Subnormality of unbounded operators via inductive limits

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KOTAC 2015, Daejeon

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Subnormality

An operator N in $\mathcal{H}$ is said to be **normal** iff N is closed, densely defined and $NN^* = N^*N$.

An operator A in $\mathcal{H}$ is **subnormal** if A is densely defined and there exists a Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Ah = Nh$ for all $h \in \mathcal{D}(S)$.

An operator A in $\mathcal{H}$ is **cosubnormal** if A is densely defined and A^* is subnormal.

Subnormality

- The creation operator of quantum mechanics:

$$a_+ = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big).$$

- Symmetric operators:

$$S \subseteq S^*.$$

Subnormality

- An operator A in $\mathcal{H}$ generates Stieltjes moment sequences iff $\{\|A^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^{\infty}(A)$
- $\mathcal{D}^{\infty}(A) := \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$

Theorem (Lambert)

A **bounded** operator A on $\mathcal{H}$ is **subnormal** iff A generates Stieltjes moment sequences.

Theorem

If A is a **subnormal** operator in $\mathcal{H}$, then A generates Stieltjes moment sequences.

Counterexamples

Theorem (Jabłoński-Jung-Stochel; B.-Jabłoński-Jung-Stochel)

There exist a **non-hyponormal** operator A which generates Stieltjes moment sequences and satisfies $\overline{\mathcal{D}^\infty(A)} = \mathcal{H}$.

Theorem (Naimark)

There exist a **symmetric** operator A such that $\overline{\mathcal{D}(A^2)} = \{0\}$.

Theorem (B.-Jabłoński-Jung-Stochel)

There exist a **subnormal** non-symmetric operator A such that $\overline{\mathcal{D}(A^2)} = \{0\}$.

Approaches

- Moment problem approach
- Consistency condition approach
- Inductive limits approach

Inductive limits

A Hilbert space $\mathcal{H}$ is the **inductive limit** of $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$, a sequence of Hilbert spaces, if there are isometries

$$\Lambda_k : \mathcal{H}_k \to \mathcal{H}, \quad \text{and} \quad \Lambda_k^l : \mathcal{H}_k \to \mathcal{H}_l, \ (k \leqslant l)$$

such that

- Λ_k^k is the identity operator on $\mathcal{H}_k$,
- $\Lambda_k^m = \Lambda_l^m \circ \Lambda_k^l$ for all $k \leqslant l \leqslant m$,
- $\Lambda_k = \Lambda_l \circ \Lambda_k^l$ for all $k \leqslant l$,
- $\mathcal{H} = \overline{\bigcup_{n\in\mathbb{N}} \Lambda_n \mathcal{H}_n}$.

We write $\mathcal{H} = \mathrm{LIM}\,\mathcal{H}_n$.

Inductive limits

$\mathcal{H} = \mathtt{LIM}\,\mathcal{H}_n$. Let C_n, $n \in \mathbb{N}$, be an operator in $\mathcal{H}_n$. Consider the subspace

$$\mathcal{D}_\infty = \bigcup_{k \in \mathbb{N}} \{\Lambda_k f \mid \exists M \geqslant k \colon \Lambda_k^m f \in \mathcal{D}(C_m) \text{ for all } m \geqslant M\}$$

and define the operator $\mathtt{lim}\, C_n$ in $\mathcal{H}$ by

$$\mathcal{D}(\mathtt{lim}\, C_n) = \{\Lambda_k f \in \mathcal{D}_\infty \colon \lim_{m \to \infty} \Lambda_m C_m \Lambda_k^m f \text{ exists}\}$$

$$(\mathtt{lim}\, C_n)\Lambda_k f = \lim_{m \to \infty} \Lambda_m C_m \Lambda_k^m f, \quad \Lambda_k f \in \mathcal{D}(\mathtt{lim}\, C_n).$$

We call $\mathtt{lim}\, C_n$ the **inductive limit** of $\{C_n\}_{n \in \mathbb{N}}$.

Inductive limits

- it is flexible
- it works in bounded and unbounded cases
- the isometricity is optional
- the inclusions $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$ are not restrictive
- it works two-ways:
  - $C \rightsquigarrow \{C_n\} \rightsquigarrow \mathtt{lim}\, C_n \rightsquigarrow C$
  - $\{C_n\} \rightsquigarrow \mathtt{lim}\, C_n \rightsquigarrow C$

Composition operators in L^2-spaces

- $(X, \mathscr{A}, \mu)$ is a σ-finite measure space,
- $\phi\colon X \to X$ is $\mathscr{A}$-measurable,
- $\mu \circ \phi^{-1} \ll \mu$, where $\mu \circ \phi^{-1}(\varDelta) := \mu\big(\phi^{-1}(\varDelta)\big)$, $\varDelta \in \mathscr{A}$.

