On Semi-cubically Hyponormal Weighted Shifts

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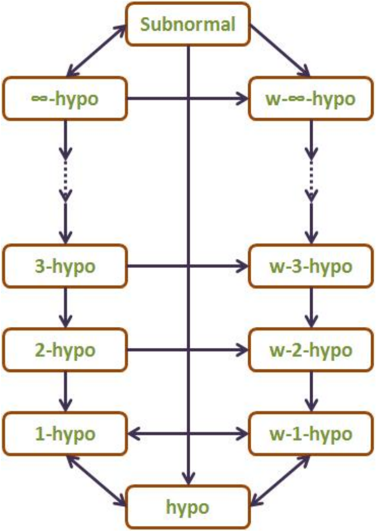
Introduction

- $\mathcal{H}$: a separable, infinite dimensional, complex Hilbert space.
- $\mathcal{L}(\mathcal{H})$: the algebra of all bounded linear operators on $\mathcal{H}$.
- **Bram and Halmos [1955, Duke Math. J. / 1960, SBM]**
  - T is subnormal $\Longleftrightarrow$ $BH(n) := ([T^{*j}, T^i])_{i,j=1}^n \geqslant 0$ for all $n \in \mathbb{N}$.
- **Definitions**
  - T is (strongly) n-hyponormal if $BH(n) \geqslant 0$.
  - T is polynomially hyponormal if $p(T)$ is hyponormal for every polynomial p.
  - T is weakly n-hyponormal if $p(T)$ is hyponormal for all polynomial p with degree n or less.
  - weakly 2 (or weakly 3)-hyponormal is referred to as quadratically (or cubically) hyponormal.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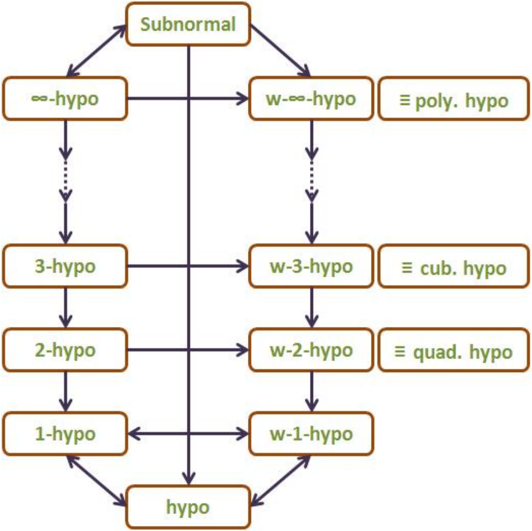
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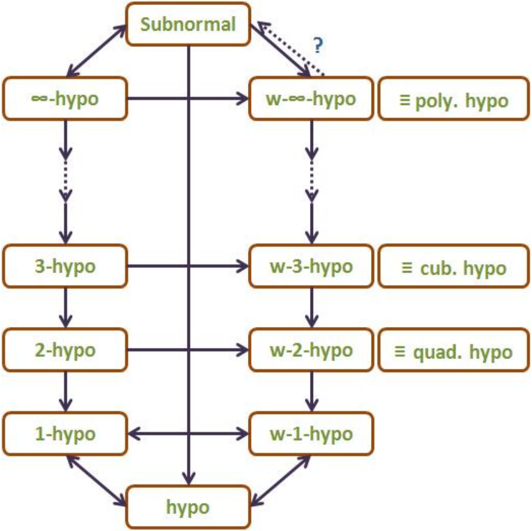
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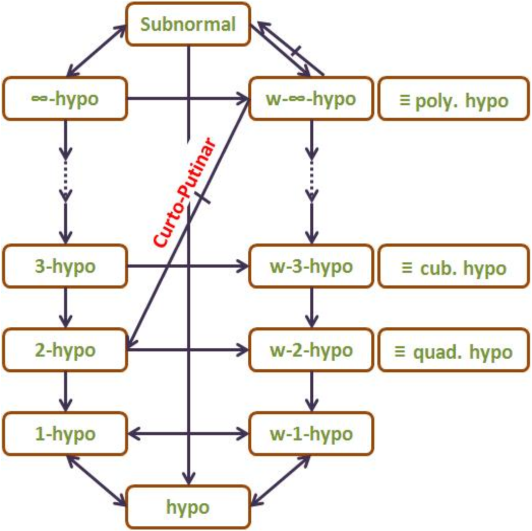
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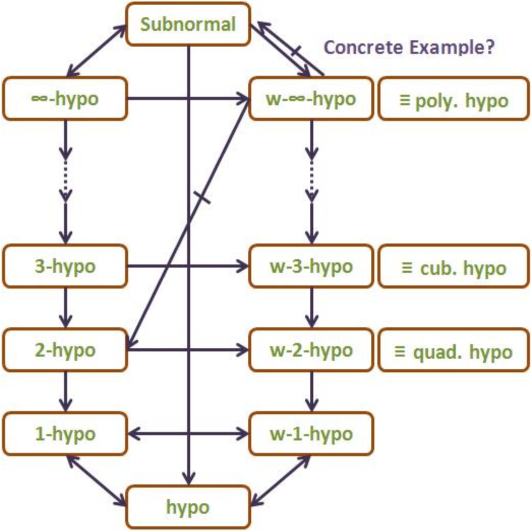
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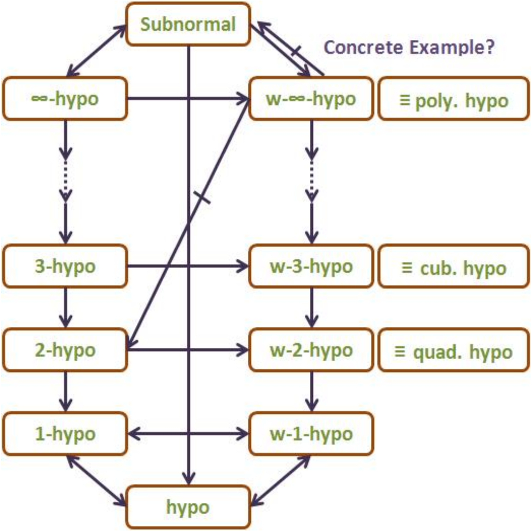
- **Open Problem**
  - Subnormal $\Longleftrightarrow$ polynomially hyponormal ?
- **Curto-Putinar [1991, Bull.AMS / 1993, J.Funct.Anal.]**
  - There exists a polynomially hyponormal operator which is not 2-hyponormal.
  - **Corollary** There exists a unilateral weighted shift that is polynomially hyponormal but not subnormal.
- **Conclusion** The answer is NO.
- **Revised research problem**
  Find concrete weighted shift s.t
  it is polynomially hyponormal but NOT subnormal.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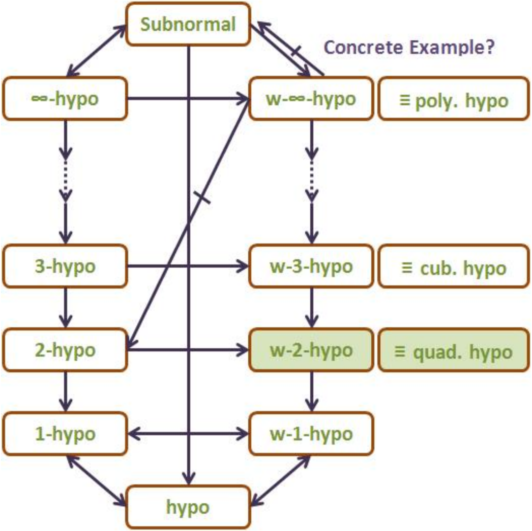
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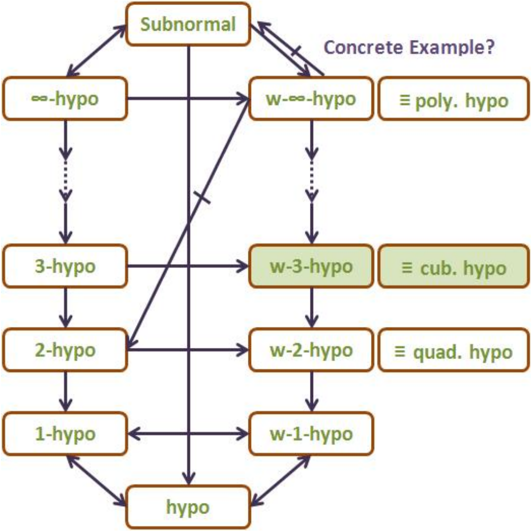

