Reconsideration of the Cubic Moment Problem

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How do we represent the following sequences?

$\beta^{(4)} : \{\beta_{ij}\} = \{5, 5, 14, 5, 14, 50\}$ $(0 \leq i + j \leq 2)$

$\Longrightarrow \beta_{00} = 2 \cdot (1^0)(1^0) + 3 \cdot (1^0)(4^0) = 5$
$\Longrightarrow \beta_{10} = 2 \cdot (1^1)(1^0) + 3 \cdot (1^1)(4^0) = 14$
$\Longrightarrow \beta_{01} = 2 \cdot (1^0)(1^1) + 3 \cdot (1^0)(4^1) = 5$
$\Longrightarrow \beta_{20} = 2 \cdot (1^2)(1^0) + 3 \cdot (1^2)(4^0) = 14$
$\Longrightarrow \beta_{11} = 2 \cdot (1^1)(1^1) + 3 \cdot (1^1)(4^1) = 5$
$\Longrightarrow \beta_{02} = 2 \cdot (1^0)(1^2) + 3 \cdot (1^0)(4^2) = 50$

Thus, we can find a formula:

$$\beta_{ij} = 2 \cdot (1^i)(1^j) + 3 \cdot (1^i)(4^j) = \int x^i y^j d\mu,$$

where $\mu = 2\,\delta_{(1,1)} + 3\,\delta_{(1,4)}$.
This is an example of a **2**-dimensional **moment problem.**

The coefficients 2 and 3 are **densities** and the two points $(1, 2)$ and $(1, 4)$ are **atoms** of the representing measure μ.

Full Moment Problem and a Solution

The **full** moment problem is to find a representing measure for an **infinite** moment sequence $\beta := \{\beta_k\}_{k \geq 0}$ such that $\beta_k = \int x^k \, d\mu$.
According to the location of the support of the measure,

supp $\mu \subseteq [a, b]$	(Hausdorff MP)	supp $\mu \subseteq \mathbb{R}$	(Hamburger MP)
supp $\mu \subseteq [0, \infty)$	(Stieltjes MP)	supp $\mu \subseteq \mathbb{T}$	(Toeplitz MP)

Theorem (Stieltjes, 1924)

$\exists$ *rep. meas.* μ *for* $\{\beta_k\}_{k=0}^{\infty}$ *s.t.* $\beta_k = \int x^k \, d\mu$, supp $\mu \in [0, \infty)$

$$\iff \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_n \\ \beta_1 & \beta_2 & \cdots & \beta_{n+1} \\ \beta_2 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \beta_n & \beta_{n+1} & \cdots & \beta_{2n} \end{pmatrix} \geq 0, \quad \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n+1} \\ \beta_2 & \beta_3 & \cdots & \beta_{n+2} \\ \beta_3 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{n+1} & \beta_{n+2} & \cdots & \beta_{2n+1} \end{pmatrix} \geq 0$$

for all $n \geq 0$.

Roughly speaking, positivity is a solution for FMP but not for TMP.

Given a doubly indexed finite sequence of real numbers, **truncated real moment sequence** (of order m), $\beta \equiv \beta^{(m)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \cdots, \beta_{m,0}, \beta_{m-1,1}, \cdots, \beta_{1,m-1}, \beta_{0,m}\}$ with $\beta_{00} > 0$, the truncated real moment problem (**TRMP**) entails seeking necessary and sufficient conditions for the existence of a positive Borel measure μ supported in the real plane $\mathbb{R}^2$ such that

$$\beta_{ij} = \int x^i y^j \, d\mu \ \ (i, j \in \mathbb{Z}_+, \ 0 \leq i + j \leq m).$$

We call μ a **representing measure** for β; if a moment sequence has such a measure, then we say the problem is **soluble** and the necessary and sufficient conditions are referred to as a **solution.**

We may also consider a problem in complex version and it is defined as follows: given a collection of complex numbers $\gamma \equiv \gamma^{(m)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \cdots, \gamma_{0,2m}, \gamma_{1,m-1}, \cdots, \gamma_{m-1,1}, \gamma_{m,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **truncated complex moment problem (TCMP)** consists of finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that $\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \quad (0 \leq i + j \leq m)$.

It is well-known that TRMP are TCMP are equivalent for an even m, and hence any techniques developed for a solution to TCMP are transferable to TRMP. Both problems are simply referred to as the **truncated moment problem (TMP).**

Even Order Moment Problems

When $m = 2n$, R. Curto and L. Fialkow have made a great contribution to various moment problems in a series of papers (complete solutions were found for $m = 2, 4$; see [1, 2, 5].)

