On Kadison's Transitive Algebra Problem: a metric property

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CONTENTS

I. INTRODUCTION

$\mathcal{H}$: a separable, infinite dimensional, complex Hilbert space.

$\mathcal{L}(\mathcal{H})$: the algebra of all bounded linear operators on $\mathcal{H}$.

$\mathbf{K} = \mathbf{K}(\mathcal{H})$: the ideal of all compact operators on $\mathcal{H}$.

$\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ is the quotent (Calkin) map.

$\sigma(T)$ $[\sigma_e(T) = \sigma(\pi(T))]$: the spectrum [essential spectrum] of T.

$\mathbb{A}$ denotes a unital norm closed, subalgebra of $\mathcal{L}(\mathcal{H})$.

$\mathbb{A}^{-W}$ denotes the closure of $\mathbb{A}$ in WOT (equiv. SOT) in $\mathcal{L}(\mathcal{H})$.

$\mathbb{A}'$ denotes the commutant of $\mathbb{A}$.

DEF. A subalgebra $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ is transitive if the only invariant subspaces for $\mathbb{A}$ are (0) and $\mathcal{H}$.

TRANSITIVE ALGEBRA PROBLEM ('55 R. Kadison)
Whether every transitive subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ satisfies
$\overline{\mathbb{A}}^{\mathrm{W}} = \mathcal{L}(\mathcal{H})$?

NOTE. An affirmative answer of TAP implies that every non-scalar operator $T \in \mathcal{L}(\mathcal{H})$ have a nontrivial hyperinvariant subspace (n.h.s.).

Burnside's Th (Finite dimensional case). If dim$\mathcal{H} < \infty$, then the only transitive subalgebra of $\mathcal{L}(\mathcal{H})$ is $\mathcal{L}(\mathcal{H})$.

■ Its direct proof can be found in 1953, Lectures in abstract algebra II, N. Jacobson.

’67 W. Arveson (First partial solutions of TAP)

• If $\mathbb{A}$ is a transitive subalgebra of $\mathcal{L}(\mathcal{H})$ and $\mathbb{A}$ contains a maximal abelian self-adjoint algebra, then $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$.

• If $\mathbb{A}$ is a transitive subalgebra of $\mathcal{L}(\mathcal{H})$ and $\mathbb{A}$ contains a unilateral shift of finite multiplicity, then $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$.

■ Several operator theorists have been studied the transitive algebra problem since 1960s, for example, C. Rickart, W. Arveson, C. Foias, C. Pearcy, R. Douglas, H. Radjavi, P. Rosenthal, E. Nordgren,...etc.

■ For more imformation, see “Pearcy(1975), Some recent developements in operator theory” or “Radjavi-Rosenthal(1972), Invariant subspces”.

A SHORT HISTORY FOR OUR STUDY

'73 V. Lomonosov Let $\mathcal{X}$ be an infinite dimensional complex Banach space. Let $\mathbb{A}$ be a transitive subalgebra of $\mathcal{L}(\mathcal{X})$ and let $0 \neq K \in \mathbf{K}$. Then there exists $A \in \mathbb{A}$ s.t. $1 \in \sigma_p(AK)$.

(Corollary) Let T be any nonscalar operator in $\mathcal{L}(\mathcal{X})$, and suppose that $TK = KT$ for some $K \neq 0$ in $\mathbf{K}$. Then T has a n.h.s.

(Corollary) If $\mathbb{A}$ is a transitive subalgebra of $\mathcal{L}(\mathcal{H})$ with a non-zero operator of finite rank, then $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$. ('72 B. Barnes)

'74 C. Pearcy If $\mathbb{A}$ is a transitive subalgebra of $\mathcal{L}(\mathcal{H})$ containing a non-zero compact operator, then $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$.

'98 Chevreau-Li-Pearcy They generalized some Lomonosov's theorems by using a "metric property":

Let $\mathcal{X}$ be a Banach space and let $\mathcal{C} \subset \mathcal{L}(\mathcal{X})$ be a set.
For y, $y_0 \in \mathcal{X}^*$, $\delta > 0$, $\alpha > 0$, we denote

$$B(y_0, \delta) := \{y \in \mathcal{X}^* : \|y - y_0\| < \delta\} \subset \mathcal{X}^*,$$

$$\Gamma^*_\alpha(y) := \{C^*y : C \in \text{conh}(\mathcal{C}),\ \|C\|_e \leq \alpha\} \subset \mathcal{X}^*.$$

"Suppose $\exists\ y_0 \in \mathcal{X}^*$ with $\|y_0\| = 2$ and $0 < \alpha_0 < \frac{1}{2}$ such that if $\|y - y_0\| \leq 1$, then $y_0 \in \Gamma^*_{\alpha_0}(y)^-$." Then $\exists\ C_0 \in \text{conh}(\mathcal{C})$ s.t.

