On spectral number theory II

Robin Harte

School of Mathematice, Trinity College Dublin hartere@gmail.com

AMS Classification: 11B25; 15B36

Key words: primes; square free numbers; polar decomposition; quadratic decomposition

1

The Fundamental Theorem of Arithmetic says ([3] Corollary 4.1.2) that every natural number is a product of primes:

$$n \in \mathbb{N} \Longrightarrow n = \prod_{p \in \mathbb{P}} \{p^{\nu_n(p)} : n \in \mathbb{N}p\} \ . \tag{1.1}$$

where

$$\mathbb{N} = \{1, 2, 3, \ldots\} = \bigcup_{n=1}^{\infty} \mathbb{N}_n$$

with $\mathbb{N}_n = \{1, 2, \ldots n\}$ and

$$\mathbb{P} = \{2, 3, 5, 7, \ldots\} = \{p \in \mathbb{N} : p \in \mathbb{N}n \Longrightarrow n \in \{1, p\}\} \setminus \{1\} \tag{1.2}$$

are the natural numbers and the primes, and

$$(n, p) \in \mathbb{N} \times \mathbb{P} \Longrightarrow \nu_n(p) = \max\{k \in \mathbb{N} : n \in \mathbb{N}p^k\} \ . \tag{1.3}$$

The primes $\mathbb{P}$ form the intersection of the sets of *prime powers*

$$\{p^k : p \in \mathbb{P}\}, k \in \{0\} \cup \mathbb{N}$$

and the *square free* numbers

$$\{n \in \mathbb{N} : \max\{\nu_n(p) : p \in \mathbb{P}\} \leq 1\} \ .$$

We remind ourselves here of a sort of "spectrum", of the natural number $n \in \mathbb{N}$, given by

$$\varpi(n) = \{p \in \mathbb{P} : n \in \mathbb{N}p\} \ , \tag{1.4}$$

which collects the prime factors of n, discarding multiplicity.

We shall find it convenient to work with the sequence, rather than the set, of primes: thus

$$\mathbf{n} = (1, 2, 3, \ldots) \ ; \ \mathbf{p} = (2, 3, 5, 7, \ldots) \in \mathbb{N}^{\mathbb{N}} \ ;$$

formally ("Sieve of Eratosthenes") for each $n \in \mathbb{N}$,

$$p_{n+1} = \min(\mathbb{N} \setminus \{1\} \setminus \mathbb{N}p_n) \ . \tag{1.5}$$

The square free numbers also form a sequence:

$$\mathbf{q} = (1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, \ldots) \tag{1.6}$$

where, for each $n \in \mathbb{N}$, q_n is a sort of "square free root" of n, and indeed participates in a sort of "polar decomposition" of n:

$$n = q'_n q_n \ , \tag{1.7}$$

and we shall for no very clear reason refer to $q'_n = n/q_n$ as the "carapace" of $n \in \mathbb{N}$. If we think of the natural number n as some kind of beetle, then its square free root consists of its legs, and the carapace rounds them off to fill out the rest of the insect. We might remark that the square free root q_n determines, and is determined by, the spectrum $\varpi(n)$.

Much of our interest in building a "spectral theory" for natural numbers has been sparked by the "Readable introduction" [3] of Rosenthal cubed; in their second edition, they report on an alternative "quadratic decomposition":

$$n = r'_n r_n \ , \tag{1.8}$$

where r_n is again square free, but now

$$r'_n = n/r_n = \max\{m^2 : n \in \mathbb{N}m^2\} \tag{1.9}$$

is a perfect square; specifically the greatest square among the factors of $n \in \mathbb{N}$. For example, if n is square free then $r'_n = 1$, while if $n = m^2$ then $r'_n = n$. In effect part of the square free root is transferred to the carapace to square it off.

