Isomorphisms of Operator Systems

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Isomorphisms are unital *completely isometric maps*.

C^*-algebras generated by operator systems

$$S_1 = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \right\}, \quad S_2 = \text{span}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right\}$$

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$$\varphi : S_2 \to S_1,\ \varphi(X) = \begin{bmatrix}1&0&0\\0&0&1\end{bmatrix} X \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix},$$

$$\varphi^{-1}(Y) = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix} Y \begin{bmatrix}1&0&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&0\\1/\sqrt{2}&0\\0&0\end{bmatrix} Y \begin{bmatrix}0&1/\sqrt{2}&0\\0&0&0\end{bmatrix}$$

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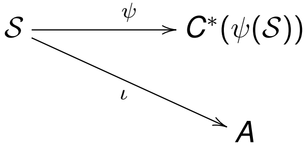

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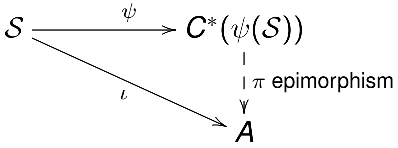

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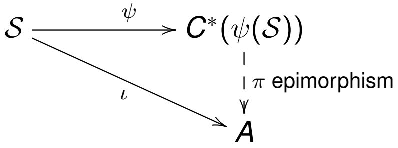

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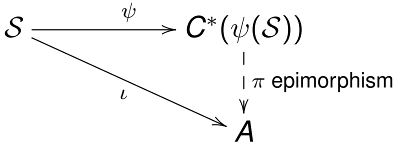

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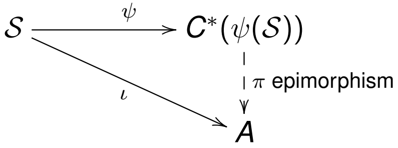

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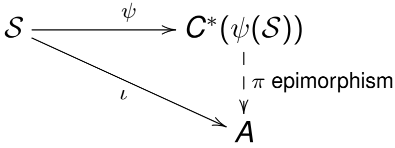

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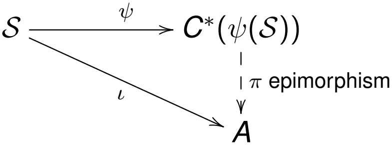

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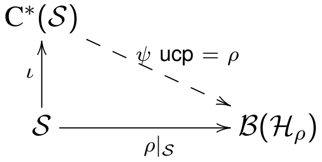

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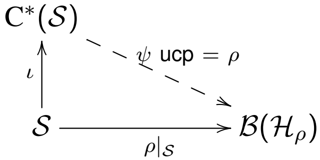

Arveson's idea:

$$\mathrm{C}^*_{\mathrm{e}}(\mathcal{S}) = C^*\left(\left(\bigoplus_{\rho \text{ boundary}} \rho\right)(\mathcal{S})\right).$$

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Claim: π_2 is not boundary. We have $\pi_2 \left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right) = b$.

Let $\phi_2 \left(\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right) = \frac{a+c}{2}$.

Then $\phi_2 \neq \pi_2$ on $C^*(\mathcal{S})$, but they agree on $\mathcal{S}$:

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$$\mathrm{C}^*_{\mathrm{e}}(\mathcal{S}) = C^*((\pi_1 \oplus \pi_3)(\mathcal{S})) = C^*((\pi_1 \oplus \pi_3)(T)) = \mathbb{C}^2.$$

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Corollary

Any two 2-dimensional operator systems are completely order isomorphic.

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For all t, $C^*(W_t) = M_2(\mathbb{C})$. Simple, so $\mathrm{C}^*_{\mathrm{e}}(\mathcal{S}_t) = M_2(\mathbb{C})$.

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So $\{\mathcal{S}_t\}_{t\in(0,1]}$ are uncountably many reduced non-isomorphic 3-dimensional operator systems in $M_2(\mathbb{C})$.

Isomorphisms of small operator systems

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TFSAE:

1. $\mathcal{S}_\lambda \simeq \mathcal{S}_\mu$;
2. *either*
   1. $|\lambda| \leq 1/2$ *and* $|\mu| \leq 1/2$, *in which case* $\mathrm{C}_\mathrm{e}^*(\mathcal{S}_\lambda) = \mathrm{C}_\mathrm{e}^*(\mathcal{S}_\mu) = M_2(\mathbb{C})$;
   2. $|\lambda| > 1/2$ *and* $|\mu| = |\lambda|$, *in which case* $\mathrm{C}_\mathrm{e}^*(\mathcal{S}_\lambda) = \mathrm{C}_\mathrm{e}^*(\mathcal{S}_\mu) = M_2(\mathbb{C}) \oplus \mathbb{C}$.