(a) $1 \in \sigma_p(C_0)$, $\|C_0\|_e \leq \alpha_0$, and $\|C_0^*\|_e \leq \alpha_0$,

(b) 1 is an isolated pt in $\sigma(C_0)$ and $\sigma(C_0^*)$,

(c) the eigen spaces in $\mathcal{X}$ and $\mathcal{X}^*$ corresponding to eigenvalue 1 of C_0 and C_0^* are finite dimensional.

'98 Chevreau-Li-Pearcy (A New Lomonosov Lemma)

Suppose $\mathbb{A} \subset \mathcal{L}(\mathcal{X})$ is transitive. Let $\{B_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}(\mathcal{X})$ such that

$$\|B_\lambda\|_e \longrightarrow 0 \text{ and } B_\lambda^{**} \xrightarrow{\text{WOT}} B_0^{**} \neq 0.$$

Then $\exists\ \{A_j\}_{j=1}^n \subset \mathbb{A}$, $\{B_{\lambda_j}\}_{j=1}^n \subset \{B_\lambda\}_{\lambda\in\Lambda}$ with $C_0 = \sum_{i=1}^n A_i B_{\lambda_i}$

such that $1 \in \sigma_p(C_0)$ and $\|C_0\|_e < \frac{1}{2}$. Consequently the same conclusions about C_0 that are stated in the above result.

■ Their results generalized several earlier results which are related to the invariant subspace problem or transitive algebra problem.

MOTIVATION FOR OUR STUDY

In the summer of 2005, as a consequence of '98 Chevreau-Li-Pearcy's construction, the following variant of the Kadison's transitive algebra problem was considered then.

PROB('05 Foias-Pearcy). If there exists a transitive subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ such that $\overline{\mathbb{A}}^{\mathrm{W}} \neq \mathcal{L}(\mathcal{H})$, is it true that $\| \ \| \cong \| \ \|_e$ (equivalent) on $\mathbb{A}$?

The existence of such transitive algebra $\mathbb{A}$ with $\overline{\mathbb{A}}^{\mathrm{W}} \neq \mathcal{L}(\mathcal{H})$ is not known yet. But Problem A is so interesting to us because of the following information.

- It is closely related to the existence of nontrivial invariant subspaces for operators of the form (subnormal)+(compact).
- It gives additional information about the existence of nontrivial invariant subspaces via a new metric property.

II. A NEW METRIC PROPERTY

Let $\mathbb{A}$ denotes a unital norm closed subalgebra of $\mathcal{L}(\mathcal{H})$.