They have introduced an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated **moment matrix.** We label the columns in the moment matrix:

$$\text{Complex:} \quad \mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^n, \bar{Z}Z^{n-1}, \ldots, \bar{Z}^n;$$
$$\text{Real:} \quad \mathit{1}, X, Y, X^2, XY, Y^2, X^n, X^{n-1}Y, \ldots, \bar{Y}^n;$$

Moment Matrices

For a moment sequence $\beta^{(2n)}$ of even order, the **moment matrix** is defined as

Complex case: $M(n)(\gamma^{(2n)}) := (\gamma_{\widehat{p}+\widehat{\bar{q}}})_{p,q\in\mathbb{C}[Z,\bar{Z}]:\,\deg p, \deg q \le n}$

Real case: $\mathcal{M}(n)(\beta^{(2n)}) := (\beta_{\mathbf{i}+\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\mathbb{Z}_+^2:\,|\mathbf{i}|,|\mathbf{j}|\le n}.$

$M(3)(\gamma)$ (Block Toeplitz)

1	Z	$\bar{Z}$	Z^2	$\bar{Z}Z$	$\bar{Z}^2$	Z^3	$\bar{Z}Z^2$	$\bar{Z}^2Z$	$\bar{Z}^3$
γ_{00}	γ_{01}	γ_{10}	γ_{02}	γ_{11}	γ_{20}	γ_{03}	γ_{12}	γ_{21}	γ_{30}
γ_{10}	γ_{11}	γ_{20}	γ_{12}	γ_{21}	γ_{30}	γ_{13}	γ_{22}	γ_{31}	γ_{40}
γ_{01}	γ_{02}	γ_{11}	γ_{03}	γ_{12}	γ_{21}	γ_{04}	γ_{13}	γ_{22}	γ_{31}
γ_{20}	γ_{21}	γ_{30}	γ_{22}	γ_{31}	γ_{40}	γ_{23}	γ_{32}	γ_{41}	γ_{50}
γ_{11}	γ_{12}	γ_{21}	γ_{13}	γ_{22}	γ_{31}	γ_{14}	γ_{23}	γ_{32}	γ_{41}
γ_{02}	γ_{03}	γ_{12}	γ_{04}	γ_{13}	γ_{22}	γ_{05}	γ_{14}	γ_{23}	γ_{32}
γ_{30}	γ_{31}	γ_{40}	γ_{32}	γ_{41}	γ_{50}	γ_{33}	γ_{42}	γ_{51}	γ_{60}
γ_{21}	γ_{22}	γ_{31}	γ_{23}	γ_{32}	γ_{41}	γ_{24}	γ_{33}	γ_{42}	γ_{51}
γ_{12}	γ_{13}	γ_{22}	γ_{14}	γ_{23}	γ_{32}	γ_{15}	γ_{24}	γ_{33}	γ_{42}
γ_{03}	γ_{04}	γ_{13}	γ_{05}	γ_{14}	γ_{23}	γ_{06}	γ_{15}	γ_{24}	γ_{33}

$M_{\mathbb{R}}(3)$ (Block Hankel)

1	X	Y	X^2	XY	Y^2	X^3	X^2Y	XY^2	Y^3
β_{00}	β_{10}	β_{01}	β_{20}	β_{11}	β_{02}	β_{30}	β_{21}	β_{12}	β_{03}
β_{10}	β_{20}	β_{11}	β_{30}	β_{21}	β_{12}	β_{40}	β_{31}	β_{22}	β_{13}
β_{01}	β_{11}	β_{02}	β_{21}	β_{12}	β_{03}	β_{31}	β_{22}	β_{13}	β_{04}
β_{20}	β_{30}	β_{21}	β_{40}	β_{31}	β_{22}	β_{50}	β_{41}	β_{32}	β_{23}
β_{11}	β_{21}	β_{12}	β_{31}	β_{22}	β_{13}	β_{41}	β_{32}	β_{23}	β_{14}
β_{02}	β_{12}	β_{03}	β_{22}	β_{13}	β_{04}	β_{32}	β_{23}	β_{14}	β_{05}
β_{30}	β_{40}	β_{31}	β_{50}	β_{41}	β_{32}	β_{60}	β_{51}	β_{42}	β_{33}
β_{21}	β_{31}	β_{22}	β_{41}	β_{32}	β_{23}	β_{51}	β_{42}	β_{33}	β_{24}
β_{12}	β_{22}	β_{13}	β_{32}	β_{23}	β_{14}	β_{42}	β_{33}	β_{24}	β_{15}
β_{03}	β_{13}	β_{04}	β_{23}	β_{14}	β_{05}	β_{33}	β_{24}	β_{15}	β_{06}

Odd Order Moment Problems

When $m = 2n + 1$, a general solution to some cases can be found in [6] and [8] together with a solution to the **truncated matrix moment problem**; a solution to the **cubic complex moment problem** (when $m = 3$) was given in [6]. In this note, we present an alternative solution to the “nonsingular” cubic complex moment sequence with anticipation that this work could contribute to pursue higher order problems.