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Semi-cubic hyponormality

- $\alpha : \alpha_0, \alpha_1, \alpha_2, \alpha_3, \cdots$: a weight sequence with $\alpha_i > 0$.
- W_α : the weighted shift on $l^2(\mathbb{N}_0)$ with an orthonormal basis $\{e_i\}_{i=0}^{\infty}$ is defined by $W_\alpha(e_j) = \alpha_j e_{j+1}, \forall j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

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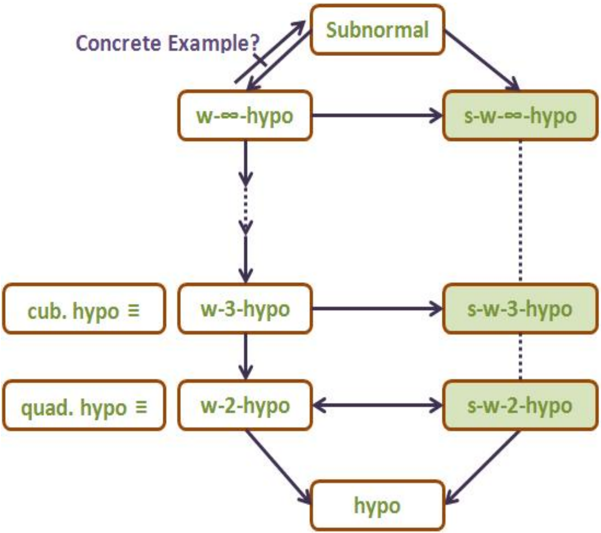
- **Definitions (Jung, Exner, 2012)**
  - W_α : semi-weakly n-hyponormal if

    $W_\alpha + sW_\alpha^n$: hyponormal, $\forall s \in \mathbb{C}$.