For y, $y_0 \in \mathcal{H}$, $\delta > 0$, we denote

$$B(y_0, \delta) := \{y \in \mathcal{H} : \|y - y_0\| < \delta\},$$

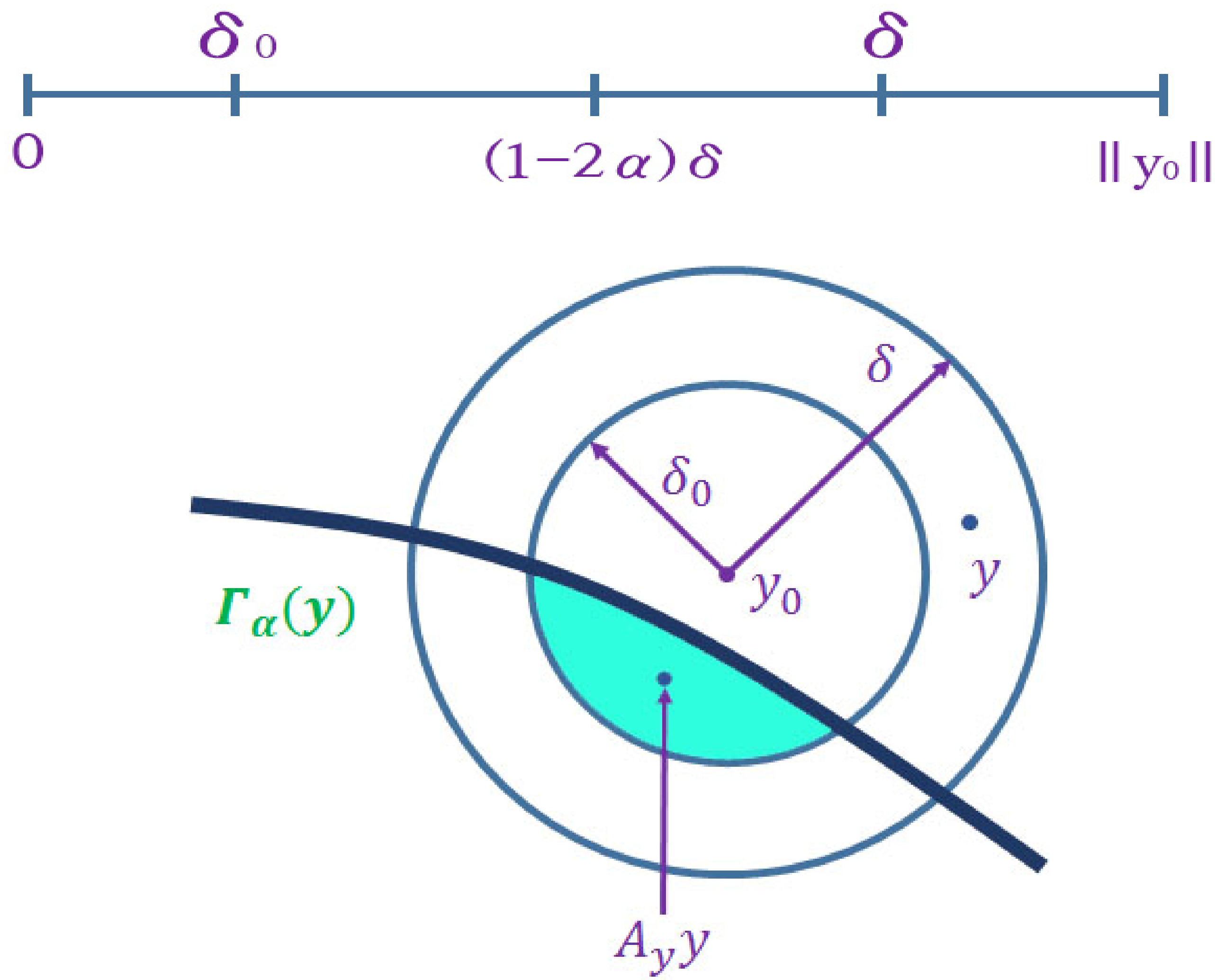
$$\Gamma_\alpha(y) := \{Ay : A \in \mathbb{A},\ \ \|A\|_e \leq \alpha\}.$$

DEF. A subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ has Property (P) if there exists $(y_0, \alpha, \delta, \delta_0)$ (implementing quadruple) such that, $y_0 \neq 0$ in $\mathcal{H}$,

P1) $0 < \alpha < 1/2$, $0 < \delta < \|y_0\|$, and $0 < \delta_0 < (1 - 2\alpha)\delta$,

P2) if $\|y_0 - y\| \leq \delta$, then $\Gamma_\alpha(y) \cap B(y_0, \delta_0)^- \neq \varnothing$, that is,

if $\|y_0 - y\| \leq \delta$, then $\exists\ A_y \in \mathbb{A}$ with $\|A_y\|_e \leq \alpha$ s.t. $\|A_y y - y_0\| \leq \delta_0$,



A_y moves y into the smaller cosed ball $B(y_0, \delta_0)^-$ centered at y_0 with radius $\delta_0 < \delta$.

PROPERTIES OF $\Gamma_\alpha(\mathbf{y})$

1. $\Gamma_\alpha(y)$ is absolutely convex, as is $\Gamma_\alpha(y)^-$.

2. If $(y_0, \alpha, \delta, \delta_0)$ implements (P) for $\mathbb{A}$, then $(ry_0, \alpha, r\delta, r\delta_0)$, $r > 0$, also implements (P) for $\mathbb{A}$.

3. $\Gamma_\alpha(y) = \alpha\Gamma_1(y)$.

