On composition operators for which $|C_\varphi^2| \geq |C_\varphi|^2$

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Outline

$\mathcal{H}$: a complex Hilbert space

$\mathcal{L}(\mathcal{H})$: the algebra of all bounded linear operators on $\mathcal{H}$

$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$

$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\}$

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3. hyponormal if $T^*T \geq TT^*$.
4. normaloid if $\|T\| = r(T)$ where $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ is the spectral radius of T.

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Normal $\Rightarrow$ Subnormal $\Rightarrow$ Hyponormal $\Rightarrow$ Normaloid.

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An operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to class A, denoted by $T \in \mathcal{A}$, if

$$|T^2| \geq |T|^2$$

where $|S| := (S^*S)^{\frac{1}{2}}$.

The invariant subspace problem (1932, J. von Neumann)

Does every operator $T \in \mathcal{L}(\mathcal{H})$ have a nontrivial invariant subspace (n.i.s)?
(i.e., $\exists$ a closed subspace $\mathcal{M} \neq \{0\}, \mathcal{H}$ such that $T\mathcal{M} \subset \mathcal{M}$?)

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- S. Brown (Int. Eq. Op. Th., 1978)
  : Subnormal operators have n.i.s.
- S. Brown (Ann. of Math., 1987)
  : Hyponormal operators with thick spectrum have n.i.s.
- Jung-Ko-Lee (Studia Math., 2010)
  : Class A operators with thick spectrum have n.i.s.

Theorem

Let T be an operator in $\mathcal{A}$, which is not a scalar multiple of the identity operator, satisfying one of the following statements:

1. T has a decomposable quasiaffine transform;
2. $\lim_{n\to\infty} \|T^n h\|^{\frac{1}{n}} < \|T\|$ for some nonzero $h \in \mathcal{H}$;
3. T is weakly or N-supercyclic for some positive integer N;
4. $\sigma_T(x) \subsetneq \sigma(T)$ for some nonzero vector $x \in \mathcal{H}$;
5. $\exists$ a nonzero vector $x \in \mathcal{H}$ such that $\|T^n x\| \leq Cr^n$ for all positive integers n, where C and r are constants with $C > 0$ and $0 < r < \|T\|$;
6. $\sigma(T)$ is not the closure of the union of all singleton components of $\sigma(T)$.

Then T has n.i.s.

$\mathbb{D}$: the open unit disk in the complex plane $\mathbb{C}$

- $H^2 = \{f : \mathbb{D} \to \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$

  The space H^2 is a Hilbert space, called the Hardy space, endowed with the inner product given by

$$\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

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- $H^\infty = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic and bounded on } \mathbb{D}\}$

Definition

For an analytic self-map φ of $\mathbb{D}$, the operator $C_\varphi : H^2 \to H^2$ defined by

$$C_\varphi h = h \circ \varphi, \ \ h \in H^2$$

is said to be a composition operator.

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Remark

The composition operator C_φ is always bounded on H^2 by Littlewood subordination theorem.

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2. For an analytic self-map φ of $\mathbb{D}$ and $f \in H^\infty$, the operator $W_{f,\varphi} := M_f C_\varphi$ is said to be a weighted composition operator on H^2.

Example

The weighted composition operator $W_{f,\varphi}$ belongs to $\mathcal{A}$ where

$$f(z) = \frac{4}{-(1-a)(1+a)(3+a)z + (3+a+a^2-a^3)},$$

and

$$\varphi(z) = \frac{(1-a+3a^2+a^3)z + (1+a)(1-a)^2}{-(1-a)(1+a)(3+a)z + (3+a+a^2-a^3)}$$

with $0 < a < 1$.

Example

The adjoint $W^*_{g,\psi}$ belongs to $\mathcal{A}$ where

$$g(z) = \frac{4}{(1+a)(1-a)^2 z + (3+a+a^2-a^3)}$$

and

$$\psi(z) = \frac{(1-a+3a^2+a^3)z + (1+a)(1-a)(3+a)}{(1+a)(1-a)^2 z + (3+a+a^2-a^3)}$$

with $0 < a < 1$.