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3. UHF C*-algebras: $\overline{\bigcup_k M_{n_k}(\mathbb{C})}$. $\quad \rightleftarrows$ sup. number $(n_1|n_2|\dots)$.

Borel reducibility, or how to measure classifications

If E is an eq. rel. on a standard Borel space X, and F on Y, we say that E is *Borel-reducible* to F if there exists $f : X \to Y$, Borel-measurable, such that

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Classification of separable C*-algebras, separable operator systems is non-smooth. How non-smooth?

Non-smooth relations

Definition

E is *classifiable by countable structures* if it is Borel reducible to isomorphism in some class of countable structures.

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E is *classifiable by orbits* (or *below a group action*) if it is Borel reducible to the orbit equivalence associated with a continuous action of a Polish group on a Polish space.

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Isomorphism of Banach spaces is not even below a group action; it is maximal among analytic equivalence relations.

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Not obvious even when acting on a finite-dimensional Hilbert space.
Not obvious even for operator systems with C*-envelope $M_2(\mathbb{C})$:

$$\mathcal{S}_t = \operatorname{span}\{I, W_t, W_t^*\}, \quad \text{where } W_t = \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix}, \quad t \in (0, 1].$$

Arveson's Invariant (2010)

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This classification is good in that it paints a picture of what operator systems acting on finite-dimensional Hilbert spaces look like, in terms of their boundary representations. But it is not really explicit!

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With the W_t above: $d = 3$, $m = 1$, $k_1 = 2$, and for example

$$\Gamma_1(\alpha,\beta,\gamma) = \begin{bmatrix} \alpha & \frac{\alpha t}{2} + \frac{t}{4}\left((-1+i)\beta-(1+i)\gamma\right) \\ \frac{\alpha t}{2} + \frac{t}{4}\left(-(1+i)\beta+(-1+i)\gamma\right) & \frac{\beta+\gamma}{2} \end{bmatrix}$$

to get $\Gamma_1(1,1,1) = I$, $\Gamma_1(1,i,-i) = W_t$, $\Gamma_1(1,-i,i) = W_t^*$.

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But U being a unitary makes every irrep a boundary representation. Indeed, if $\pi : C^*(U) \to \mathbb{C}$ is an irrep and $\psi : C^*(U) \to \mathbb{C}$ is ucp with $\psi(U) = \pi(U)$, then

$$I = \pi(U)^*\pi(U) = \psi(U)^*\psi(U) \leq \psi(U^*U) = \psi(I) = I.$$

So U is in the multiplicative domain of ψ, and $\psi = \pi$ on $C^*(U)$, and So $\mathrm{C}^*_{\mathrm{e}}(U) = C^*(U) = C(\sigma(U))$.

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Is this condition sufficient?

Isomorphism of Operator Systems generated by unitaries

Theorem (A.-Coskey-Kalantar-Kennedy-Lupini-Sabok, 2014)

Let U, V be unitaries with $|\sigma(U)| = |\sigma(V)| \leq 3$. Then $\mathcal{OS}_y(U) \simeq \mathcal{OS}_y(V)$.

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Let U, V be unitaries with $|\sigma(U)| = |\sigma(V)| \geq 5$. TFSAE:

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So for unitaries with finite spectrum of at least 5 points, the distance between eigenvalues is an invariant of the corresponding operator systems.

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$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad W = \frac{1}{2}\,U + i\,\frac{\sqrt{3}}{2}\,U^*.$$

Then $\mathrm{C}^*_\mathrm{e}(U) = \mathrm{C}^*_\mathrm{e}(V) = \mathrm{C}^*_\mathrm{e}(W) = \mathbb{C}^4$.

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Then $\mathrm{C}^*_\mathrm{e}(U) = \mathrm{C}^*_\mathrm{e}(V) = \mathrm{C}^*_\mathrm{e}(W) = \mathbb{C}^4$.

More rigidity than the case of $|\sigma(U)| \leq 3$, but less than $|\sigma(U)| \geq 5$:

Proposition (ACKKLS 2014, A. 2015)

1. $\mathcal{OS}_y(U) \not\simeq \mathcal{OS}_y(V)$.
2. $\mathcal{OS}_y(U) = \mathcal{OS}_y(W)$, *but* $\sigma(W)$ *is not a rigid deformation of* $\sigma(U)$.

Thank you!