4. $\lambda y \in \Gamma_\alpha(y)$, $\forall\lambda \in \mathbb{C}$ with $|\lambda| \le \alpha$.

■ **Invariant**

5. $A(\Gamma_\alpha(y)) \subset \Gamma_\alpha(y)$, $\forall A \in \mathbb{A}$ with $\|A\|_e \le 1$.

6. $\mathbb{A}\,(\vee\{\Gamma_\alpha\,(y)\}) \subset \vee\{\Gamma_\alpha\,(y)\}$.

■ Conditions for $\Gamma_\alpha(y)^- = \mathcal{H}$

7. If $\Gamma_\alpha\,(y)$ contains any real line, i.e., any set of the form $\{x + rz : r \in \mathbb{R}\}$ where $x \in \Gamma_\alpha(y)$ and $0 \neq z \in \mathcal{H}$, then

- $\mathbb{A}z \subset \Gamma_\alpha(y)$,
- $\Gamma_\alpha(y)$ contains the line $\mathbb{R}z$ through the origin,
- if z is a cyclic vector for $\mathbb{A}$, then $\Gamma_\alpha\,(y)^- = \mathcal{H}$.

8. If $\Gamma_\alpha(y)^-$ contains some βy with $|\beta| > \alpha$, then

- $\mathbb{A}y \subset \Gamma_\alpha(y)^-$.
- if y is a cyclic vector for $\mathbb{A}$, $\Gamma_\alpha(y)^- = \mathcal{H}$.

9. If $\mathbb{A} = \mathbb{A}^{-W}$, then

$\exists\ \alpha > 0$ such that $\Gamma_\alpha(y) = \mathcal{H}$ $(\forall y \neq 0) \iff \mathbb{A} = \mathcal{L}(\mathcal{H})$.

III. TRANSITIVITY AND (P)

TH A. Every $\mathbb{A}$ with (P) contains a nonzero idempotent of finite rank.

Poof(Sketch). Let $(y_0, \alpha, \delta, \delta_0)$ be an implementing quadruple with respect to (P).

Step 1. Fix β with $\alpha < \beta < (\delta - \delta_0)/2\delta$. For every $y \in B(y_0, \delta)^-$,

$\exists\ A_y \in \mathbb{A}$ s.t. $\|A_y\|_e \leq \alpha,\ \|A_y y - y_0\| \leq \delta_0$.

We have $A_y = T_y + K_y$ with $\|T_y\| < \beta,\ K_y \in \mathbf{K}$.

Step 2. Put $\delta_1 := \delta - (\delta_0 + 2\beta\delta) > 0$. For every $y \in B(y_0, \delta)^-$,

$\mathcal{V}_y(\delta_1) := \{w \in B(y_0,\delta)^- : \|K_y(w-y)\| < \delta_1\}$ (weakly open).

Then $B(y_0,\delta)^- \subset \cup_{y\in B(y_0,\delta)^-}\mathcal{V}_y(\delta_1)$. So $B(y_0,\delta)^- \subset \cup_{j=1}^n \mathcal{V}_{y_j}(\delta_1)$.

By the partition of unity, $\exists \{f_j\}_{j=1}^n$ weakly cont on $B(y_0,\delta)^-$ such that $\sum_{k=1}^n f_k(x) = 1$ and supp$f_j \subset \mathcal{V}_{y_j}(\delta_1)$.

Step 3. Construct a map g defined on $B(y_0,\delta)^-$ by

$g(w) := \sum_{j=1}^n f_j(w)A_{y_j}w, \quad w \in B(y_0,\delta)^-.$

By the construction of g, we see that $g(B(y_0,\delta)^-) \subset B(y_0,\delta)$.

By the Schauder-Tychonoff fixed point theorem,
$\exists\ w_0 \in B(y_0,\delta)^-$ with $w_0 \neq 0$ such that $g(w_0) = w_0$.

Step 4. Define an operator in $\mathcal{L}(\mathcal{H})$ by $A_0 = \sum_{j=1}^{n} f_j(w_0)A_{y_j}$.

Then $A_0 \in \mathbb{A}$, $\|A_0\|_e \le \alpha < 1/2$, and $A_0 w_0 = w_0$.

Step 5. Thus $1 \in \sigma_p(A_0)\backslash\sigma_e(A_0)$, and 1 is an isolated eigenvalue of A_0 of finite multiplicity.