Define $C_\phi\colon L^2(\mu) \supseteq \mathcal{D}(C_\phi) \to L^2(\mu)$ by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu)\colon \int |f \circ \phi|^2 d\mu < \infty\},$$
$$C_\phi f = f \circ \phi, \quad f \in \mathcal{D}(C_\phi).$$

CO's with matical symbols

- $\kappa \in \mathbb{N}$
- ϕ – an **invertible linear** transformation of $\mathbb{R}^\kappa$
- $\mathscr{E}_+$ – the set of all entire functions γ of the form

$$\gamma(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C},$$

  where $a_n \geqslant 0$ for all $n \in \mathbb{Z}_+$ and $a_k > 0$ for some $k \geqslant 1$
- $|\cdot|$ – Hilbert norm on $\mathbb{R}^\kappa$
- $d\mu_\gamma = \gamma(\|\cdot\|^2) dm_\kappa$

CO's with matrical symbols

Theorem (Stochel)

(i) If $\gamma \in \mathscr{E}_+$ is a **polynomial**, then ϕ induces bounded composition operator on $L^2(\mu_\gamma)$ and on $L^2(\mu_{1/\gamma})$.

(ii) If $\gamma \in \mathscr{E}_+$ is **not a polynomial**, then ϕ induces bounded composition operator on $L^2(\mu_\gamma)$ (resp. on $L^2(\mu_{1/\gamma})$) if and only if $\|\phi^{-1}\| \leqslant 1$ (resp. $\|\phi\| \leqslant 1$).

Theorem (Stochel)

Let C_ϕ be **bounded** on $L^2(\mu_\gamma)$. Then C_ϕ is subnormal if and only if ϕ is **normal** in $(\mathbb{R}^\kappa, |\cdot|)$.

CO's with matical symbols

- C_ϕ is densely defined in $L^2(\mu_\gamma)$ and in $L^2(\mu_{1/\gamma})$

Theorem (B.-Jabłoński-Jung-Stochel; B.-Dymek-Płaneta)

Let $\gamma \in \mathscr{E}_+$. If ϕ is **normal** in $(\mathbb{R}^\kappa, |\cdot|)$, then

(i) C_ϕ is subnormal in $L^2(\mu_\gamma)$,

(ii) C_ϕ is cosubnormal in $L^2(\mu_{1/\gamma})$.

Sketch of the proof

- $\gamma = \sum_{k=0}^{\infty} a_k z^k \quad \rightsquigarrow \quad \gamma_n = \sum_{k=0}^{n} a_k z^k$
- $\big(C_\phi, L^2(mu_\gamma)\big) \quad \rightsquigarrow \quad \Big\{\big(C_\phi, L^2(\mu_{\gamma_n})\big)\Big\}$
- use the characterization of bounded CO's with matrical symbols
- use the criterion for subnormality of unbounded op-s due to Cichoń, Stochel, and Szafraniec
- use the eqivalence of C_ϕ in $L^2(\mu_\gamma)$ and $C_{\phi^{-1}}$ in $L^2(\mu_{1/\gamma})$

CO's with infinite matrical symbols

- $\phi\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ is induced by an infinite matrix
- $\mu = \mu_{\mathrm{G}} = \mu_{\mathrm{G},1} \otimes \mu_{\mathrm{G},1} \otimes \ldots$, where $d\mu_{\mathrm{G},1} = \mathrm{e}^{-\frac{x^2}{2}}\, dm$

Theorem (B.-Płaneta)

Let ϕ be a transformation of $\mathbb{R}^\infty$ induced by a matrix $(\phi_{ij})_{i,j\in\mathbb{N}}$. Let $\phi_{(n)}$, $n \in \mathbb{N}$, be the linear transformation of $\mathbb{R}^n$ induced by the matrix $(\phi_{ij})_{i,j=1}^n$. If the following conditions are satisfied:

(i) $\inf_{n\in\mathbb{N}} |\det \phi_{(n)}| > 0$,

(ii) for every $j \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that $\phi_{j,k} = 0$ for all $k \geqslant K$,

(iii) $\sup_{n\in\mathbb{N}} \|\phi_{(n)}\| \leqslant 1$,

then C_ϕ is bounded on $L^2(\mu_{\mathrm{G}})$. Moreover, C_ϕ is the limit in the strong operator topology of $\{C_{\phi_{(n)}} \otimes I_n\}_{n\in\mathbb{N}}$.