C.Bayer and J. Teichmann proved if a moment sequence admits one or more representing measures, one of them must be finitely atomic.

Thus, if a real sequence $\beta^{(2n)} = \{\beta_{00}, \beta_{10}, \beta_{01}, \ldots, \beta_{2n,0}, \ldots, \beta_{0,2n}\}$ has a representing measure, then it can be finitely atomic, that is, we may write

$$\mu := \sum_{k=1}^{\ell} \rho_k \delta_{w_k},$$

where $\ell \leq \dim \mathcal{P}_{2n}$ ($\mathcal{P}_{2n}$ is the set of two variable polynomials whose degree $\leq 2n$.)

We try to find positive numbers $\rho_1, \ldots, \ldots, \rho_k$ called **densities** and points $(x_1, y_1), \ldots, (x_k, y_k)$ called **atoms** of the measure such that

$$\beta_{ij} = \rho_1 x_1^i y_1^j + \cdots + \rho_k x_k^i y_k^j = \int x^i y^j d\mu \qquad 0 \leq i + j \leq n.$$

List of Necessary Conditions

- Positivity (positive semidefinite): $M(n) \geq 0$
  (Use the nested determinant test or eigenvalues)

- Recursively Generated:
  $p, q, pq \in \mathcal{P}_n$, $p(X, Y) = 0 \Longrightarrow (pq)(X, Y) = \mathbf{0}$.

- Variety Condition: rank $M(n) \leq$ card $\mathcal{V}(M(n))$
  (Algebraic variety: $\mathcal{V} \equiv \mathcal{V}(M(n)) = \cap_{p(X,Y)=\mathbf{0}} \mathcal{Z}(p)$,
  where $\mathcal{Z}(p)$ is the zero set of the polynomial $p(x, y) = 0$.)

  Note. In the presence of a measure μ, the following must be true:

$$\text{supp } \mu \subseteq \mathcal{V}.$$

(RG)-property.

Example

Consider a recursively generated $M(2)$ with a column relation $X = \mathit{1}$:

$$\begin{pmatrix} 1 & X & Y & X^2 & XY & Y^2 \\ \hline \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix}$$

To have a representing measure, $M(2)$ must have additional column relations:

$$X^2 = \mathit{1}, \quad XY = Y.$$

Variety Condition $r \le v$

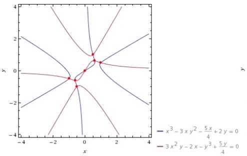
Example. Consider a complex moment matrix $M(3)$

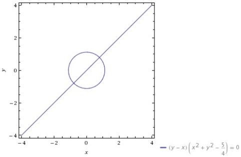
$$\begin{pmatrix}
224 & 0 & 0 & 176i & 208 & -176i & 0 & 0 & 0 & 0 \\
0 & 208 & -176i & 0 & 0 & 0 & 196i & 236 & -196i & -92 \\
0 & 176i & 208 & 0 & 0 & 0 & -92 & 196i & 236 & -196i \\
-176i & 0 & 0 & 236 & -196i & -92 & 0 & 0 & 0 & 0 \\
208 & 0 & 0 & 196i & 236 & -196i & 0 & 0 & 0 & 0 \\
176i & 0 & 0 & -92 & 196i & 236 & 0 & 0 & 0 & 0 \\
0 & -196i & -92 & 0 & 0 & 0 & 277 & -227i & -97 & -61i \\
0 & 236 & -196i & 0 & 0 & 0 & 227i & 277 & -227i & -97 \\
0 & 196i & 236 & 0 & 0 & 0 & -97 & 227i & 277 & -227i \\
0 & -92 & 196i & 0 & 0 & 0 & 61i & -97 & 227i & 277
\end{pmatrix}$$

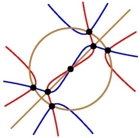
$M(3)$ is positive semidefinite and has three column relations

$$q_7(z,\bar{z}) = z^3 - 2iz - (5/4)\bar{z} = 0, \qquad q_{LC}(z,\bar{z}) := (\bar{z} + iz)(\bar{z}z - (5/4)) = 0,$$

and $\bar{q}_7(z,\bar{z}) = 0$. Now solve the system of polynomials Re $q_7 = 0$, Im $q_7 = 0$, Re $q_{LC} = 0$, Im $q_{LC} = 0$; there are 7 zeros to the system. Thus, we see that rank $M(3) = v = 7$.