So the corresponding Riesz idempotent E_1 belongs to $\mathbb{A}$. ■

COR B. If $\mathbb{A}$ is a transitive algebra with (P), then $\overline{\mathbb{A}}^{\mathsf{W}} = \mathcal{L}(\mathcal{H})$.

COR C. (a) If $\mathbb{A}$ has (P), then $\mathbb{A}'$ is intransitive.
(b) If $\mathbb{A}$ is transitive, then $\mathbb{A}'$ does not have (P).

SOME EXAMPLES

EXA 1. Let $\mathbb{A}$ be the C^*-algebra acting on $L^2(\mathbb{T})$ generated by $\{M_\varphi : \varphi \in L^\infty(\mathbb{T})\}$ and $R_\alpha \in \mathcal{L}(L^2(\mathbb{T}))$ (a rotation) defined by $(R_\alpha f)(e^{i\theta}) = f(e^{i(\theta+\alpha)})$, where $\alpha(\text{mod } 2\pi)$ is irrational. Then

(a) $\overline{\mathbb{A}}^{\mathsf{W}} = \mathcal{L}(\mathcal{H})$,

(b) $\mathbb{A}$ is transitive,

(c) $\mathbb{A}$ does not have (P).

NOTE. The reverse implication of Corollary C is not always true.

EXA 2. Supose $\dim\mathcal{H} < \aleph_0$. Then $\mathcal{L}(\mathcal{H})$ has Property (P).

EXA 3. If $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ is transitive, then
$\mathbb{A}$ has $(P) \iff \mathbb{A}$ contains a nonzero idempotent E of finite rank.

EXA 4. If $\mathbb{A}$ is abelian and transitive, then $\mathbb{A}$ does not have (P).

IV. EQUIVALENCE OF TWO NORMS

NOTATION. For $T \in \mathcal{L}(\mathcal{H})$, we denote

$$\mathbb{A}_{T_p} = \{p(T) : p \in \mathbb{C}[z]\}^{-\| \, \|}, \ \mathbb{A}_{T_r} = \{r(T) : r \in \mathcal{R}(\sigma(T))\}^{-\| \, \|},$$

where $\mathbb{C}[z]$ is the set of complex polynomials and $\mathcal{R}(\sigma(T))$ is the set of rational functions with poles off $\sigma(T)$.

Define $\mathbb{A}_T$ as either $\mathbb{A}_{T_r}$ or $\mathbb{A}_{T_p}$.

TH D. If $\| \, \|_e \cong \| \, \|$ on $\mathbb{A}_T$ and $\sigma_e(T)$ is a K-spectral set for $\pi(T)$ for some $K \geq 1$, then $\mathbb{A}_T$ has a n.i.s.

COR E. Suppose $T \in \mathcal{L}(\mathcal{H})$ is invertible and $\sigma_e(T)$ is a K-spectral set for $\pi(T)$. If $\| \, \|_e \cong \| \, \|$ on $\mathbb{A}_{T_r}$, then T and T^{-1} have a common n.i.s.

$\sigma(T)$ is a K-spectral set for T if $\exists K \geq 1$ s.t. $\|r(T)\| \leq K \sup_{\delta \in \sigma(T)} |r(\zeta)|, \ \forall r \in \mathcal{R}(\sigma(T)),$

COR F. Let $T = S + K \in \mathcal{L}(\mathcal{H})$, where S is subnormal and $K \in \mathbf{K}$. If $\| \ \|_e \cong \| \ \|$ on $\mathbb{A}_T$, then $\mathbb{A}_T$ is n.i.s.

TH G. Let $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ be as above and $\alpha > 0$ fixed. Then $\Gamma_\alpha(y)$ is bounded, $\forall y \in \mathcal{H} \Longleftrightarrow \| \ \|_e \cong \| \ \|$ on $\mathbb{A}$.

TH H. Suppose $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$, $\alpha > 0$, $\Gamma_\alpha(y)^-$ is bounded for some $y \in \mathcal{H}$, and $(\Gamma_\alpha(y)^-)^\circ \neq \emptyset$. Then,

(a) $\| \ \|_e \cong \| \ \|$ on $\mathbb{A}$,

(b) $\{\Gamma_\alpha(y)^- : y \in B(0,1)^-\}$ is uniformly bounded,

(c) y is a strictly cyclic vector for $\overline{\mathbb{A}}^{\mathsf{W}}$, i.e., $\overline{\mathbb{A}}^{\mathsf{W}} y = \mathcal{H}$,

(d) if $\mathbb{A} = \overline{\mathbb{A}}^{\mathsf{W}}$, then $\mathbb{A}$ has a n.i.s.