  - W_α : semi-cubically hyponormal if

    $W_\alpha + sW_\alpha^3$: hyponormal, $\forall s \in \mathbb{C}$.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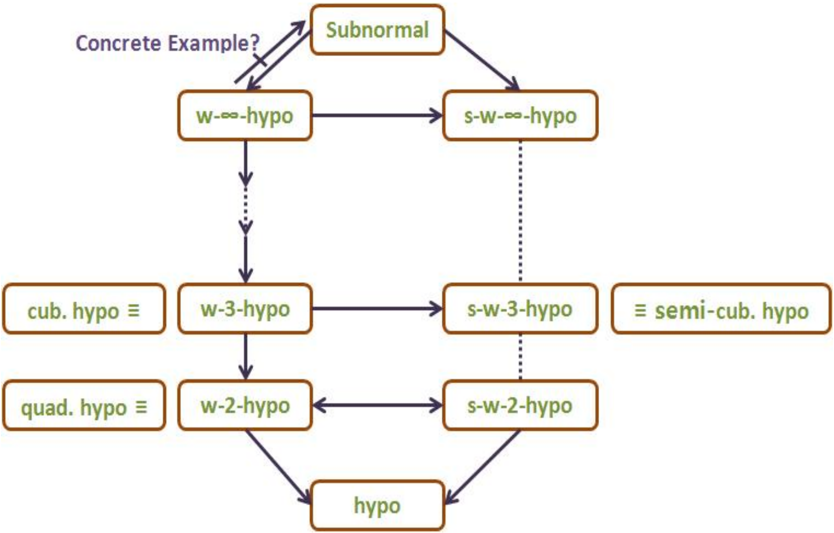
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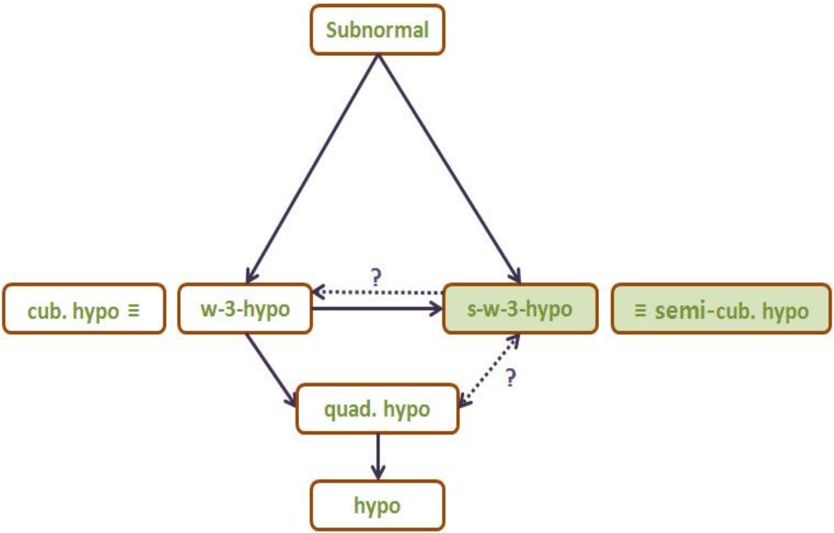
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- P_n : the orthogonal projection onto $\vee_{k=0}^{n}\{e_k\}$.
- For $s \in \mathbb{C}$ and $n \geqslant 0$, define

$$D_n := D_n(s) = P_n\left[(W_\alpha + sW_\alpha^3)^*, W_\alpha + sW_\alpha^3\right]P_n, \forall s \in \mathbb{C}$$
$$= \begin{pmatrix} q_0 & 0 & z_0 & 0 & \cdots & 0 \\ 0 & q_1 & 0 & z_1 & \ddots & \vdots \\ \bar{z}_0 & 0 & q_2 & \ddots & \ddots & 0 \\ 0 & \bar{z}_1 & \ddots & \ddots & \ddots & z_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{z}_{n-2} & 0 & q_n \end{pmatrix},$$

where $q_k = u_k + v_k|s|^2$, $z_k = \sqrt{w_k}\bar{s}$,

$$u_k = \alpha_k^2 - \alpha_{k-1}^2,$$
$$v_k = \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 - \alpha_{k-3}^2\alpha_{k-2}^2\alpha_{k-1}^2,$$
$$w_k = \alpha_k^2\alpha_{k+1}^2(\alpha_{k+2}^2 - \alpha_{k-1}^2)^2.$$

- W_α : semi-cubically hyponormal

  $\Longleftrightarrow D_n(s) \geqslant 0, \forall s \in \mathbb{C}, \forall n \in \mathbb{N}_0.$

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- **Stampfli's method [1966, Pacific J. Math.]**
  - For a given numbers $0 < \alpha_0 < \alpha_1 < \alpha_2$, define

    $\alpha_n^2 = \Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}$, for all $n \geqslant 2$,

    where $\Psi_0 = -\frac{\alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}$, $\quad \Psi_1 = \frac{\alpha_1^2 (\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}$.
  - α is written as $\alpha : (\alpha_0, \alpha_1, \alpha_2)^\wedge$.
  - $\alpha := \{\alpha_n\}_{n=0}^\infty$ is bounded.
  - $\lim_{n\to\infty} \alpha_n = L := \sqrt{\frac{1}{2}\left(\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}\right)}$.

Main Results

Lemma 1

For $\alpha : \sqrt{x}, \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)^{\wedge}$,

$u_n v_{n+2} = w_n, n \geqslant 2.$

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- Define a sequence $\{\eta_n\}_{n=0}^{\infty}$ as $\eta_n = \frac{v_n}{u_n}$.

Lemma 2

For $\alpha : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^{\wedge}$, it holds that

$$\eta_{n+1} \geqslant \eta_n, \ \forall n \geqslant 4,$$
$$\lim_{n\to\infty} \eta_n = \frac{\left(\Psi_1^2 + \Psi_0\right)^2}{\Psi_0^2} L^4 =: Q.$$

Lemma 3

For $\alpha : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$, suppose a positive integer $l \geqslant 2$. Then the following assertion holds ;

$$q_l - \cfrac{z_l^2}{q_{l+2} - \cfrac{z_{l+2}^2}{\ddots \cfrac{\ddots}{q_{l+2k-2} - \frac{z_{l+2k-2}^2}{q_{l+2k}}}}}$$

$$= v_l t + \frac{u_l}{1 + \eta_{l+2} t + \eta_{l+2}\eta_{l+4} t^2 + \cdots + \eta_{l+2}\eta_{l+4}\cdots\eta_{l+2k} t^k}$$

where $t = |s|^2$ and $k \geqslant 1$.