Main Approaches to TMP

- Rank-preserving Positive Moment Matrix Extension;
  (Need to define proper higher order moments)
- Positive Extension of Riesz Functional;
  (Need to define proper higher order moments)
- Consistency for Extremal Cases;
  (Need to find a representation theorem for certain polynomials)
- Rank-one Decomposition.

Flat Extension Theorem

The extension $M(n)$ of $M(n-1)$ is said to be **flat** if rank $M(n-1)$ $=$ rank $M(n)$; (this case subsumes all previous results for the Hamburger, Stieltjes, Hausdorff, and Toeplitz TMP's.)

Theorem (RC-L. Fialkow, Mem. AMS, 1996)

If $\beta^{(2n)}$ has a rank $M(n)$-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.

To build a flat extension moment matrix

$$M(n+1) = \begin{bmatrix} M(n) & B(n+1) \\ B(n+1)^* & C(n+1) \end{bmatrix},$$

we need to allow new moment $\beta_{n,0}, \beta_{n-1,1}, \ldots, \beta_{0,n}$ with keeping recursiveness and then check if $C(n+1)$ is Hankel.
For example, if $M(1)$ has a column relation $X = 1$, then $M(2)$ must have the column relations $X^2 = X$ and $XY = Y$; thus,

$$\begin{bmatrix} M(1) & B(2) \end{bmatrix} = \left[\begin{array}{c|cc|ccc} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \hline \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \end{array}\right]$$

Note that the only new moment is β_{03}.

How to Find a Measure Explicitly

Positivity of Block Matrices (Tool to find an Extension).

Theorem (Smul'jan, 1959)

$$\tilde{A} := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \iff \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^* AW \end{cases}.$$

Moreover, rank $\tilde{A}$ =rank $A \iff C = W^* AW$.

How to Find Densities.

Let r be the rank of $M(n)$ and let $\mathcal{V} = \{(x_1, y_1), \ldots, (x_v, y_v)\}$ be the algebraic variety of $M(n)$. Also, denote the Vandermonde matrix V as

$$V = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & \cdots & x_1^n & \cdots & y_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_v & y_v & x_v^2 & x_v y_v & y_v^2 & \cdots & x_v^n & \cdots & y_v^n \end{pmatrix}$$

If $M(n)$ admits a flat extension $M(n+1)$, then the moment sequence has r-atomic representing measure and all the points in $\mathcal{V}$ serve as atoms of the measure. If $\mathcal{B}$ is the basis for the column space of $M(n)$ and if $V_{\mathcal{B}}$ is the submatrix of V with columns in $\mathcal{B}$. Then we can find the densities by solving:

$$V_{\mathcal{B}}^T \begin{pmatrix} \rho_1 & \rho_2 & \cdots & \rho_r \end{pmatrix}^T = \begin{pmatrix} \beta_{10} & \beta_{20} & \cdots & \beta_{r0} \end{pmatrix}^T.$$

The Quadratic Moment Problem: $M(1)$

Recall $M(1) = \begin{bmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} \\ \gamma_{01} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{02} & \gamma_{11} \end{bmatrix}$.

Theorem (RC-L. Fialkow, 1996)

$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$; *$r :=$ rank $M(1)$. Then TFAE:*

- *γ has a rep. meas.;*
- *γ has an r-atomic rep. meas.;*
- *$M(1) \geq 0$.*

In this case,

- $r = 1 \Longrightarrow \exists$ a unique rep. meas.;
- $r = 2 \Longrightarrow \exists$ 2-atomic rep. meas. parameterized by a line;
- $r = 3 \Longrightarrow \exists$ 3-atomic rep. meas. contain a sub-param. by a circle.