2

This suddenly is serious business: Rosenthal[3] are able to deploy it to prove ([3] Theorem 13.7.8) that

$$\sum_{p\in\mathbb{P}} 1/p = \sum_{n=1}^{\infty} 1/p_n = \infty \ ; \tag{2.1}$$

the prime reciprocals combine to form a divergent series. Begin with a little counting: if $m \in \mathbb{N}$ then

$$\#\{r'_n : n \in \mathbb{N}_m\} \leq \sqrt{m} \ , \tag{2.2}$$

since

$$n \in \mathbb{N}_m \Longrightarrow (r'_n)^2 \leq n \leq m \ .$$

Now with

$$M(m, p) = \mathbb{N}_m \cap \mathbb{N}p \ ,$$

observe that

$$\#M(m, p) \leq m/p \ . \tag{2.3}$$

Also with

$$M'(m, p) = \mathbb{N}_m \setminus \mathbb{N}p \ ,$$

write

$$M'_k(m) = \bigcap_{k<j} M'(m, p_j) \ ;$$

we claim that

$$\#\{r_n : n \in M'_k(m)\} \leq 2^k \ .$$

It follows therefore that

$$\#M'_k(m) \leq 2^k \sqrt{m} \ . \tag{2.4}$$

Now if, for a contradiction, the series of prime reciprocals were to converge, then there would have to be $k \in \mathbb{N}$ for which

$$\sum_{n=k+1}^{\infty} 1/p_n < 1/2 \ . \tag{2.5}$$

With $N(m) = \#M'(m, p_{k+1})$ we have successively

$$m - N(m) \leq m(1/p_{k+1} + 1/p_{k+2} + \ldots) < m/2 \ ,$$

and hence

$$m/2 < N(m) \leq 2^k \sqrt{m} \tag{2.6}$$

and so (for arbitrary $m \in \mathbb{N}$)

$$\sqrt{m} < 2^{k+1} \ ! \tag{2.7}$$

3

Much of this can be repeated for monic complex polynomials

$$p \equiv p(z) \equiv z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n \in \text{Poly}_1 \subseteq \mathbb{C}[z] \subseteq \mathbb{C}^{\mathbb{C}} \ ,$$

with $z : \mathbb{C} \to \mathbb{C}$ the complex coordinate, $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}^n$. The "primes" are now the polynomials of degree 1, and the "spectrum" given by

$$\varpi(p) = \{z - \lambda : \lambda \in p^{-1}(0) \subseteq \mathbb{C}\} \ , \tag{3.1}$$

and the fundamental theorem of arithmetic is replaced by the fundamental theorem of algebra:

$$p = \prod_{\lambda \in \mathbb{C}} (z - \lambda)^{\nu_p(\lambda)} = \prod_{j=1}^{n} z - \lambda_j \ . \tag{3.2}$$

Here $\nu_p(\lambda)$ is given by the analogue of (1.3), and the first product is over λ for which $\nu_p(\lambda) \neq 0$, and there may be repetitions in the second.

When the degree of a polynomial p is greater than or equal to 5 , it is difficult to determine the λ_j: but the test of square free status is that a polynomial and its derivative are mutually prime, which by the Euclidean axiom will say that

$$1 \in \mathbb{C}[z]p + \mathbb{C}[z]dp/dz \ .$$

More generally, writing

$$p \equiv p^\vee \cdot p^\wedge \tag{3.3}$$

for the "polar decomposition", with square free root $p^\wedge$ and carapace $p^\vee$, we claim

$$p^\vee = \text{hcf}(p, dp/dz) \ . \tag{3.4}$$

To see this suppose

$$p = q \cdot r$$

with

$$\text{hcf}(q, r) = 1 \ ;$$

then $p^\vee = q^\vee \cdot r^\vee$ and $\varpi(q) \cap \varpi(r) = \emptyset$, and proceed by induction on $\#\varpi(p)$.

For a rather simple example, if

$$p \equiv z^2(z - 1) \ ,$$

with square free root $z(z - 1)$ and carapace z, we have

$$dp/dz \equiv 3z^2 - 2z \ ,$$

and hence

$$3z^2 \equiv z dp/dz - 3p \ ,$$

and finally, for (3.4),

$$2z \equiv (3z - 1)dp/dz - 9p \ .$$

4 References

[1] Harte, R.E. Spectral dsjointness and the Euclidean algorithm *Math. Proc. Royal Irish Acad.* 118A (2018) 65-69

[2] Harte, R.E. On spectral number theory *Funct. Anal. Approx. Comp.* 13 (2021) 1-6

[3] Rosenthal, D., Rosenthal, D. and Rosenthal, P. A readable introduction to real mathematics *Springer Undergraduate Texts in Mathematics* (second edition) (2021)