- For $t(=|s|^2) \geqslant 0$ and for each n, define,

$$A_n(t) = \begin{pmatrix} q_0 & 0 & -\sqrt{w_0 t} & 0 & \cdots & 0 \\ 0 & q_1 & 0 & -\sqrt{w_1 t} & \ddots & \vdots \\ -\sqrt{w_0 t} & 0 & q_2 & 0 & \ddots & 0 \\ 0 & -\sqrt{w_1 t} & 0 & q_3 & \ddots & -\sqrt{w_{n-2} t} \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\sqrt{w_{n-2} t} & 0 & q_n \end{pmatrix}$$

where the q_i, u_i, v_i, and w_i are for D_n.

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Lemma 4

Let W_α be a weighted shift. Then for each n,
$D_n(s)$ is a (complex) positive quadratic form over $\mathbb{C}^{n+1}$ for all $s \in \mathbb{C}$
if and only if
$A_n(t)$ is a (real) positive quadratic form over $\mathbb{R}^{n+1}$ for all $t \geqslant 0$.

- Define, for a vector $(x_0, \cdots, x_n)$ in $\mathbb{R}_+^{n+1}$,

$$F_n(x_0, \ldots, x_n, t) := \langle A_n(t)(x_0, \cdots, x_n)^T, (x_0, \cdots, x_n)^T \rangle$$
$$= \sum_{i=0}^{n} u_i x_i^2 - 2\sqrt{t} \sum_{i=0}^{n-2} \sqrt{w_i} x_i x_{i+2} + t \sum_{i=0}^{n} v_i x_i^2.$$

- Put $f_2 \equiv f_2(x_0, x_1, x_2, x_3, t)$

$$:= \sum_{i=0}^{1} (u_i + t v_i) x_i^2 + t \sum_{i=2}^{3} v_i x_i^2 - 2\sqrt{t} \sum_{i=0}^{1} \sqrt{w_i} x_i x_{i+2}.$$

- $F_n(x_0, x_1, \cdots, x_n, t)$

$$= f_2 + u_{n-1} x_{n-1}^2 + u_n x_n^2 + \sum_{i=2}^{n-2} \left(\sqrt{u_i} x_i - \sqrt{t v_{i+2}} x_{i+2} \right)^2.$$

Lemma 5

Let $\alpha(x) : \sqrt{x}, \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)^{\wedge}$ and $n \geqslant 4$. Then T.F.A.E:

(i) $F_n(x_0, x_1, \cdots, x_n, t) \geqslant 0$ for any $x_i,\ t \in \mathbb{R}_+\ (i = 0, 1, \ldots, n)$;

(ii) $f_2 + P(2; n)x_2^2 + P(3; n)x_3^2 \geqslant 0$, for any $x_i, t \in \mathbb{R}_+\ (i = 0, 1, 2, 3)$ where

$$P(l; n) = \frac{u_l}{1+\eta_{l+2}t+\eta_{l+2}\eta_{l+4}t^2+\cdots+\eta_{l+2}\eta_{l+4}\cdots\eta_{l+2[(n-l)/2]}t^{[(n-l)/2]}}$$

with $l \geqslant 2$.

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with $l \geqslant 2$.

Lemma 6

Suppose $n \geqslant 4$.

$F_n(x_0, x_1, ..., x_n, t) \geqslant 0$ for any $x_i \in \mathbb{R}_+$ and $t > \frac{1}{Q}$

if and only if

$f_2(x_0, ..., x_3, t) \geqslant 0$ for any $x_0, ..., x_3 \in \mathbb{R}_+$ and $t > \frac{1}{Q}$.

- $\widehat{h}_3 = \min\{\sqrt{a}, \sqrt{\Theta}\}$ where
  $\Theta := \frac{ab\{(c-a)^2(c-b)Q + c(-a^2b^2 + a^2bc + a^2c^2 + 2ab^2c - 4abc^2 + bc^3)\}}{a^2bc(b-a)^2 + (c-a)^2(a^2 + bc - 2ab)Q} > 0.$

Lemma 7

$\sup\{x : f_2(x_0, x_1, x_2, x_3, t) \geqslant 0, t > \frac{1}{Q}\} \leqslant (\widehat{h}_3)^2.$

- **Proof of Lemma 7 (Setch)**

Using Ψ_0 and Ψ_1, we can have that

$$\alpha_4^2 = \frac{1}{bc - ac}\left(ab^2 - 2abc + bc^2\right),$$

$$\alpha_5^2 = \frac{-bc^3 + a^2b^2 - a^2c^2 + 4abc^2 - 2ab^2c - a^2bc}{(a-b)(ab - 2ac + c^2)}.$$

We can have the corresponding symmetric matrix $\Omega(t)$ with respect to $f_2(x_0, x_1, x_2, x_3, t)$ such that $f_2(x_0, x_1, x_2, x_3, t) = \langle \Omega(t)(x_0, \cdots, x_3)^T, (x_0, \cdots, x_3)^T \rangle$, namely,

$$\Omega(t) = \begin{pmatrix} x + abxt & 0 & -\sqrt{txab^2} & 0 \\ 0 & a - x + abct & 0 & -\sqrt{tab(c-x)^2} \\ -\sqrt{txab^2} & 0 & tbc\alpha_4^2 & 0 \\ 0 & -\sqrt{tab(c-x)^2} & 0 & t\left(c\alpha_4^2\alpha_5^2 - xab\right) \end{pmatrix}.$$