Example. Revisit the example at the beginning:

$$\beta^{(4)} : \{\beta_{ij}\} = \{5, 5, 14, 5, 14, 50\} \Longrightarrow M(1) = \begin{bmatrix} 5 & 5 & 14 \\ 5 & 5 & 14 \\ 14 & 14 & 50 \end{bmatrix}$$

Note that $M(1)$ has a column relation $X = \mathit{1}$.
To build a flat $M(2)$, we impose on $M(2)$ to have $X^2 = \mathit{1}$ and $XY = 1$:

$$\begin{bmatrix} M(1) & B(2) \end{bmatrix} = \begin{bmatrix} 5 & 5 & 14 & 5 & 14 & 50 \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \end{bmatrix}$$

We now find W such that $M(1)W = B(2)$ and get, for $k_1, k_2, k_3 \in \mathbb{R}$,

$$W = \begin{bmatrix} 1 - k_1 & -k_2 & (-7\,\beta_{03} - 27\,k_3 + 1250)/27 \\ k_1 & k_2 & k_3 \\ 0 & 1 & (5\beta_{03} - 700)/54 \end{bmatrix}.$$

Example (Continued)

We then evaluate $C(2) = W^* M(1) W$ and get

$$\begin{bmatrix} 5 & 5 & 14 & 5 & 14 & 50 \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \\ 5 & 5 & 14 & 5 & 14 & 50 \\ 14 & 14 & 50 & 14 & 50 & \beta_{03} \\ 50 & 50 & \beta_{03} & 50 & \beta_{03} & 5(s03^2 - 280\beta_{03} + 25000)/54 \end{bmatrix}$$

The column relations in $M(2)$ are $X = \mathit{1}, X^2 = \mathit{1}, XY = Y$, and

$$Y^2 = \frac{-7\beta_{03} - 27k + 1250}{27}\mathit{1} + kX + \frac{(5\beta_{03} - 700)}{54}Y$$

for some $k \in \mathbb{R}$. If we take $\beta_{03} = 194$, then the algebraic variety $\mathcal{V} = \{(1,1),(1,4)\}$.

If we take $\beta_{03} = 194$, then the algebraic variety $\mathcal{V} = \{(1,1),(1,4)\}$.

To find the densities, solve the Vandermonde equation:

$$\begin{bmatrix} 1 & 1 \\ y_1 & y_2 \end{bmatrix}\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} \beta_{00} \\ \beta_{01} \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix}.$$

Thus, we get $\rho_1 = 2$ and $\rho_2 = 3$ and a representing measure is $\mu = 2\,\delta_{(1,1)} + 3\,\delta_{(1,4)}$.

Invariance under a Degree-one Transformation

For $a, b, c, d, e, f \in \mathbb{R}$ with $bf \neq ce$, let
$\Psi(x, y) \equiv (\Psi_1(x, y), \Psi_2(x, y)) := (a + bx + cy, d + ex + fy)$ for $x, y \in \mathbb{R}$.
We now can build a new equivalent moment sequence $\tilde{\beta}^{(2n)} : \left\{\tilde{\beta}_{ij}\right\}$
with the definition $\tilde{\beta}_{ij} := L_\beta(\Psi_1^i \Psi_2^j)$ $(0 \leq i + j \leq 2n)$. We can immediately check that $L_{\tilde{\beta}}(p) = L_\beta(p \circ \Psi)$ for every $p \in \mathcal{P}_n$.

Proposition

[2, cf. Proposition 1.7] (Invariance under degree-one transformations.) *Let $\mathcal{M}(n)$ and $\tilde{\mathcal{M}}(n)$ be the moment matrices associated with γ and $\tilde{\beta}$, and let $J\hat{p} := \widehat{p \circ \Psi}$ $(p \in \mathcal{P}_n)$. Then the following are true:*

(i) $\tilde{\mathcal{M}}(n) = J^* \mathcal{M}(n) J$;

(ii) $\tilde{\mathcal{M}}(n) \geq 0 \iff \mathcal{M}(n) \geq 0$;

(iii) $\text{rank } \tilde{\mathcal{M}}(n) = \text{rank } \mathcal{M}(n)$;

(iv) *$\mathcal{M}(n)$ admits a flat extension if and only if $\tilde{\mathcal{M}}(n)$ admits a flat extension.*

Singular Quartic Moment Problem

Consider a quartic moment sequence $\gamma : \gamma_{00}, \gamma_{01}, \dots, \gamma_{04}, \dots, \gamma_{40}$.
Note that if a singular $M(2)(\gamma)$ with a conic column relation $p(Z, \bar{Z}) = 0$ has a measure μ, then supp $\mu \subseteq \mathcal{Z}(p)$.
Via the equivalence of TMP under degree-one transformations, one can reduce the study to cases corresponding to the following four real conics:

$(a)\ \bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$	parabola: $y = x^2$
$(b)\ \bar{W}^2 = -4i1 + W^2$	hyperbola: $yx = 1$
$(c)\ \bar{W}^2 = W^2$	pair of intersect. lines: $yx = 0$
$(d)\ \bar{W}W = 1$	unit circle: $x^2 + y^2 = 1$;
$(e)\ (W + \bar{W})(W + \bar{W} - 2) = 0$	pair of parallel lines: $x(x-1) = 0$.