Theorem (Schwartz)

Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is normal if and only if $\varphi(z) = \lambda z$ with $|\lambda| \leq 1$.

- C. C. Cowen, *Linear fractional composition operators on H^2*, Int. Eq. Op. Th. **11**(1988), 151-160.
- C. C. Cowen and T. Kriete, *Subnormality and composition operators on H^2*, J. Funct. Anal. **81**(1988), 298-319.
- K. W. Dennis, *Co-hyponormality of composition operators on the Hardy space*, Acta Sci. Math. (Szeged), **68**(2002), 401-411.

Goal

To characterize composition operators C_φ in $\mathcal{A}$ and adjoints C_φ^* in $\mathcal{A}$!

Definition

Let φ be an analytic self-map of $\mathbb{D}$. We say that $a \in \overline{\mathbb{D}}$ is a fixed point of φ if $\lim_{r \to 1^-} \varphi(ra) = a$. We write $\varphi'(a) = \lim_{r \to 1^-} \varphi'(ra)$.

In the above definition, the limit $\lim_{r \to 1^-} \varphi'(ra)$ exists if $|a| < 1$. Moreover, if $|a| = 1$, then this limit exists and $0 < \varphi'(a) \leq \infty$ by Julia-Carathéodory-Wolff.

An automorphism φ of $\mathbb{D}$ is called elliptic if it has two fixed points such that one is inside $\partial\mathbb{D}$ and another is outside $\partial\mathbb{D}$.

Theorem (Denjoy-Wolff Theorem)

If φ is an analytic self-map of $\mathbb{D}$, not an elliptic automorphism, then there exists a point a of $\overline{\mathbb{D}}$ so that the iterates $\{\varphi_n\}$ of φ converges uniformly to a on compact subsets of $\mathbb{D}$. In addition, a is the unique fixed point of φ in $\overline{\mathbb{D}}$ for which $|\varphi'(a)| \leq 1$.

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Definition

The unique fixed point a in Denjoy-Wolff Theorem is called the Denjoy-Wolff point of φ.

For an analytic self-map φ of $\mathbb{D}$, one of the following statements holds:

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2. φ has Denjoy-Wolff point $a \in \mathbb{D}$;
3. φ has Denjoy-Wolff point $a \in \partial\mathbb{D}$ with $0 < \varphi'(a) < 1$;
4. φ has Denjoy-Wolff point $a \in \partial\mathbb{D}$ with $\varphi'(a) = 1$.

Lemma (Cowen-Kriete)

Let φ be an analytic self-map of $\mathbb{D}$ with Denjoy-Wolff point a. If $0 < |a| < 1$ or if $|a| = 1$ and $\varphi'(a) = 1$, then C_φ is not normaloid.

Theorem

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1. 0 is a fixed point of φ.
2. $|\varphi(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|\varphi'(0)| \leq 1$.

Theorem

Let φ be an analytic self-map of $\mathbb{D}$. If $C_\varphi \in \mathcal{A}$, then the following statements hold:

1. 0 is a fixed point of φ.
2. $|\varphi(z)| \leq |z|$ for all $z \in \mathbb{D}$ and $|\varphi'(0)| \leq 1$.
3. If $|\varphi(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D} \setminus \{0\}$ or $|\varphi'(0)| = 1$, then C_φ is unitary.

Example

If $\varphi(z) = sz + t$ with $s, t \neq 0$ and $|s| + |t| \leq 1$, then $C_\varphi \notin \mathcal{A}$.

For a subset Δ of $\mathbb{C}$, let iso(Δ) denote the set of all isolated points in Δ and let $\Delta^* = \{\overline{\lambda} : \lambda \in \Delta\}$.

