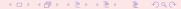
The products of Hankel operators

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Motivation

Consider two $N \times N$ Hankel matrices

$$B_{1} = \begin{pmatrix} a_{1} & a_{2} & \cdots & a_{N} \\ a_{2} & a_{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2N-2} \\ a_{N} & \cdots & a_{2N-2} & a_{2N-1} \end{pmatrix}, B_{2} = \begin{pmatrix} b_{1} & b_{2} & \cdots & b_{N} \\ b_{2} & b_{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{2N-2} \\ b_{N} & \cdots & b_{2N-2} & b_{2N-1} \end{pmatrix}$$

When
$$B_1B_2=\begin{pmatrix}c_0&c_{-1}&\cdots&c_{-N+1}\\c_1&c_0&\ddots&\vdots\\\vdots&\ddots&\ddots&c_{-1}\\c_{N-1}&\cdots&c_1&c_0\end{pmatrix}$$
 : Toeplitz matrix ?

We first discuss this question in the infinite-dimensional case and answer this question.



Definitions

 H^2 : the Hardy space on \mathbb{T} .

P: the orthogonal projection from L^2 onto H^2 .

 $J:(H^2)^\perp\to H^2$ is given by $J(z^{-n})=z^{n-1},\,n\ge 1.$

Definition

Given $\varphi\in L^\infty$, define the *Toeplitz operator* T_φ and the *Hankel operator* H_φ on H^2 by

$$f \in H^2 \quad \Rightarrow \quad T_{\varphi}f := P(\varphi f), \quad H_{\varphi}f := J(I - P)(\varphi f).$$

Note that I-P is the orthogonal projection from L^2 onto $(H^2)^{\perp}$.



Let
$$\varphi = \sum_{n=-\infty}^\infty a_n z^n \in L^\infty$$
 and $H^2 = \vee \{1, z, z^2, \cdots \}$. Then

$$T_{\varphi} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} \\ a_1 & a_0 & a_{-1} & \ddots \\ a_2 & a_1 & \ddots & \ddots \\ & \ddots & \ddots & \end{pmatrix}, \ H_{\varphi} = \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & \ddots & \\ a_{-3} & \ddots & \ddots & \\ \vdots & & & & \vdots \end{pmatrix}$$

It is interesting to ask that

They are solved as

- $\bullet \ \, T_{\varphi}T_{\psi}=T_{\Phi} \quad \Longleftrightarrow \quad \text{either ψ or $\overline{\varphi}$ is analytic. In this case $\Phi=\varphi\psi$,}$
- \bullet $H_{\varphi}H_{\psi}\neq T_{\Phi}$ for any $\varphi,\,\psi\in L^{\infty}$.

Definitions of TTO and THO

For an inner function u on \mathbb{T} , let

$$K_u^2:=H^2\ominus uH^2.$$

For example,

$$u=z^N \implies uH^2=\vee\{z^N,z^{N+1},\cdots\}$$
 so that
$$K_u^2=\vee\{1,z,z^2,\cdots,z^{N-1}\}.$$

Let P_u be the orthogonal projection from H^2 onto K_u^2 .

Definition

Given $\varphi \in L^{\infty}$, define the truncated Toeplitz operator(TTO) A_{φ} and the truncated Hankel operator(THO) B_{φ} on K_u^2 by

$$f \in K_u^2 \quad \Rightarrow \quad A_{\varphi}f := P_u(T_{\varphi}f), \quad B_{\varphi}f := P_u(H_{\varphi}f).$$

Suppose
$$u=z^N$$
 and $\varphi=\sum_{n=-\infty}^\infty a_n z^n.$ Then

$$A_{\varphi} = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{1-N} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{N-1} & \cdots & a_1 & a_0 \end{pmatrix}, B_{\varphi} = \begin{pmatrix} a_{-1} & a_{-2} & \cdots & a_{-N} \\ a_{-2} & a_{-3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2-2N} \\ a_{-N} & \cdots & a_{2-2N} & a_{1-2N} \end{pmatrix}$$

However, if u is a general inner function (ex: $u = \prod_{i=1}^N \frac{z - \lambda_i}{1 - \overline{\lambda_i} z}$, $\lambda \in \mathbb{D}$), we cannot guarantee the Toeplitz or Hankel matrix form.

Notations

For $f \in K_u^2$, define \tilde{f} by

$$\tilde{f}(z) := u(z)\overline{zf(z)}.$$

Then \sim is a conjugate linear isometry of K_n^2 onto itself.

For $\lambda\in\mathbb{D}$, let $k^u_\lambda(z):=rac{1-\overline{u(\lambda)}u(z)}{1-ar{\lambda}z}\in K^2_u.$ Then

- $\ \ {}^{2}\ \widetilde{k_{\lambda}^{u}}=\tfrac{u(z)-u(\lambda)}{z-\lambda}\in K_{u}^{2}.$

In particular, $u=z^N,\,\lambda=0\implies k_0^u=1,\,\widetilde{k_0^u}=z^{N-1}.$



Let $S_u := A_z$ the compression of the unilateral shift on H^2 onto K_u^2 .

For
$$c \in \mathbb{C}$$
, let $S_{u,c} := S_u + c(k_0^u \otimes \widetilde{k_0^u})$.