Using the Flat Extension Theorem, we know all the cases except (c) have a flat extension $M(3)$, and so admit a rank $M(2)$-atomic (5-atomic) measure.
However, (c) may allow a 6-atomic measure.

Nonsingular Quartic Moment Problem: $M(2)$

Theorem (L. Fialkow-J. Nie, JFA, 2009)

If $M(2)(\beta) > 0$, then β has a representing measure.

Proof. Based on convex analysis of a positive linear functional.

Theorem (R. Curto-S. Yoo, cond. to appear in PAMS)

If $M(2)(\beta) > 0$, then β has a 6-atomic representing measure.

Proof. Based on Rank-one decomposition.

More General Solutions to TMP

(i) The column $\bar{Z}$ is a linear combination of the columns 1 and Z;

(ii) For some $k \leq [n/2] + 1$, the analytic column Z^k is a linear combination of columns corresponding to monomials of lower degree;

(iii) The analytic columns of $M(n)$ are linearly dependent and span $\mathcal{C}_{M(n)}$, the column space of $M(n)$;

(iv) $M(n)$ is *extremal* (rank $M(n) = \text{card } \mathcal{V}$, where $\mathcal{V}$ is the algebraic variety of the moment sequence) and consistent;

(v) $M(n)$ is *recursively determinate*, that is, if $M(n)$ has only column dependence relations of the form

$$X^n = p(X, Y) \quad (p \in \mathcal{P}_{n-1});$$
$$Y^m = q(X, Y) \quad (q \in \mathcal{P}_m,\ q \text{ has no } y^m \text{ term, } m \leq n),$$

where $\mathcal{P}_k$ denotes the subspace of polynomials in $\mathbb{R}[x, y]$ whose degree is less than or equal to k.

Normalized $\mathcal{M}(1)$

We need to have a "normalized" $\mathcal{M}(2)$, that is, whose $\mathcal{M}(1)$ is the identity matrix. Without loss of generality, we may assume $\beta_{00} = 1$. Let d_i denote the leading principal minors of $\mathcal{M}(2)$. In particular,

$$\begin{aligned} d_2 &= -\beta_{10}^2 + \beta_{20} \\ d_3 &= -\beta_{02}\beta_{10}^2 + 2\beta_{01}\beta_{10}\beta_{11} - \beta_{11}^2 - \beta_{01}^2\beta_{20} + \beta_{02}\beta_{20}. \end{aligned}$$

Choose a degree one transformation:

$$\Psi(x, y) = (a + bx + cy, d + ex + fy),$$

where $a = \frac{\beta_{01}\beta_{20} - \beta_{10}\beta_{11}}{\sqrt{d_2 d_3}}$, $b = \frac{\beta_{11} - \beta_{01}\beta_{10}}{\sqrt{d_2 d_3}}$, $c = -\sqrt{\frac{d_2}{d_3}}$, $d = -\frac{\beta_{10}}{\sqrt{d_2}}$, $e = \frac{1}{\sqrt{d_2}}$, and $f = 0$. Note that $bf - ce = -\sqrt{\frac{1}{d_3}} \neq 0$. Through this transformation, any positive semidefinite $\mathcal{M}(2)$ with a nonsingular $\mathcal{M}(1)$ can be translated to

$$\begin{pmatrix} 1 & 0 & 0 & \tilde\beta_{20} & \tilde\beta_{11} & \tilde\beta_{02} \\ 0 & 1 & 0 & \tilde\beta_{30} & \tilde\beta_{21} & \tilde\beta_{12} \\ 0 & 0 & 1 & \tilde\beta_{21} & \tilde\beta_{12} & \tilde\beta_{03} \\ \tilde\beta_{20} & \tilde\beta_{30} & \tilde\beta_{21} & \tilde\beta_{40} & \tilde\beta_{31} & \tilde\beta_{22} \\ \tilde\beta_{11} & \tilde\beta_{21} & \tilde\beta_{12} & \tilde\beta_{31} & \tilde\beta_{22} & \tilde\beta_{13} \\ \tilde\beta_{02} & \tilde\beta_{12} & \tilde\beta_{03} & \tilde\beta_{22} & \tilde\beta_{13} & \tilde\beta_{04} \end{pmatrix}.$$

Recursively Determinate Moment Problems.

