Abrahamse's Theorem for Matrix-Valued Symbols

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Motivation of the talk

Why Toeplitz operators?

(1) Toeplitz operators are of importance in connection with a variety of problems in physics, probability theory, information and control theory and several other fields.

(2) Toeplitz operators constitute one of the most important classes of non-self adjoint operators and they are a fascinating example of the fruitful interplay between such topics as operator theory, function theory and the theory of Banach algebras.

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Subnormality

In 1950, Paul Halmos introduced the notion of subnormality of operators. Nowadays, the theory of subnormal operators is an extensive and highly developed area.

Subnormality

$\mathcal{H}$:=an infinite dimensional separable complex Hilbert space
$T \in \mathcal{B}(\mathcal{H}) :=$ the set of all bounded linear operators acting on $\mathcal{H}$

1. $[T^*, T] \equiv T^*T - TT^* =$ the self-commutator of T.
2. T is called *normal* if $[T^*, T] = 0$ and *hyponormal* if $[T^*, T] \geq 0$;
3. T is called *subnormal* if there exists a normal operator N acting on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $T = N|_{\mathcal{H}}$, i.e.,

$$\exists \text{ a normal } N = \begin{pmatrix} T & * \\ 0 & * \end{pmatrix};$$

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In order to determine the subnormality by definition, we should find a normal extension of the operator. However, it is not a constructive method to find such an extension.

There are a couple of constructive methods to determine the subnormality. The best one of them is the Bram-Halmos characterization of subnormality.

Halmos's Problem 5

Bram-Halmos Characterization of Subnormality. An operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \quad \text{(for all } k \geq 1)$$

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The Bram-Halmos criterion is tractable step by step. But it is also impossible to determine the positivity of the above matrix for *all* positive integers k.

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Halmos's Problem 5 (1970, Bull. AMS) Is every subnormal Toeplitz operator either normal or analytic ?

Toeplitz operators

Definition. The *Toeplitz operator with symbol* $\varphi \in L^\infty(\mathbb{T})$ is the operator T_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2 \text{ and } P := \text{the projection of } L^2 \text{ onto } H^2)$$

If $\varphi = \sum_{n=-\infty}^{\infty} a_n z^n$ then

$$T_\varphi = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ \vdots & a_2 & a_1 & a_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}$$

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Note. If $\varphi \in H^\infty \equiv H^2 \cap L^\infty$ then $\forall h \in H^2$,

$$T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h.$$

Thus the multiplication operator M_φ on L^2 is a normal extension of T_φ; i.e., T_φ is subnormal if $\varphi \in H^\infty$.

Consequently, Halmos's Problem 5 is to ask whether or not every non-analytic subnormal Toeplitz operator is exactly normal.

Cowen and Long's Theorem

In 1984, Carl Cowen and John Long gave a negative answer to the Halmos's Problem 5.

Cowen and Long's Theorem (1984, Crelle's J) $\exists\psi \in H^\infty$ such that $T_{\psi+\alpha\overline{\psi}}$ ($0 < \alpha < 1$) is subnormal

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Notation. For $\varphi \in L^\infty$, write

$$\varphi_+ := P(\varphi) \in H^2 \quad \text{and} \quad \varphi_- = \overline{P^\perp(\varphi)} \in zH^2.$$

Thus we can write $\varphi = \overline{\varphi_-} + \varphi_+$.

Abrahamse's Theorem

Some authors gave interesting sufficient conditions for the answer to the Halmos's Problem 5 to be affirmative. Among them, the Abrahamse's theorem is the most interesting result.

Definition. A function $\varphi \in L^\infty$ is said to be *in the Nevanlinna class* $\mathcal{N}$ (or bounded type) if

$$\varphi = \frac{\psi_1}{\psi_2} \qquad (\psi_j \in H^\infty(\mathbb{D}) \text{ for } j = 1, 2).$$

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Fact. If $\varphi \in L^\infty$ is in the Nevanlinna class $\mathcal{N}$, then we can write

$$\varphi_- = \theta \bar{a}\,,$$

where θ is inner, $a \in H^2$, and θ and a are *coprime*, in the sense that there does not exist a common inner divisor of θ and a.

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Abrahamse's Theorem (1976, Duke Math. J) Let $\varphi, \overline{\varphi} \in \mathcal{N}$. If T_φ is subnormal then T_φ is normal or analytic.

In other words, the answer to the original Halmos's Problem 5 is affirmative when the symbol is in the Nevanlinna class $\mathcal{N}$.

Block Toeplitz operators

Definition.