If $u = z^N$, then $S_{u,c}$ is of the form

$$\begin{pmatrix} 0 & & & c \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in M_N.$$

Characterization of TTO and THO

Theorem (Sarason, 2007)

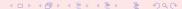
 $A \in \mathcal{L}(K_u^2)$ is TTO \iff there exist $f,g \in K_u^2$ such that

$$A - S_u A S_u^* = (f \otimes k_0^u) + (k_0^u \otimes g).$$

Theorem (Gu, 2014)

 $B \in \mathcal{L}(K_u^2)$ is THO \iff there exist $f,g \in K_u^2$ such that

$$B - S_u^* B S_u^* = (f \otimes k_0^u) + (\widetilde{k_0^u} \otimes g).$$



The case of TTO

Indeed, it is natural to ask

if $A_1,\,A_2$ are Toeplitz matrices (or generally TTO),

 A_1A_2 is Toeplitz matrix (or TTO) \iff ?

N. Sedlock gave the nice solution to this question.

Theorem (Sedlock, 2011)

Suppose A_1, A_2 are TTO such that A_1 or $A_2 \neq \lambda I$. Then

$$A_1A_2: \ TTO \iff A_1, A_2 \in \{S_{u,c}\}' \ \text{for some } c \in \mathbb{C}.$$

In this case, if $u=z^N$, then A_1,A_2 are of the form

$$\begin{pmatrix} a_0 & ca_{N-1} & \cdots & ca_1 \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & ca_{N-1} \\ a_{N-1} & \cdots & a_1 & a_0 \end{pmatrix}, \quad c \in \mathbb{C}.$$

which is called the "generalized circulant form".



Main Question

Now, let me go back to main question.

Question

Suppose B_1 , B_2 are Hankel matrices (or generally THO).

 B_1B_2 is Toeplitz matrix (or TTO) \iff ?

Lemma

For a function f, define \hat{f} by $\hat{f}(z) = \overline{f(\overline{z})}$.

For example, if $f(z)=a_0+a_1z+\cdots$, then $\hat{f}(z)=\overline{a_0}+\overline{a_1}z+\cdots$.

Lemma (Gu, 2014)

Suppose $u=\hat{u}$ and $D:=B_{\bar{u}}.$ Then

- $oldsymbol{0}$ D is self-adjoint and unitary,
- ② if $B \in \mathcal{L}(K_u^2)$ is THO, then both BD and DB are TTO.

Results

Theorem (Main)

Suppose $B_1,\ B_2$ are THO such that B_1 or $B_2 \neq \lambda D$. If $u=\hat{u}$, then $B_1B_2:\ TTO\ \Leftrightarrow\ S_{u,c}B_1=B_1S_{u,\bar{c}}^*,\ S_{u,\bar{c}}^*B_2=B_2S_{u,c}\ \text{for some }c\in\mathbb{C}.$

In this case, the shape of the symbol of the THO B_1, B_2 and the resulting TTO B_1B_2 can be concretely determined.

Indeed, there exist $f_1,f_2\in K_u^2$ such that $B_1=B_{\overline{u}\phi_1(\overline{z})}$, where $\phi_1=f_1+\frac{c}{1+cu(0)}\overline{S_u\tilde{f}_1}$ and $B_2=B_{\overline{u}\phi_2}$, where $\phi_2=f_2+\frac{c}{1+cu(0)}\overline{S_u\tilde{f}_2}$. Moreover, if $B_1B_2=A$, then the symbol of A is $A_{\phi_1}f_2$.

Finally, let me introduce the answer of the initial question.

Assume $u=z^N$ and write $B_1=B_{\varphi}$ and $B_2=B_{\psi}$, where

$$\varphi=\sum_{n=-\infty}^{\infty}a_{-n}z^n,\;\psi=\sum_{n=-\infty}^{\infty}b_{-n}z^n.\;\text{Then}$$

$$B_{1} = \begin{pmatrix} a_{1} & a_{2} & \cdots & a_{N} \\ a_{2} & a_{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2N-2} \\ a_{N} & \cdots & a_{2N-2} & a_{2N-1} \end{pmatrix}, B_{2} = \begin{pmatrix} b_{1} & b_{2} & \cdots & b_{N} \\ b_{2} & b_{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{2N-2} \\ b_{N} & \cdots & b_{2N-2} & b_{2N-1} \end{pmatrix}.$$

Theorem (Matrix version)

Suppose B_1 , B_2 are Hankel matrices as the above. Then B_1B_2 : Toeplitz matrix \iff

$$B_{1} = \begin{pmatrix} 0 & 0 & \cdots & a_{N} \\ 0 & & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{2N-2} \\ a_{N} & \cdots & a_{2N-2} & a_{2N-1} \end{pmatrix}, B_{2} = \begin{pmatrix} b_{1} & b_{2} & \cdots & b_{N} \\ b_{2} & b_{3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{N} & \cdots & 0 & 0 \end{pmatrix}$$

or

or
$$B_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_2 & & \ddots & \frac{1}{c}a_1 \\ \vdots & \ddots & \ddots & \vdots \\ a_N & \frac{1}{c}a_1 & \cdots & \frac{1}{c}a_{N-1} \end{pmatrix}, B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & & \ddots & cb_1 \\ \vdots & \ddots & \ddots & \vdots \\ b_N & cb_1 & \cdots & cb_{N-1} \end{pmatrix}.$$

We can say the second form is the "generalized skew-circulant form".



Thank you!