Theorem

Let φ be an analytic self-map of $\mathbb{D}$. If $C_\varphi \in \mathcal{A}$, then

$$\text{iso}(\sigma(C_\varphi)) \subset \sigma_p(C_\varphi) \subset \{(\varphi'(0))^n : n = 0, 1, 2, \cdots\} \subset \sigma(C_\varphi),$$

and

$$\sigma_p(C_\varphi)^* \subset \{\overline{(\varphi'(0))^n} : n = 0, 1, 2, \cdots\} \subset \sigma_p(C_\varphi^*).$$

Example

If $\varphi(z) = \frac{z}{2-z}$, then $C_\varphi \in \mathcal{A}$ and

$$\sigma_p(C_\varphi) \subsetneqq \{(\varphi'(0))^n : n = 0, 1, 2, \cdots\}.$$

We say that $T \in \mathcal{L}(\mathcal{H})$ is binormal if T^*T commutes with TT^*.

Theorem

Let φ is an analytic self-map of $\mathbb{D}$. Assume that $C_\varphi^* C_\varphi$ is a diagonal matrix with respect to the basis $\{z^n\}_{n=0}^{\infty}$. If

1. φ is a polynomial function, or
2. C_φ is binormal and $\varphi'(0) \neq 0$,

then

$$C_\varphi \in \mathcal{A} \iff C_\varphi \text{ is subnormal.}$$

Example

Let $\varphi(z) = cz^k$ where $k > 1$ is an integer and $|c| \leq 1$. Then $C_\varphi^* C_\varphi$ is diagonal with respect to the basis $\{z^n\}_{n=0}^{\infty}$. Therefore, $C_\varphi \in \mathcal{A}$ if and only if C_φ is subnormal. Indeed,

$$C_\varphi \in \mathcal{A} \iff |c| = 1.$$

Let φ be a linear fractional self-map of $\mathbb{D}$, namely $\varphi(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$. If $C_\varphi \in \mathcal{A}$, then $b = 0$. Thus $\varphi(z) = \frac{z}{uz+v}$ where $u = \frac{c}{a}$ and $v = \frac{d}{a}$.

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Let $\varphi(z) = \frac{z}{uz+v}$ for some $u, v \in \mathbb{C}$ with $u \neq 0$ and $|v| \geq 1 + |u|$. Then the following statements are equivalent.

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2. C_φ belongs to class A.

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1. C_φ is subnormal.
2. C_φ belongs to class A.
3. C_φ is binormal.
4. $v > 1$ and $|u| = v - 1$.

Corollary

If φ is an automorphism of $\mathbb{D}$ (i.e., C_φ is invertible), then

$$C_\varphi \in \mathcal{A} \iff C_\varphi \text{ is unitary.}$$

Corollary

Let φ be a linear fractional self-map of $\mathbb{D}$ such that $C_\varphi^* C_\varphi$ has a diagonal matrix with respect to the standard basis $\{z^n\}_{n=0}^\infty$. Then

$$C_\varphi \in \mathcal{A} \iff C_\varphi \text{ is normal.}$$

Theorem

Let φ be an analytic self-map of $\mathbb{D}$ with Denjoy-Wolff point a. If $C_\varphi^* \in \mathcal{A}$, then either $|a| = 1$ and $0 < \varphi'(a) < 1$ or else C_φ is normal.

Example

If $\varphi(z) = \frac{z}{uz+v}$ with $u \neq 0$ and $|v| \geq 1 + |u|$, then $\varphi(0) = 0$ and thus $C_\varphi^* \notin \mathcal{A}$.

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Example

If $\varphi(z) = \frac{(2-t)z+t}{-tz+(2+t)}$ for some complex number t with $\mathrm{Re}(t) \geq 0$, then $\varphi(1) = 1$ and $\varphi'(1) = 1$. Hence $C_\varphi^* \notin \mathcal{A}$.

Corollary

Let φ be an analytic self-map of $\mathbb{D}$. Then

$$C_\varphi, C_\varphi^* \in \mathcal{A} \iff C_\varphi \text{ is normal.}$$

Theorem

Let φ be a nonconstant analytic self-map of $\mathbb{D}$. If $C_\varphi^* \in \mathcal{A}$ is injective, then φ is univalent.