From the elementary operations of determinants,

$$\det \Omega(t) = d_1(t) \cdot d_2(t),$$

where

$$d_1(t) = \det \begin{pmatrix} x + abxt & -\sqrt{txab^2} \\ -\sqrt{txab^2} & tbc\alpha_4^2 \end{pmatrix},$$

$$d_2(t) = \det \begin{pmatrix} a - x + abct & -\sqrt{tab(c-x)^2} \\ -\sqrt{tab(c-x)^2} & t\left(c\alpha_4^2\alpha_5^2 - xab\right) \end{pmatrix}.$$

By direct computations,

$$d_1(t) = \frac{a^2 - 2ac + c^2 + t\left(abc^2 - 2a^2bc + a^2b^2\right)}{b-a} > 0, \quad \text{for all } t > 0,$$

and

$$d_2(t) = \frac{bt\left(A(x)t + B(x)\right)}{(b-a)^2},$$

where

$$A(x) =$$

$$abc\left(-a(b-a)^2 x - a^2b^2 + a^2bc + a^2c^2 + 2ab^2c - 4abc^2 + bc^3\right)$$

and

$$B(x) = (c-a)^2\left(ab(c-b) + x(2ab - bc - a^2)\right).$$

Then

$d_2(t) \geqslant 0$ for all $t > \frac{1}{Q} \iff A(x) \geqslant 0$ and $-\frac{B(x)}{A(x)} \leqslant \frac{1}{Q}$.

And we can have that

$$A(x) \geqslant 0 \iff x \leqslant a \quad \text{and} \quad -\frac{B(x)}{A(x)} \leqslant \frac{1}{Q} \iff x \leqslant \Theta.$$

Since $0 < x < a$, we have $x \leqslant (\widehat{h}_3)^2$. □

Positively Semi-cubic hyponormality

Recall W_α is semi-cubically hyponormal $\Longleftrightarrow \mathrm{D}_n(s) \geqslant 0$, $s \in \mathbb{C}$, $n \in \mathbb{N}_0$.

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Recall W_α is semi-cubically hyponormal $\iff D_n(s) \geqslant 0,\ s \in \mathbb{C},\ n \in \mathbb{N}_0$.

W_α is semi-cubically hyponormal
$\iff D_n^{(1)}(t) \geqslant 0,\ D_n^{(2)}(t) \geqslant 0,\ t \geqslant 0,\ n \in \mathbb{N}_0$, where

$$D_n^{(1)} := \begin{pmatrix} q_0 & r_0 & 0 & & \\ r_0 & q_2 & r_2 & \ddots & \\ 0 & r_2 & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & r_{2n-2} \\ & & 0 & r_{2n-2} & q_{2n} \end{pmatrix},\ D_n^{(2)} := \begin{pmatrix} q_1 & r_1 & 0 & & \\ r_1 & q_3 & r_3 & \ddots & \\ 0 & r_3 & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & r_{2n-1} \\ & & 0 & r_{2n-1} & q_{2n+1} \end{pmatrix},$$

$t = |s|^2,\ q_k = u_k + t v_k,\ r_k = \sqrt{t w_k}$.

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$t = |s|^2,\ q_k = u_k + tv_k,\ r_k = \sqrt{tw_k}$.

Put $d_n^{(j)}(t) := \det\left(D_n^{(j)}(t)\right),\ j = 1, 2$.

$$\implies d_n^{(j)}(t) = \sum_{i=0}^{n+1} c^{(j)}(n,i)t^i.$$

If $c^{(j)}(n,i) \geqslant 0$, for $0 \leqslant i \leqslant n$ and $c^{(j)}(n,n+1) > 0,\ n \geqslant 0,\ j = 1, 2$, then W_α is semi-cubically hyponormal.

Definition

W_α is positively semi-cubically hyponormal if

$c^{(j)}(n,i) \geqslant 0$, for $0 \leqslant i \leqslant n$ and $c^{(j)}(n,n+1) > 0$, $n \geqslant 0$, $j = 1, 2$.

Definition

W_α is positively semi-cubically hyponormal if

$c^{(j)}(n, i) \geqslant 0$, for $0 \leqslant i \leqslant n$ and $c^{(j)}(n, n+1) > 0$, $n \geqslant 0$, $j = 1, 2$.

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Lemma 8

Let $\alpha : \sqrt{x}, \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)^{\wedge}$.
$x \leqslant (\widehat{h}_3)^2 \Longrightarrow W_{\alpha(x)}$ is positively semi-cubically hyponormal.

Theorem 9

Let $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^{\wedge}$ with $x \leqslant a < b < c$ and let $W_{\alpha(x)}$ be the associated weighted shift. Then T.F.A.E:

(i) $W_{\alpha(x)}$ is semi-cubically hyponormal,

(ii) $F_n(x_0, x_1, \cdots, x_n, t) \geqslant 0$ for all $x_0, x_1, \cdots, x_n, t$ in $\mathbb{R}_+$ and all $n \geqslant 2$,

(iii) $F_n(x_0, x_1, \cdots, x_n, t) \geqslant 0$ for all $x_0, x_1, \cdots, x_n$ in $\mathbb{R}_+$, $t > \frac{1}{Q}$ and all $n \geqslant 4$,

(iv) $f_2(x_0, x_1, x_2, x_3, t) \geqslant 0$ for all x_0, x_1, x_2, x_3 in $\mathbb{R}_+$ and all $t > \frac{1}{Q}$,

(v) $\sqrt{x} \leqslant \widehat{h}_3$,

(vi) $W_{\alpha(x)}$ is positively semi-cubically hyponormal.

