \mathbb{D}$, let $k_\lambda^u(z) := \frac{1-\overline{u(\lambda)}u(z)}{1-\overline{\lambda}z} \in K_u^2$. Then

1. $\langle f, k_\lambda^u \rangle = f(\lambda)$ for $f \in K_u^2$,
2. $\widetilde{k_\lambda^u} = \frac{u(z)-u(\lambda)}{z-\lambda} \in K_u^2$.

In particular, $u = z^N$, $\lambda = 0 \implies k_0^u = 1$, $\widetilde{k_0^u} = z^{N-1}$.

Let $S_u := A_z$ the compression of the unilateral shift on H^2 onto K_u^2.

For $c \in \mathbb{C}$, let $S_{u,c} := S_u + c(k_0^u \otimes \widetilde{k_0^u})$.

If $u = z^N$, then $S_{u,c}$ is of the form

$$\begin{pmatrix} 0 & & & c \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in M_N.$$

Theorem (Sarason, 2007)

$A \in \mathcal{L}(K_u^2)$ *is TTO* $\iff$ *there exist* $f, g \in K_u^2$ *such that*

$$A - S_u A S_u^* = (f \otimes k_0^u) + (k_0^u \otimes g).$$

Theorem (Gu, 2014)

$B \in \mathcal{L}(K_u^2)$ *is THO* $\iff$ *there exist* $f, g \in K_u^2$ *such that*

$$B - S_u^* B S_u^* = (f \otimes k_0^u) + (\widetilde{k_0^u} \otimes g).$$

The case of TTO

Indeed, it is natural to ask

if A_1, A_2 are Toeplitz matrices (or generally TTO),

$A_1 A_2$ is Toeplitz matrix (or TTO) $\quad\Longleftrightarrow\quad$?

N. Sedlock gave the nice solution to this question.

Theorem (Sedlock, 2011)

Suppose A_1, A_2 *are TTO such that* A_1 *or* $A_2 \neq \lambda I$. *Then*

$$A_1 A_2 : \textit{TTO} \iff A_1, A_2 \in \{S_{u,c}\}' \text{ for some } c \in \mathbb{C}.$$

In this case, if $u = z^N$, then A_1, A_2 are of the form

$$\begin{pmatrix} a_0 & ca_{N-1} & \cdots & ca_1 \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & ca_{N-1} \\ a_{N-1} & \cdots & a_1 & a_0 \end{pmatrix}, \quad c \in \mathbb{C}.$$

which is called the "generalized circulant form".

Main Question

Now, let me go back to main question.

Question

Suppose B_1, B_2 are Hankel matrices (or generally THO).
$B_1 B_2$ is Toeplitz matrix (or TTO) $\iff$?

Lemma

For a function f, define $\hat{f}$ by $\hat{f}(z) = \overline{f(\overline{z})}$.

For example, if $f(z) = a_0 + a_1 z + \cdots$, then $\hat{f}(z) = \overline{a_0} + \overline{a_1} z + \cdots$.

Lemma (Gu, 2014)

Suppose $u = \hat{u}$ and $D := B_{\bar{u}}$. Then

1. D is self-adjoint and unitary,
2. if $B \in \mathcal{L}(K_u^2)$ is THO, then both BD and DB are TTO.

Theorem (Main)

Suppose B_1, B_2 are THO such that B_1 or $B_2 \neq \lambda D$. If $u = \hat{u}$, then

$$B_1 B_2 : \text{ TTO } \Leftrightarrow S_{u,c} B_1 = B_1 S^*_{u,\bar{c}},\ S^*_{u,\bar{c}} B_2 = B_2 S_{u,c} \text{ for some } c \in \mathbb{C}.$$

In this case, the shape of the symbol of the THO B_1, B_2 and the resulting TTO $B_1 B_2$ can be concretely determined.

Indeed, there exist $f_1, f_2 \in K_u^2$ such that $B_1 = B_{\overline{u}\phi_1(\overline{z})}$, where $\phi_1 = f_1 + \frac{c}{1+cu(0)} \overline{S_u \tilde{f}_1}$ and $B_2 = B_{\overline{u}\phi_2}$, where $\phi_2 = f_2 + \frac{c}{1+cu(0)} \overline{S_u \tilde{f}_2}$. Moreover, if $B_1 B_2 = A$, then the symbol of A is $A_{\phi_1} f_2$.

Finally, let me introduce the answer of the initial question.

Assume $u = z^N$ and write $B_1 = B_\varphi$ and $B_2 = B_\psi$, where $\varphi = \sum_{n=-\infty}^{\infty} a_{-n} z^n$, $\psi = \sum_{n=-\infty}^{\infty} b_{-n} z^n$. Then

$$B_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2N-2} \\ a_N & \cdots & a_{2N-2} & a_{2N-1} \end{pmatrix}, B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{2N-2} \\ b_N & \cdots & b_{2N-2} & b_{2N-1} \end{pmatrix}.$$

Theorem (Matrix version)

Suppose B_1, B_2 *are Hankel matrices as the above. Then*
B_1B_2 : *Toeplitz matrix* $\iff$

$$B_1 = \begin{pmatrix} 0 & 0 & \cdots & a_N \\ 0 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2N-2} \\ a_N & \cdots & a_{2N-2} & a_{2N-1} \end{pmatrix}, \; B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_N & \cdots & 0 & 0 \end{pmatrix}$$

or

$$B_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_2 & & \ddots & \frac{1}{c}a_1 \\ \vdots & \ddots & \ddots & \vdots \\ a_N & \frac{1}{c}a_1 & \cdots & \frac{1}{c}a_{N-1} \end{pmatrix}, \; B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & & \ddots & cb_1 \\ \vdots & \ddots & \ddots & \vdots \\ b_N & cb_1 & \cdots & cb_{N-1} \end{pmatrix}.$$

We can say the second form is the "generalized skew-circulant form".

Thank you !