EXA 4. Let $D \in \mathcal{L}(\mathcal{H})$ be any Donoghue shift operator and let $\mathbb{A}_D$ be the (unital, norm-closed) subalgebra of $\mathcal{L}(\mathcal{H})$ generated by D. Then
(a) $\Gamma_\alpha(e_0) = \{\sum_{n\in\mathbb{N}_0} \zeta_n e_n \in \mathcal{H} : |\zeta_0| \leq \alpha\}$ is a closed set, $\forall\alpha > 0$,
(b) $\Gamma_\alpha(e_0)$ contains the ball $B(0,\alpha)^-$ and is unbounded.
(Note that e_0 is a strictly cyclic vector for $\mathbb{A}_D$.)

Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis for $\mathcal{H}$, and let D be defined by $De_n = \alpha_n e_{n+1}$, $n = 0, 1, \ldots$, where $\{\alpha_n\}_{n=0}^\infty$ is a strictly decreasing sequence of positive numbers such that $\{\alpha_n\}_{n=0}^\infty \in l^p$ for $1 \leq p < \infty$. Then D is called Donoghue shift operator.

EXA 5. If $V \in \mathcal{L}(L^2([0,1]))$ is the Volterra integral operator defined by $(Vf)(x) = \int_0^x f(t)\ dt,\ x \in [0,1],\ f \in L^2([0,1])$.
Then $\Gamma_\alpha(f_0)^- = \mathcal{H}$, $\forall\alpha > 0$.
(Note that $f_0 \equiv 1$ is a cyclic vector for $\mathbb{A}_V$.)

A REMARK ON INVARIANT SUBSPACE PROBLEM

TH I (Lomonosov). Suppose $\mathbb{A} = \overline{\mathbb{A}}^{\mathrm{W}} \neq \mathcal{L}(\mathcal{H})$ and $\mathbb{A}$ contains $\{A_\lambda\}$ s.t. $A_\lambda \overset{\mathrm{WOT}}{\to} A_0 \neq 0$ and $\|A_\lambda\|_e \to 0$. Then $\mathbb{A}$ has a n.i.s.

PROOF. Suppose that $\mathbb{A}$ is transitive.

Claim. $\mathbb{A}$ has (P) with some $(y_0, \alpha, \delta, \delta_0)$.

By a well-known fact in the convexity, we can obtain a net $\{A_\mu\}$ s.t. $A_\mu \overset{\mathrm{SOT}}{\to} A_0$ and $\|A_\mu\|_e \to 0$. Let $y_0 \in \mathcal{H}$ with $A_0 y_0 \neq 0$.

Choose $0 < \delta < \frac{\|A_0 y_0\|}{2\|A_0\|}$, $0 < \alpha < 1/2$, $0 < \delta_0 < (1 - 2\alpha)\delta$.

Let $y \in B(y_0, \delta)$. Observe $\|A_0 y\| > \|A_0 y_0\| / 2 > 0$.

By transitivity, $\exists\ A_1 \in \mathbb{A}$ such that $\|A_1(A_0 y) - y_0\| < \delta_0$.

Then $\|A_1 A_{\mu_0} y - y_0\| < \delta_0$ and $\|A_1 A_{\mu_0}\|_e < \alpha$ for suff far μ_0.

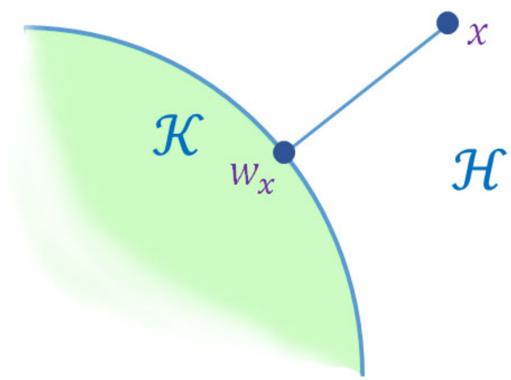
So $\Gamma_\alpha(y) \cap B(y_0, \delta_0)^- \neq \emptyset$. Thus $\mathbb{A}$ has (P).