- Put $h_2 := \left(\sup\{x | W_{\alpha(x)} \text{is quadratically hyponormal}\}\right)^{\frac{1}{2}}$.
- From Curto and Fialkow,

$$h_2 = \min\left\{\sqrt{a}, \left(\frac{a^2b^2c+ab^2(c-a)K+ab(c-b)K^2}{a^3b+ab(c-a)K+(a^2-2ab+bc)K^2}\right)^{\frac{1}{2}}\right\}$$

where $K := -\frac{\Psi_1^2}{\Psi_0^2}L^2$.

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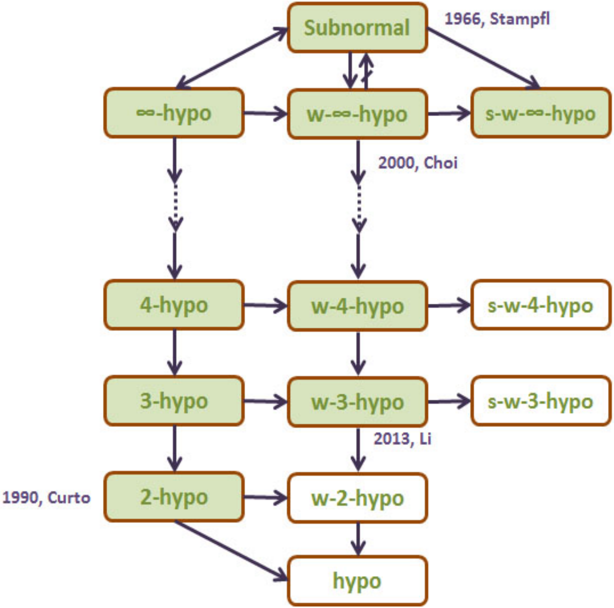
where $K := -\frac{\Psi_1^2}{\Psi_0^2}L^2$.

Example 10

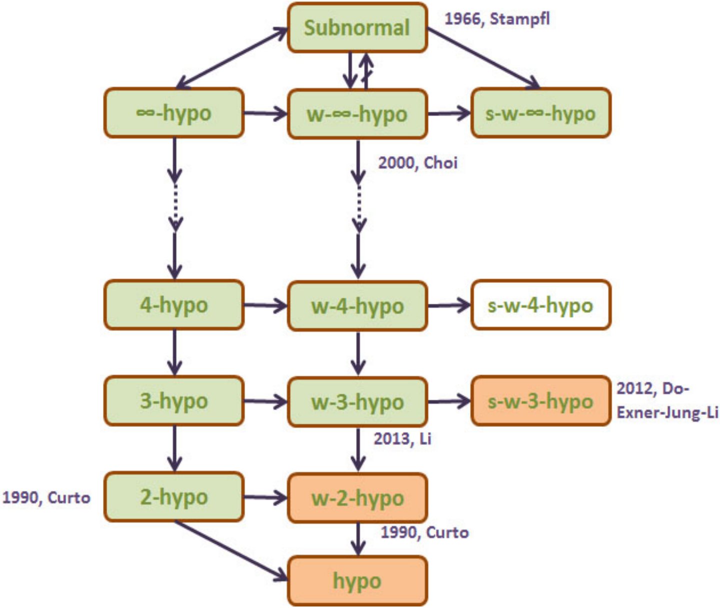
Let a weight sequence $\alpha(x) : \sqrt{x}, (1, \sqrt{2}, \sqrt{3})^{\wedge}$ with $0 < x \leqslant 1$.
$\Rightarrow \Psi_0 = -2, \Psi_1 = 4,\ K = 8\sqrt{2} + 16$ and $Q = 49(\sqrt{2} + 2)^2$.
$\Rightarrow h_2 \approx 0.85628$, and $\widehat{h}_3 \approx 0.82520$.

Flatness of Semi-cubic Hyponormality

- **Stampfli (1966, Pacific J. Math.)**
  W_α is subnormal and $\alpha_n = \alpha_{n+1}$ for $n \in \mathbb{N}_0 \Longrightarrow \alpha_1 = \alpha_2 = \cdots$.
- **Curto (1990, IEOT)**
  W_α is 2-hypo. and $\alpha_n = \alpha_{n+1}$ for $n \in \mathbb{N}_0 \Longrightarrow \alpha_1 = \alpha_2 = \cdots$.
- **Choi (2000, Bull. KMS)**
  W_α is poly. hypo. and $\alpha_0 = \alpha_1 \Longrightarrow \alpha_0 = \alpha_1 = \alpha_2 = \cdots$.
- **Curto (1990, IEOT)**
  $\exists\ \alpha$ with $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geqslant 2)$ s.t. W_α is quad. hypo.
- Weighted shift W_α has Flatness property if
  $\alpha_1 = \alpha_2 \Longrightarrow \alpha_1 = \alpha_n, \forall n \in \mathbb{N}$.