CO's with infinite matrical symbols

Theorem (B.-Płaneta)

Let ϕ be a transformation of $\mathbb{R}^\infty$ induced by a matrix $(\phi_{ij})_{i,j\in\mathbb{N}}$. Let $\phi_{(n)}$, $n \in \mathbb{N}$, be the linear transformation of $\mathbb{R}^n$ induced by the matrix $(\phi_{ij})_{i,j=1}^n$. If the following conditions are satisfied:

(i) for every $n \in \mathbb{N}$, $\phi_{(n)}$ is invertible,

(ii) for every $j \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that $\phi_{j,k} = 0$ for all $k \geqslant K$,

(iii) there exists $\varepsilon > 0$ such that

$$\left\{ \left| \det \phi_{(n)}^{-1} \right| \cdot \exp \tfrac{1}{2} \left(\| \cdot \|^2 - \| \phi_{(n)}^{-1}(\cdot) \|^2 \right) \right\}_{n\in\mathbb{N}}$$

is uniformly in $L^{1+\epsilon}(\mu_{\mathbb{G}})$

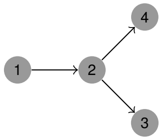
then C_ϕ is densely defined operator in $L^2(\mu_{\mathbb{G}})$ and $C_\phi = \overline{\mathtt{lim}\, C_{\phi_{(n)}}}$.

Sketch of the proofs

- $L^2(\mu_{\mathrm{G}}) = \mathtt{LIM}\, L^2(\mu_{\mathrm{G},n})$, with $d\mu_{\mathrm{G},n} = \frac{1}{(\sqrt{2\pi})^n} \exp(-\frac{x_1^2+\ldots+x_n^2}{2}) dm_n$
- $\big(C_\phi, L^2(\mu_{\mathrm{G}})\big) \quad \rightsquigarrow \quad \Big\{\big(C_{\phi_{(n)}}, L^2(\mu_{\mathrm{G},n})\big)\Big\}$
- $\mathtt{lim}\, C_{\phi_{(n)}}$ has “nice” properties
- compare $\mathtt{lim}\, C_{\phi_{(n)}}$ with C_ϕ

Weighted shifts on directed trees

- $\mathscr{T} = (V, E)$ – directed tree,
- $\mathrm{par}(u) = \{v \in V \colon (v, u) \in E\}$ – parent of $u \in V$,
- $\mathrm{Chi}(u) = \{v \in V \colon (u, v) \in E\}$ – children of $u \in V$,
- root – the root of $\mathscr{T}$ (provided it exists),
- $V^\circ = V \setminus \{\mathrm{root}\}$,



- $\mathrm{par}(2) = 1$, $\mathrm{par}(3) = \mathrm{par}(3) = 2$,
- $\mathrm{Chi}(2) = \{3, 4\}$, $\mathrm{Chi}(3) = \mathrm{Chi}(4) = \varnothing$,
- $\mathrm{root} = 1$,
- $V^\circ = \{2, 3, 4\}$,

WS's on directed trees

- $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$

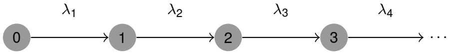
Define $\mathsf{S}_{\boldsymbol{\lambda}} =: \ell^2(V) \subseteq \mathcal{D}(\mathsf{S}_{\boldsymbol{\lambda}}) \to \ell^2(V)$ by

$$\begin{aligned} \mathcal{D}(\mathsf{S}_{\boldsymbol{\lambda}}) &= \{f \in \ell^2(V) \colon \Lambda_{\mathscr{T}} f \in \ell^2(V)\}, \\ \mathsf{S}_{\boldsymbol{\lambda}} f &= \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(\mathsf{S}_{\boldsymbol{\lambda}}), \end{aligned}$$

where

$$(\Lambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathrm{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathrm{root}. \end{cases}$$

WS's on directed trees



$$S_{\boldsymbol{\lambda}} e_k = \lambda_{k+1} e_{k+1}, \quad k \in \mathbb{Z}_+.$$

If $e_u \in \mathcal{D}(S_{\boldsymbol{\lambda}})$, with $e_u = \chi_{\{u\}}$, then

$$S_{\boldsymbol{\lambda}} e_u = \sum_{v \in \mathrm{Chi}(u)} \lambda_v e_v.$$

WS's on directed trees

Theorem (B.-Jabłoński-Jung-Stochel)

Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T} = (V, E)$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$ such that $\mathscr{E}_V := \{e_v \colon v \in V\} \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$. If there exist a family $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and a family $\{\varepsilon_v\}_{v\in V} \subseteq \mathbb{R}_+$ satisfying the **consistency condition**:

$$\mu_u(\sigma) = \sum_{v \in \mathrm{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{t}\mu_v(dt) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ u \in V,$$

then $S_{\boldsymbol\lambda}$ is subnormal.

Sketch of the proof

- $\boldsymbol{\lambda} \quad \rightsquigarrow \quad \boldsymbol{\lambda}_{(n)}$
- $S_{\boldsymbol{\lambda}_{(n)}}$ are bounded and satisfy the consistency condition
- $(S_{\boldsymbol{\lambda}}, \ell^2(V)) \quad \rightsquigarrow \quad \left\{(S_{\boldsymbol{\lambda}_{(n)}}, \ell^2(V))\right\}$
- use the characterization of bounded subnormal WS's on directed trees through the consistency condition
- use the criterion for subnormality due to CIchoń, Stochel, ans Szafraniec

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