Commutators on the Dirichlet space

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Notations

- Open unit disc: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
  Unit ball: $\mathbb{B} = \{z \in \mathbb{C}^d : |z| < 1\}$

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- Hardy space: $H^2(\mathbb{D}) = \{f \in Hol(\mathbb{D}) : \|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty, f(z) = \sum_{n=0}^{\infty} a_n z^n\}$
  Bergman space:
  $L_a^2(\mathbb{D}) = \{f \in Hol(\mathbb{D}) : \|f\|_{L_a^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty\}$

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- Dirichlet space: $D = \{f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f'|^2 dA < \infty\}$
  norm: $\|f\|_D^2 = \|f\|_{H^2}^2 + \int_{\mathbb{D}} |f'|^2 dA$
  kernel: $K_\zeta(z) = \frac{1}{\bar{\zeta} z} \log \frac{1}{1 - \bar{\zeta} z}$

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- In 2011, Fang and Xia proved that: Let $f \in M(H_d^2)$. Then $[M_f^*, M_{z_i}] \in S_p, p > 2d, i = 1, \cdots, d$.
  They also showed that: There is a $\varphi \in M(H_d^2)$ such that M_φ is not essentially hyponormal, i.e. $\pi(M_f)$ is not hyponormal in $B(H_d^2)/\mathcal{K}(H_d^2)$.

Similarities

Similar properties between H^2_d and D:

- $M(H^2_d) \subsetneq H^\infty(\mathbb{B})$
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- The corona theorem is true for H^2_d and D.

Main Results

Theorem 2.1 (L)

Let $p > 1, f \in M(D)$. Then the commutator $[M_f^*, M_z]$ is in the Schatten p-class, i.e. there exists a constant $c(p)$ depending on p such that

$$\|[M_f^*, M_z]\|_p \leq c(p)\|M_f\|.$$

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Theorem 2.2 (L)

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In $L_a^2(\mathbb{D})$, let $\varphi \in L^\infty(\mathbb{D})$, let $T_\varphi = PM_\varphi P$ be the Toeplitz operator and $H_\varphi = P^\perp M_\varphi P$ be the Hankel operator, where $P : L^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is the orthogonal projection.

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Then

$$\begin{aligned}[M_z^*, M_f] &= [T_{\overline{z}}, T_f] \\ &= T_{\overline{z}} T_f - T_f T_{\overline{z}} = P\overline{z}P^\perp fP = H_z^* H_f,\end{aligned}$$

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thus

$$\begin{aligned}[M_z^*, M_f][M_z^*, M_f]^* &= H_z^* H_f H_f^* H_z \\ &\leq \|H_f^*\|^2 H_z^* H_z = \|H_f^*\|^2 [M_z^*, M_z].\end{aligned}$$

Let $N \geq 3$ be a fixed odd number, $u_\alpha(z) = \left(\frac{1-|\alpha|^2}{1-\overline{\alpha}z}\right)^{(N+1)}$, $\alpha \in \mathbb{D}$, and $Q = \int_{\mathbb{D}} u_\alpha \otimes u_\alpha \frac{dA(\alpha)}{(1-|\alpha|^2)^2}$.

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Let $p > 1, f \in M(D)$. Then the commutator $[M_z^*, M_f Q]$ is in the Schatten p-class.

Note that $c(N)I \leq Q \leq d(N)I$ and

$$[M_z^*, M_f] = [M_z^*, M_f QQ^{-1}] = [M_z^*, M_f Q]Q^{-1} + M_f[Q, M_z^*]Q^{-1},$$

Theorem 2.1 follows from the above theorem.

Sketch of the proof of Theorem 2.3

$u_\alpha(z) = \left(\frac{1-|\alpha|^2}{1-\overline{\alpha}z}\right)^{(N+1)}$

$$
\begin{aligned}
Q &= \int_{\mathbb{D}} u_\alpha \otimes u_\alpha \frac{dA(\alpha)}{(1-|\alpha|^2)^2} \\
&= \int_0^1 \int_{\partial\mathbb{D}} u_{r\eta} \otimes u_{r\eta} \frac{|d\eta|}{\pi} \frac{rdr}{(1-r^2)^2} \\
&= \int_0^1 T_s \frac{ds}{s},
\end{aligned}
$$

where $s = (1-r^2)^{\frac{1}{2}}$, $T_s = \frac{1}{s^2} \int_{\partial\mathbb{D}} u_{(1-s^2)^{\frac{1}{2}}\eta} \otimes u_{(1-s^2)^{\frac{1}{2}}\eta} \frac{|d\eta|}{\pi}$.

$$[M_z^*, M_f Q] = \int_0^1 [M_z^*, M_f T_s] \frac{ds}{s}.$$

If

$$\|[M_z^*, M_f T_s]\|_p \leq c(p) \|M_f\| s^{2-2/p},$$

then

$$\|[M_z^*, M_f Q]\|_p \leq \widetilde{c}(p) \|M_f\|.$$

For every $0 < s < 1$, $T_s = \frac{1}{s^2}\int_{\partial\mathbb{D}} u_{(1-s^2)^{\frac{1}{2}}\eta} \otimes u_{(1-s^2)^{\frac{1}{2}}\eta} \frac{|d\eta|}{\pi}$.