Subnormal
1966, Stampfl
∞-hypo
w-∞-hypo
s-w-∞-hypo
2000, Choi
4-hypo
w-4-hypo
s-w-4-hypo
3-hypo
w-3-hypo
s-w-3-hypo
2013, Li
1990, Curto
2-hypo
w-2-hypo
hypo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hypo

- $W_{\alpha(x,y)}$ with $\alpha(x,y) : 1, (1, \sqrt{x}, \sqrt{y})^{\wedge}$ and $1 < x < y$
  - $\Psi_0 = x\frac{x-y}{x-1}, \quad \Psi_1 = x\frac{y-1}{x-1}$
  - $Q = \left(\frac{\left(y+xy^2-3xy+x^2\right)\left(\sqrt{x^2(y-1)^2+4x(x-1)(x-y)}+x(y-1)\right)}{2(x-y)(x-1)^2} \right)^2$
  - From Lemma 7,
    - $d_1(t) = d_1(x,y,t) = \det \begin{pmatrix} 1+tx & -\sqrt{tx} \\ -\sqrt{tx} & \frac{tx^2(y^2-2y+x)}{x-1} \end{pmatrix}$
    - $d_2(t) = d_2(x,y,t)$
      $= \det \begin{pmatrix} 1+tx & -\sqrt{tx}(y-1) \\ -\sqrt{tx}(y-1) & \frac{tx(y-1)(xy^2+y(1-3x)+2x^2-2x+1)}{(x-1)^2} \end{pmatrix}.$

- $d_1(x,y,t) = \frac{tx^2(1-2y+y^2+t(x^2-2xy+xy^2))}{x-1}$

  $= \frac{tx^2\{(y-1)^2+t((x-y)^2+y^2(x-1))\}}{x-1}$

  $> 0, \forall t > 0.$
- $d_2(x,y,t) = \frac{tx(y-1)(A(x,y)t+B(x,y))}{(x-1)^2}$,

  where $A(x,y) := xy(1-2x+2x^2+y-3xy+xy^2)$

  and $B(x,y) := 1-2x+x^2-y+2xy-x^2y.$
- $d_2(x,y,t) \geqslant 0, \forall t > \frac{1}{Q} \iff A(x,y) \geqslant 0$ and $-\frac{B(x,y)}{A(x,y)} \leqslant \frac{1}{Q}$

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- $d_2(x,y,t) \geqslant 0, \forall t > \frac{1}{Q} \iff A(x,y) \geqslant 0$ and $-\frac{B(x,y)}{A(x,y)} \leqslant \frac{1}{Q}$

▶ W.L.O.G, $x = 1+h$ and $y = 1+h+k$ for $h > 0, k > 0$.

- $A(x,y) = 2h^2 + h^3 + hk + 2h^2k + k^2 + hk^2 > 0$ for all $h, k > 0$
- For all $h > 0, k > 0$,
  $-\frac{B(1+h,1+h+k)}{A(1+h,1+h+k)} \leqslant \frac{1}{Q} \Longleftrightarrow \widehat{p}_3(h,k) := \sum_{i=0}^{9} \xi_i k^i \leqslant 0$, where

$$
\begin{aligned}
\xi_0 &= 2h^9 (h+1)^4, \xi_1 = h^8 (16h+7)(h+1)^3, \\
\xi_2 &= 4h^6 \left(3h + 14h^2 + 14h^3 - 1\right)(h+1)^2, \\
\xi_3 &= h^5 (h+1)\left(3h + 98h^2 + 190h^3 + 112h^4 - 4\right), \\
\xi_4 &= h^4 \left(2h + 109h^2 + 322h^3 + 356h^4 + 140h^5 - 5\right), \\
\xi_5 &= 2h^3 (h+1)\left(5h + 46h^2 + 88h^3 + 56h^4 - 1\right), \\
\xi_6 &= h^2 (h+1)\left(13h + 64h^2 + 104h^3 + 56h^4 - 1\right), \\
\xi_7 &= h^2 (h+1)\left(34h + 42h^2 + 16h^3 + 9\right), \\
\xi_8 &= 2h\left(4h + h^2 + 2\right)(h+1)^2, \text{ and } \xi_9 = (h+1)^3.
\end{aligned}
$$

Theorem 11

Let $\alpha(x,y) : 1, \left(1, \sqrt{x}, \sqrt{y}\right)^{\wedge}$ with $1 < x < y$. Then $W_{\alpha(x,y)}$ is semi-cubically hyponormal if and only if $\widehat{p}_3\,(h,k) \leqslant 0$ for $h > 0$ and $k > 0$.

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Question

$$\text{semi-cubic hyponormality} \overset{?}{\longleftrightarrow} \text{quadratic hyponormality}$$

- **Recall from [Curto and Jung, 2000, IEOT]**
  - $W_{\alpha(x,y)}$ is quadratically hyponormal $\Leftrightarrow p_2(h,k) = \sum_{i=0}^{7} \rho_i k^i \leqslant 0$,

    where $\rho_0 = h^7 (h+2)(h+1)^3$,

    $\rho_1 = h^6 \left(16h + 6h^2 + 7\right)(h+1)^2$,

    $\rho_2 = h^4 \left(5h + 53h^2 + 96h^3 + 66h^4 + 15h^5 - 4\right)$,

    $\rho_3 = h^3 (h+1) \left(5h + 52h^2 + 65h^3 + 20h^4 - 4\right)$,

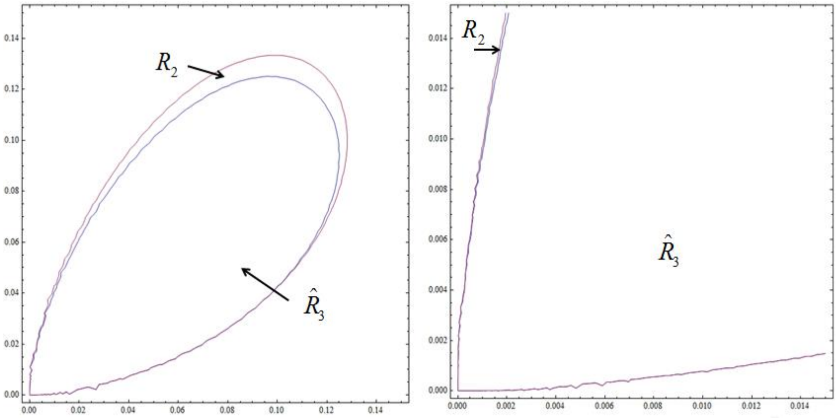
    $\rho_4 = h^2 \left(8h + 35h^2 + 15h^3 - 1\right)(h+1)^2$,

    $\rho_5 = 3h^2 \left(6h + 2h^2 + 3\right)(h+1)^2$,

    $\rho_6 = h(h+5)(h+1)^3$,

    and $\rho_7 = (h+1)^3$, for $h, k > 0$.

