An operator-theoretical proof for the second-order phase transition in superconductivity

Shuji Watanabe
Division of Mathematical Sciences, Gunma University
Email: shuwatanabe@gunma-u.ac.jp

In 1911, Onnes found out that the electrical resistance in mercury suddenly becomes zero just as the temperature decreases and reaches about 4.2 Kelvin. Zero electrical resistance means that once an electric current occurs, the electric current never vanishes and remains the same. This surprising phenomenon is called superconductivity, and a superconducting material is called a superconductor. For this discovery, Onnes received the Nobel Prize in physics. After that, superconductivity has been found out experimentally in many materials at very low temperatures, and it has been observed in many experiments that the transition from a normal conducting state to a superconducting state is a second-order phase transition in terms of thermodynamics. On the other hand, the BCS-Bogoliubov model (based on quantum field theory) by Bardeen, Cooper and Schrieffer, and independently by Bogoliubov has been established and has found out to be a successful theory of superconductivity. For this model, Bardeen, Cooper and Schrieffer also received the Nobel Prize in physics.

In the BCS-Bogoliubov model, one partially differentiates the solution to the BCS-Bogoliubov gap equation with respect to the temperature twice so as to show that the transition is a second-order phase transition. Here, the BCS-Bogoliubov gap equation is a nonlinear integral equation. But, in the physics literature, one never shows that the solution to the BCS-Bogoliubov gap equation is partially differentiable with respect to the temperature. Therefore, if the solution were not partially differentiable with respect to the temperature, then the BCS-Bogoliubov model could not show that the transition is a second-order phase transition. Moreover, the BCS-Bogoliubov model could not deal with the specific heat and the critical magnetic field. This is why we need to show that the solution is partially differentiable with respect to the temperature twice as well as showing its existence and uniqueness.

Motivated by this, I show that the solution to the BCS-Bogoliubov gap equation is indeed partially differentiable with respect to the temperature twice and give a proof of the statement that the transition is a second-order phase transition on the basis of fixed-point theorems in my paper (Kyushu J. Math. **74** (2020), 177-196, https://doi.org/10.2206/kyushujm.74.177). In this way, from the viewpoint of operator theory, I solved the long-standing problem of the second-order phase transition left unsolved for sixty-two years since the discovery of the BCS-Bogoliubov model.

In my another paper (*An operator-theoretical study of the specific heat and the critical magnetic field in the BCS-Bogoliubov model of superconductivity*, Scientific Reports **10** (2020), 9877, https://doi.org/10.1038/s41598-020-65456-5 (Springer Nature)), on the basis of the results above, I study the temperature dependence of the specific heat and the critical magnetic field in the model from the viewpoint of operator theory. I give the exact and explicit expression for the gap in the specific heat divided by the specific heat. I then show that it does not depend on superconductors and is a universal constant. Moreover, I show that the critical magnetic field is smooth with respect to the temperature, and pointed out the behavior of both the critical magnetic field and its derivative.