$$T_s = \frac{c(s)}{|B(y,s)|}\int_{B(y,s)} \sum_{k=1}^{L} X_k(\eta)\frac{|d\eta|}{\pi},$$

where $B(y,s) = \{x \in \partial\mathbb{D} : d(y,x) = |1 - y\overline{x}|^{1/2} < s\}$, $c(s) \le \frac{1}{2}$, $L \le 50\pi + 1$, and

$$X_k(\eta) = \sum_{i \in I_k} \chi_{E_i}(O_i\eta) u_{(1-s^2)^{\frac{1}{2}} O_i\eta} \otimes u_{(1-s^2)^{\frac{1}{2}} O_i\eta}.$$

O_i are unitary transformations on $\mathbb{C}$, $\bigcup_{k=1}^{L}\bigcup_{i\in I_k} E_i = \partial\mathbb{D}$, I_k has the property that $B(O_i\eta, s) \cap B(O_j\eta, s) = \emptyset, i, j \in I_k, i \ne j$.

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$$[M_z^*, M_f T_s] = \frac{c(s)}{|B(y,s)|} \int_{B(y,s)} \sum_{k=1}^{L} [M_z^*, M_f X_k(\eta)] \frac{|d\eta|}{\pi}.$$

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Let $\alpha_i(\eta) = (1 - s^2)^{\frac{1}{2}} O_i\eta$, then

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thus

$$[M_z^*, M_f X_k(\eta)] = \sum_{i\in I_k} \chi_{E_i}(O_i\eta)(M_z^* - \overline{\alpha_i(\eta)}) f u_{\alpha_i(\eta)} \otimes u_{\alpha_i(\eta)} -$$

$$\sum_{i\in I_k} \chi_{E_i}(O_i\eta) f u_{\alpha_i(\eta)} \otimes (M_z - \alpha_i(\eta)) u_{\alpha_i(\eta)}$$

$$:= A - B$$

Let $\{e_i : i \in I_k\}$ be an orthonormal set, then

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Show $\|G\|_p \leq c(p)\|M_f\|s^{2-2/p}, \|H\| \leq C$. Then

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Similarly,

$$\|B\|_P \le \widetilde{c}(p)\|M_f\| s^{2-2/p}.$$

Essentially Hyponormal Operators

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There is an $f \in M(D)$ such that M_f is not essentially hyponormal.

- A bounded operator T on $\mathcal{H}$ is called hyponormal, if $[T^*, T] = T^*T - TT^* \geq 0$.

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- A bounded operator T on $\mathcal{H}$ is called essentially hyponormal, if $[\pi(T)^*, \pi(T)] \geq 0$, where $\pi : B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Proposition

If $f \in M(D)$, and M_f is essentially hyponormal, then $\|\pi(M_f)\| \leq \|f\|_{H^\infty}$.

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Proof.

M_f essentially hyponormal $\Rightarrow \|\pi(M_f)\| = r_\sigma(\pi(M_f)) \leq r_\sigma(M_f)$
$\sigma(M_f) = \mathrm{ran}\ \Gamma(M_f)$, where $\Gamma : M(D) \to C(\mathcal{M})$ Gelfand transform
$\mathbb{D}$ dense in $\mathcal{M} \Rightarrow \sigma(M_f) = \mathrm{clos} f(\mathbb{D}) \Rightarrow r_\sigma(M_f) = \|f\|_{H^\infty}$. □

Theorem 2.6

There exists a sequence $\{g_j\}_{j\geq 1} \subseteq M(D)$ *such that*

$$\inf_{j\geq 1} \|\pi(M_{g_j})\| > 0 \quad \textit{and} \quad \lim_{j\to\infty} \|g_j\|_{H^\infty(\mathbb{D})} = 0.$$

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Idea: Let $h_\alpha(z) = \frac{1-|\alpha|^2}{1-\overline{\alpha}z}, \alpha \in \mathbb{D}$, then
$\|h_\alpha\|_D = 1, \|h_\alpha\|_{H^\infty} = \|\pi(M_{h_\alpha})\| = 1 + |\alpha|$, and

$$\|M_{h_\alpha}\| \asymp \left(\frac{1}{|\alpha|^2} \log \frac{1}{1-|\alpha|^2}\right)^{1/2} \to \infty \quad \text{as } |\alpha| \to 1.$$

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Let $f_i = \frac{h_{\alpha_i}}{\|M_{h_{\alpha_i}}\|}$, show that there exists a sequence of natural numbers $\{k(r)\}_{r\geq 1}$ such that the sequence $\{g_j = \sum_{r=j}^{\infty} f_{k(r)}\}_{j\geq 1}$ satisfies the theorem.

Thank You!