- $\mathcal{R}_2 = \{(h,k) : p_2(h,k) \leqslant 0 \text{ for } h,k > 0\}$
- $\widehat{\mathcal{R}}_3 = \{(h,k) : \widehat{p}_3(h,k) \leqslant 0 \text{ for } h,k > 0\}$



Theorem 12

Let $W_{\alpha(x,y)}$ be a weighted shift with a weight sequence $\alpha(x,y) : 1, \left(1, \sqrt{x}, \sqrt{y}\right)^{\wedge}$ for $1 < x < y$.
Then $\mathcal{R}_2 \backslash \widehat{\mathcal{R}}_3$, $\widehat{\mathcal{R}}_3 \backslash \mathcal{R}_2$ and $\mathcal{R}_2 \cap \widehat{\mathcal{R}}_3$ are all non-empty sets.

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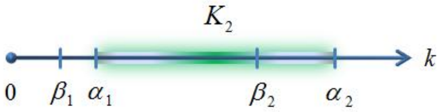
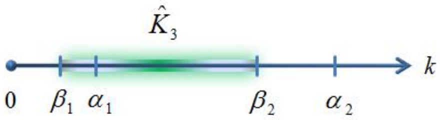
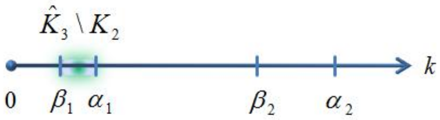
Proof of Theorem 12 (Sketch)

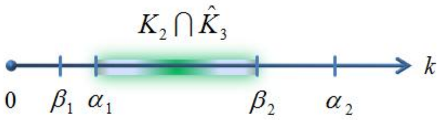
W.L.O.G, $x = 1 + h, y = 1 + h + k$ for $h, k > 0$.
Consider $h = \frac{1}{100}$.
Let $\mathcal{K}_2 = \{k > 0 : p_2(k) \equiv p_2\left(1/100, k\right) \leqslant 0\}$,
and $\widehat{\mathcal{K}}_3 = \{k > 0 : \widehat{p}_3(k) \equiv \widehat{p}_3\left(1/100, k\right) \leqslant 0\}$.
Then $\exists\ \beta_1 < \alpha_1 < \beta_2 < \alpha_2$ such that $\mathcal{K}_2 = [\alpha_1, \alpha_2]$ and $\widehat{\mathcal{K}}_3 = [\beta_1, \beta_2]$.
(Note $\alpha_1 = 0.000787776068\cdots$, $\alpha_2 = 0.0422764016\cdots$,
$\beta_1 = 0.000786885627\cdots$ and $\beta_2 = 0.0402782805\cdots$)


K_2
k
0
β_1
α_1
β_2
α_2

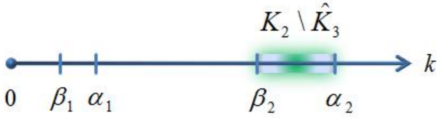

$\hat{K}_3$
k
0
β_1
α_1
β_2
α_2



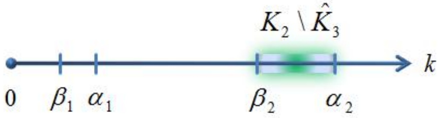
For $k \in \widehat{\mathcal{K}}_3 \backslash \mathcal{K}_2 = (\beta_1, \alpha_1)$, $(\frac{1}{100}, k) \in \widehat{\mathcal{R}}_3 \backslash \mathcal{R}_2$.



For $k \in \widehat{\mathcal{K}}_3 \backslash \mathcal{K}_2 = (\beta_1, \alpha_1)$, $(\frac{1}{100}, k) \in \widehat{\mathcal{R}}_3 \backslash \mathcal{R}_2$.
For $k \in \widehat{\mathcal{K}}_3 \cap \mathcal{K}_2 = (\alpha_1, \beta_2)$, $(\frac{1}{100}, k) \in \widehat{\mathcal{R}}_3 \cap \mathcal{R}_2$.



For $k \in \widehat{\mathcal{K}}_3 \backslash \mathcal{K}_2 = (\beta_1, \alpha_1)$, $(\frac{1}{100}, k) \in \widehat{\mathcal{R}}_3 \backslash \mathcal{R}_2$.
For $k \in \widehat{\mathcal{K}}_3 \cap \mathcal{K}_2 = (\alpha_1, \beta_2)$, $(\frac{1}{100}, k) \in \widehat{\mathcal{R}}_3 \cap \mathcal{R}_2$.
For $k \in \mathcal{K}_2 \backslash \widehat{\mathcal{K}}_3 = (\beta_2, \alpha_2)$, $(\frac{1}{100}, k) \in \mathcal{R}_2 \backslash \widehat{\mathcal{R}}_3$.



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For $k \in \mathcal{K}_2 \backslash \widehat{\mathcal{K}}_3 = (\beta_2, \alpha_2)$, $(\frac{1}{100}, k) \in \mathcal{R}_2 \backslash \widehat{\mathcal{R}}_3$.

Therefore the proof is complete. □

THANK YOU