$\mathcal{M}(n)$ is *recursively determinate*, that is, if $\mathcal{M}(n)$ has only column dependence relations of the form

$$X^n = p(X, Y) \quad (p \in \mathcal{P}_{n-1});$$
$$Y^m = q(X, Y) \quad (q \in \mathcal{P}_m,\ q \text{ has no } y^m \text{ term}, m \leq n),$$

where $\mathcal{P}_k$ denotes the subspace of polynomials in $\mathbb{R}[x, y]$ whose degree is less than or equal to k. We may summarize the main results in [3] as follows:

> *$\mathcal{M}(n)$ admits an flat extension $\mathcal{M}(n+1)$ $\iff$ $\mathcal{M}(n)$ is positive and recursively determinate for an even n; additionally, it is required to check simply whether $\mathcal{M}(n+1)$ is positive or not for an odd n.*

Cubic Moment Problem

We may assume that $\beta \equiv \beta^{(3)} : \{1, 0, 0, 1, 0, 1, a_1, a_2, a_3, a_4\}$ and we may write an extension as

$$\mathcal{M}(2) := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & a_2 & a_3 & a_4 \\ 1 & a_1 & a_2 & \beta_{40} & \beta_{31} & \beta_{22} \\ 0 & a_2 & a_3 & \beta_{31} & \beta_{22} & \beta_{13} \\ 1 & a_3 & a_4 & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix}, \tag{1}$$

where $\beta_{40}, \beta_{31}, \beta_{22}, \beta_{13}$, and β_{04} are undetermined new moments. We refer $\beta^{(3)}$ with an invertible $\mathcal{M}(1)$ as a nonsingular cubic moment sequence. Our strategy is to show that an extended sequence $\beta^{(4)}$ of $\beta^{(3)}$ admits a flat extension so that both sequences has a 4-atomic representing measure.

Cubic Moment Problem (Part 1)

Using the Smul'jan's Theorem, we compute when $\mathcal{M}(2)$ is flat. Indeed, if we take $W := B(2)$, then $C(2) = W^T \mathcal{M}(1) W$ is

$$\begin{pmatrix} 1 + a_1^2 + a_2^2 & a_1 a_2 + a_2 a_3 & 1 + a_1 a_3 + a_2 a_4 \\ a_1 a_2 + a_2 a_3 & a_2^2 + a_3^2 & a_2 a_3 + a_3 a_4 \\ 1 + a_1 a_3 + a_2 a_4 & a_2 a_3 + a_3 a_4 & 1 + a_3^2 + a_4^2 \end{pmatrix} \tag{2}$$

Consequently, $\mathcal{M}(2)$ is a flat extension of $\mathcal{M}(1)$ if and only if $1 + a_1 a_3 + a_2 a_4 - a_2^2 - a_3^2 = 0$, which is equivalent to the commutativity of the matrices defined in [7]. If it is the case, then since $\mathcal{M}(2)$ has a unique 3-atomic representing measure, $\beta^{(3)}$ has a minimal 3-atomic measure. For the coming references, let $k := 1 + a_1 a_3 + a_2 a_4 - a_2^2 - a_3^2$. We proved the following theorem just now:

Theorem

If $\mathcal{M}(1)$ of $\beta^{(3)} : \{1, 0, 0, 1, 0, 1, a_1, a_2, a_3, a_4\}$ is positive definite and $k = 0$, then $\beta^{(3)}$ admits a 3-atomic representing measure.

Part 2

Theorem

If $\mathcal{M}(1)$ of $\beta^{(3)}: \{1,0,0,1,0,1,a_1,a_2,a_3,a_4\}$ is positive definite and $k \neq 0$, then $\beta^{(3)}$ admits a 4-atomic representing measure.

Proof. Our strategy is to show $\beta^{(3)}$ can be extended to a $\beta^{(4)}$ whose $\mathcal{M}(2)$ admits a flat extension.