Theorem

Let $\varphi(z) = sz + t$ where $s \neq 0$ and $|s| + |t| \leq 1$. Then

$$C_\varphi^* \in \mathcal{A} \iff C_\varphi^* \text{ is subnormal.}$$

Theorem

For $0 < s < 1$ and $|r| \leq 1$, let $\varphi(z) = \frac{(r+s)z+(1-s)}{r(1-s)z+(1+rs)}$ be a linear fractional self-map of $\mathbb{D}$ with Denjoy-Wolff point 1 and $0 < \varphi'(1) = s < 1$. If $C_\varphi^* \in \mathcal{A}$, then

$$|r - \frac{1}{3}| \leq \frac{2}{3}.$$

For $0 < s < 1$ and $-1 \le r \le 1$, let $\varphi(z) = \frac{(r+s)z+(1-s)}{r(1-s)z+(1+rs)}$ is a linear fractional self-map of $\mathbb{D}$ with Denjoy-Wolff point 1 and $0 < \varphi'(1) = s < 1$.

1. $C_\varphi^* \in \mathcal{A} \Longrightarrow -\frac{1}{3} \le r \le 1$.
2. C_φ^* is hyponormal $\Longrightarrow 2-\sqrt{5} \le r \le 1$ by Dennis.
3. C_φ^* is subnormal $\Longleftrightarrow 0 \le r \le 1$ by Cowen.

We expect that there maybe exist

1. $r \in [-\frac{1}{3}, 2-\sqrt{5})$ such that C_φ^* is a non-hyponormal operator in $\mathcal{A}$.
2. $r \in [2-\sqrt{5}, 0)$ such that C_φ^* is hyponormal but not subnormal.

For an operator $T \in \mathcal{L}(\mathcal{H})$, define the commutant $\{T\}'$ of T as to be the collection of all operators in $\mathcal{L}(\mathcal{H})$ commuting with T.

Proposition

Let φ be an analytic self-map of $\mathbb{D}$ such that C_φ is a non-normal operator in $\mathcal{A}$. For $f \in H^\infty$,

$$M_f \in \{C_\varphi\}' \iff f \text{ is constant on } \mathbb{D}.$$

Theorem

Suppose that φ is an analytic self-map of $\mathbb{D}$ such that $C_\varphi \in \mathcal{A}$. If ψ is an analytic self-map of $\mathbb{D}$ such that $C_\psi \in \{C_\varphi\}'$, then zH^2 is a nontrivial invariant subspace for C_ψ.

Theorem

Suppose that φ is an analytic self-map of $\mathbb{D}$ such that $C_\varphi \in \mathcal{A}$. If ψ is an analytic self-map of $\mathbb{D}$ such that $p(C_\psi) \in \{C_\varphi\}'$ for some polynomial p of degree 2, then at least one of the following statements holds:

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1. zH^2 is a nontrivial invariant subspace for C_ψ or C_ψ^2;

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1. zH^2 is a nontrivial invariant subspace for C_ψ or C_ψ^2;
2. $\{f \in H^2 : f(\psi(0)) = 0\} = \{K_{\psi(0)}\}^\perp$ is a nontrivial invariant subspace for C_ψ;

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2. $\{f \in H^2 : f(\psi(0)) = 0\} = \{K_{\psi(0)}\}^\perp$ is a nontrivial invariant subspace for C_ψ;
3. $\varphi(\psi(0)) \neq (\varphi \circ \psi)(\psi(0))$.

Remark

Suppose that φ is an analytic self-map of $\mathbb{D}$ such that $C_\varphi \in \mathcal{A}$. Let ψ be an analytic self-map of $\mathbb{D}$ such that $p(C_\psi) \in \{C_\varphi\}'$ for some polynomial p of degree 2. If $\varphi = \varphi \circ \psi$, i.e., C_ψ has φ as an eigenfunction corresponding to the eigenvalue 1, then C_ψ or C_ψ^2 has a nontrivial invariant subspace.

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Thank you for your attention!!