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L^2_{\mathbb{C}^n} &\equiv L^2_{\mathbb{C}^n}(\mathbf{T}) = L^2(\mathbf{T}) \otimes \mathbb{C}^n \cong L^2 \oplus \cdots \oplus L^2 \\
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Definition. For a matrix-valued function $\Phi \in L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbf{T})$, the *(block) Toeplitz operator* $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ *with (matrix-valued) symbol* Φ is defined by

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If $\Phi \in L^\infty_{M_n}$ then we can write

$$\Phi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{pmatrix} \qquad (\varphi_{ij} \in L^\infty)$$

and

$$T_\Phi = \begin{pmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{pmatrix}$$

For $\Phi \equiv [\varphi_{ij}] \in L^\infty_{M_n}$, we say that Φ is in the Nevanlinna class $\mathcal{N}$ [*rational*] if each entry φ_{ij} is in $\mathcal{N}$ [rational].

A matrix-valued version of Halmos's Problem 5

In light of the original Halmos's Problem 5 on scalar-valued Toeplitz operators, we would like to ask the following question:

Problem. Which subnormal Toeplitz operators with matrix-valued symbols are either normal or analytic ?

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However, Abrahamse's Theorem is liable to fail for matrix-valued symbols (even for matrix-valued trigonometric polynomial symbols). For instance, take

$$\Phi \equiv \begin{pmatrix} \overline{z} + z & 0 \\ 0 & z \end{pmatrix}.$$

Then

$$T_\Phi \equiv \begin{pmatrix} U^* + U & 0 \\ 0 & U \end{pmatrix} \quad \text{(where } U \text{ is the unilateral shift)}$$

is neither normal nor analytic although T_Φ is subnormal.

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Question. What causes this fail for matrix-valued cases ?

It seems to be so hard to recognize the core of this phenomenon. To overcome this example, we should get a new idea.

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If $\varphi \in L^\infty$ is in the Nevanlinna class $\mathcal{N}$, then we can write

$$\varphi_- = \omega \overline{a} \quad (\omega \text{ inner; } \omega \text{ and } a \text{ are coprime})$$

Since $\varphi = \frac{a}{\omega} + \varphi_+$, the singularities of φ come from ω. Thus we have

$$\begin{aligned}\varphi \text{ has a singularity} &\Longleftrightarrow \exists \text{ an inner } \theta \text{ such that } \theta \text{ is an inner divisor of } \omega \\ &\Longleftrightarrow \omega H^2 \subset \theta H^2 \\ &\Longleftrightarrow \ker H_\varphi \subset \theta H^2.\end{aligned}$$

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Definition. Let $\Phi \in L^\infty_{M_n}$ be in the Nevanlinna class $\mathcal{N}$. Then Φ is said to have a *matrix singularity* if

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Lemma. Let $\Phi \in L^\infty_{M_n}$ be in the Navanlinna class $\mathcal{N}$. Thus we may write

$$\Phi = A\Theta^* \quad \text{(right coprime factorization).}$$

Then the following are equivalent:

1. Φ has a matrix singularity;
2. Θ has a nonconstant diagonal-constant inner divisor.

Main Theorem

Main Theorem (Curto, Hwang, and Lee, 2015)

Let $\Phi \in L^\infty_{M_n}$ be such that $\Phi, \Phi^* \in \mathcal{N}$.
Assume that Φ has a matrix singularity.
If T_Φ is sunnormal then T_Φ is normal or analytic.

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(1) If $n = 1$, then $\Theta = \theta \in H^\infty$ is vacuously diagonal-constant, so that our main theorem reduces to the original Abrahamse's Theorem.

(2) The assumption "Φ has a matrix singularity" is essential in the main theorem. Let

$$\Phi := \begin{pmatrix} \overline{z} + z & 0 \\ 0 & z \end{pmatrix}.$$

Then T_Φ is neither normal nor analytic. Observe that

$$\ker H_\Phi = \ker H_{\begin{pmatrix} \overline{z} & 0 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} H^2_{\mathbb{C}^n},$$

which shows that $\Theta \equiv \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$ does not have any diagonal-constant inner divisor, so that Φ does not have a matrix singularity.

References

1. (1970) P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. 76.

2. (1976) M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. 43.

3. (2001) R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Memoirs Amer. Math. Soc. (vol 712).

4. (2012) R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. 230.

5. (2012) R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. 263.

6. (2014) R.E. Curto, I.S. Hwang, D. Kang and W.Y. Lee, *Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols*, Adv. Math. 255.

7. (2015) R.E. Curto, I.S. Hwang and W.Y. Lee, *Abrahamse's Theorem for matrix-valued symbols* (preprint).

Thank you for your attention

Appendix

Note.
We may define the matrix singularity for $\Phi \in L^\infty_{M_n}$ by the singularity of some entry of Φ: in other words, we say that Φ has a singularity at $\alpha \in \mathbb{D}$ if some entry of Φ has a singularity at $z = \alpha$. This is not equivalent to our definition.

Example.
Let

$$\Phi := \begin{pmatrix} \frac{1}{z} + z & 0 \\ 0 & z \end{pmatrix}.$$

As we saw in the preceding, Φ does not have any matrix singularity. However the entry $\frac{1}{z} + z$ of Φ has a pole at $z = 0$.