Case 1. $k > 0$

The calculation of $C(2)$ in (2) promotes us to select

$$\begin{aligned} \beta_{40} &:= 1 + a_1^2 + a_2^2, & \beta_{31} &:= a_1a_2 + a_2a_3, \\ \beta_{22} &:= 1 + a_1a_3 + a_2a_4, & \beta_{13} &:= a_2a_3 + a_3a_4, \\ \beta_{04} &:= 1 + a_3^2 + a_4^2. \end{aligned}$$

For positivity of $\mathcal{M}(2)$, we need β_{22} to be positive, which follows from

$$k > 0 \Longrightarrow 1 + a_1a_3 + a_2a_4 > a_2^2 + a_3^2 \geq 0. \qquad (3)$$

$\Longrightarrow$ rank $\mathcal{M}(2) = 4$ with $X^2 = 1 + a_1X + a_2Y$ and $Y^2 = 1 + a_3X + a_4Y$.
$\Longrightarrow$ Nested determinants of the compression of $\mathcal{M}(2)$: $1, 1, 1$, and k.
$\Longrightarrow$ $\mathcal{M}(2)$ is positive semidefinite and recursively determinate.
$\Longrightarrow$ $\mathcal{M}(2)$ has a 4-atomic measure, so does $\beta^{(3)}$.

Proof (Continued)

Case 2. $k < 0$
With a similar reason, let

$$\beta_{31} := a_1a_2 + a_2a_3, \qquad \beta_{22} := a_2^2 + a_3^2, \qquad \beta_{13} := a_2a_3 + a_3a_4.$$

In order for positivity of $\mathcal{M}(2)$ and to have rank $\mathcal{M}(2) = 4$, set

$$\begin{aligned}\beta_{40} &:= 2 + a_1^2 + a_2^2 \\ \beta_{04} &:= 2 + a_2^4 + 2a_1a_3 + a_1^2a_3^2 + 2a_2^2a_3^2 + a_3^4 + 2a_2a_4 + 2a_1a_2a_3a_4 + a_4^2 + a_2^2a_4^2 \\ &\quad -2a_2^2 - 2a_1a_2^2a_3 - a_3^2 - 2a_1a_3^3 - 2a_2^3a_4 - 2a_2a_3^2a_4\end{aligned}$$

so that the two columns XY and Y^2 in $\mathcal{M}(2)$ are linearly dependent.
Claim. $\beta_{04} > 0$ for $\mathcal{M}(2) \geq 0$.
Note that if a function $f \in \mathcal{P}_{2d}$ is a sum of squares if and only if $f = \mathbf{z}^T Q\mathbf{z}$ for some $Q \geq 0$, where $\mathbf{z}$ is a vector of monomials of degree less than or equal to d. The semidefinite program for β_{04} has a following solution:

$$\beta_{04} = \mathbf{z}^T \begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix} \mathbf{z} \tag{4}$$

where $\mathbf{z} = (1, a_3, a_4, a_2^2, a_3^2, a_1a_3, a_2a_4)^T$. Due to positivity of the matrix in the above, we know that $\beta_{04} > 0$.

Proof (Continued)

Since all the nested determinants of the compression of $\mathcal{M}(2)$ are 1 and both β_{40} and β_{04} are positive, it follows that $\mathcal{M}(2) \geq 0$.
Approach 1. Show: card $\mathcal{V}(M(2)) = 4$. Very difficult!
Approach 2. Translate the problem into the complex version and to apply a previous result: Let

$$L := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & i & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2i & 0 & -2i \\ 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad M(2) = L^* \mathcal{M}(2) L. \tag{5}$$

Then we know rank $M(2) = 4$ with on column relation
$\bar{Z}Z = 2 1 + A(a_1, a_2, a_3, a_4)Z + B(a_1, a_2, a_3, a_4) + (k-1)/(-k-1)Z^2$,
where A and B are some polynomials in a_1, a_2, a_3, and a_4 with complex coefficients.
Note that the determinant of the compression of $M(2)$ is $4(-k-1)^2$, which means $-k-1 \neq 0$. Since $k < 0$, $k-1 < 0$; we know $(k-1)/(-k-1) \neq 0$. By Theorem 3.1 in [2], $M(2)$ has a 4-atomic measure if and only if there is a new moment γ_{32} satisfying

$$\gamma_{32} \equiv \bar{\gamma}_{23} = 2\gamma_{21} + A\gamma_{22} + B\gamma_{31} + \frac{k-1}{-k-1}\gamma_{23}. \tag{6}$$

A calculation shows this equation has a solution, and thus the proof is complete.

Future Study.

- Solve quintic moment problems;
- Solve sextic moment problems with a single cubic column relation;
- Solve nonsingular sextic moment problems;
- Find more connections between the moment problem and operator theory focusing on, e.g., subnormal completion problem for 2-variable weighted shifts and the study of quadratic hyponormality for unilateral weighted shifts;

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