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Positivity of operators discussed by Dragomir

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Abstract.

This talk is based on the joint work with R. Nakamoto.

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space H. An operator A is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. In paticular, we denote $A > 0$ if A is positive and invertible. The positivity of operators induces the order $A \geq B$ among selfadjoint operators by $A - B \geq 0$. A continuous function f on an interval I is said to be operator monotone if f is order preserving, i.e., $A \geq B$ for A, B whose spectra are included in I implies $f(A) \geq f(B)$.

Now, motivated by a recent work due to Dragomir, we discuss the positivity of

$$(B - A)(f(B) - f(A))$$

for $A, B > 0$ and operator monotone function f on $(0, \infty)$.

We note that the simplest numerical arithmetic-geometric mean inequality is $x + \frac{1}{x} \geq 2$ for $x > 0$. We pose an operator version of it:

Proposition. For invertible positive operators A and B, if $AB^{-1} + BA^{-1}$ is selfadjoint, then

$$AB^{-1} + BA^{-1} \geq 2.$$

Now we claim that Proposition implies the following theorem by birtue of the integral representation for operator montone functions.

Theorem. Let f be an operator monotone function on $(0, \infty)$ and $A, B > 0$ such that $A(B + s)^{-1} + B(A + s)^{-1}$ is selfadjoint for all $s \geq 0$. Then

$$(B - A)\{f(B) - f(A)\} \geq 0.$$

References

[1] S. S. Dragomir, *Some inequalities for operator monotone functions*, RGMIA, V23a57.

[2] S. S. Dragomir, *Several inequalities for operator monotone functions on finite intervals*, RGMIA, V23a58.