By Cor B, $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$. Contradiction!. ■

(Recall) Suppose $\mathcal{C} \subset \mathcal{L}(\mathcal{X})$ is a convex set and there exists $\{A_\lambda\}$ in $\mathcal{C}$ s.t. $A_\lambda \overset{\text{WOT}}{\to} A_0 \neq 0$ and $\|A_\lambda\|_e \to 0$, then there exists another net $\{A_\mu\}$ in $\mathcal{C}$ s.t. $A_\mu \overset{\text{WOT}}{\to} A_0 \neq 0$ and $\|A_\mu\|_e \to 0$.

BOUNDED $\Gamma_\alpha(y)$'S

(Recall) If $\varnothing \neq \mathcal{K} \subset \mathcal{H}$ absolutely convex and norm-closed in $\mathcal{H}$, then for every $x \in \mathcal{H}$, $\exists\, w_x$ such that $\|x - w_x\| = \min\{\|z - x\| : z \in \mathcal{K}\}$.



Define $\Phi_{\mathcal{K}} : \mathcal{H} \to \mathcal{H}$ by $\Phi_{\mathcal{K}}(x) = x - w_x$. Then

- $\|\Phi_{\mathcal{K}}(x_1) - \Phi_{\mathcal{K}}(x_2)\| \leq \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{H}$, (Lipschitz)
- $\langle \Phi_{\mathcal{K}}(x_1) - \Phi_{\mathcal{K}}(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in \mathcal{H}$, (monotonicity)
  (cf. '68 G. Minty)

PROP J. With $\mathcal{K}$ and $\Phi_{\mathcal{K}}$ as above,
if $\lim_{\|x\|\to\infty} \|x - w_x\| = \infty$, then $\Phi_{\mathcal{K}}(\mathcal{H}) = \mathcal{H}$.

Since $\Gamma_\alpha(y)^-$ is absolutely convex, we may consider $\mathcal{K} = \Gamma_\alpha(y)^-$.

REMARK. Let $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$, $y \in \mathcal{H}\backslash(0)$ and $\alpha > 0$. Then
$\Gamma_\alpha(y)^-$ is bounded $\iff \lim_{\|x\|\to\infty} \|x - w_x\| = \infty$.

PROP L. Let $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$, $y \in \mathcal{H}\backslash(0)$ and $\alpha > 0$. Then TFAE

(a) $\Gamma_\alpha(y)^-$ is bounded

(b) there exists a hyperplane $\mathcal{M}$ (i.e., a translation of a subspace of codimension 1) such that $\lim_{\substack{\|x\|\to\infty \\ x\in\mathcal{M}}} \|x - w_x\| = \infty$.

(In fact, $\mathcal{M} = 2y + \{y - w_y\}^\perp$.)

In this case, $\Phi_{\mathcal{K}}(\mathcal{H}) = \mathcal{H}$.

CONNECTION: $\|A_n\|_e \to 0$ AND $A_n \to 0$(WOT)

TH M. Let $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$. Suppose there exist $y \in \mathcal{H}\backslash(0)$ and $\alpha > 0$ such that $\Gamma_\alpha(y)^-$ is bounded. Let $\{A_n\} \subset \mathbb{A}$ s.t. $\|A_n\|_e \to 0$. Then
(a) if y is a cyclic vector for $\mathbb{A}$ and $\sup_n \|A_n\| < \infty$, $A_n \overset{\text{WOT}}{\to} 0$,
(b) if y is a strictly cyclic vector for $\mathbb{A}$, then $A_n \overset{\text{WOT}}{\to} 0$.

TH N. Let $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ be transitive. Suppose there exist $y \in \mathcal{H}\backslash(0)$ and $\alpha > 0$ such that $\Gamma_\alpha(y)^- \neq \mathcal{H}$. If $\{A_n\} \subset \mathbb{A}$ with $\|A_n\|_e \to 0$ and $\sup_n \|A_n\| < \infty$, then $A_n \overset{\text{WOT}}{\to} 0$.

REMARK. There are some results in our paper about the following:

- more properties of (P),
- Property (P) in the finite dimensional case.

This talk is based on the paper, Transitivity and structure of operator algebras with a metric property, Indagationes Mathematicae 25 (2014), 1-23 (with C. Foias , E. Ko, C. Pearcy).

Thank You