Conditional positive definiteness in operator theory

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Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. Recall that a sequence $\{\gamma_n\}_{n=0}^{\infty} \subset \mathbb{R}$ is conditionally positive definite (CPD for brevity) if $\sum_{i,j=0}^{k} \gamma_{i+j}\lambda_i\bar{\lambda}_j \geq 0$ for all finite sequence $\{\lambda_k\}_{k=0}^{n} \subset \mathbb{C}$ such that $\sum_{i=0}^{k} \lambda_i = 0$. An operator $T \in B(\mathcal{H})$ is said to be conditionally positive definite if $\{\|T^n h\|^2\}_{n=0}^{\infty}$ is CPD for all $h \in \mathcal{H}$. Every subnormal operator is CPD obviously, but the converse is not. In particular, if $T \in B(\mathcal{H})$ is a contractive CPD operator, then T is subnormal. In 1991 J. Stochel suggested a question: is it true that, for a given positive integer $j \geq 2$, $T \in B(\mathcal{H})$ is subnormal if and only if $\{\|T^n h\|^{2j}\}_{n=0}^{\infty}$ is positive definite for all $h \in \mathcal{H}$?, which remains unsolved for more than 30 years. The study of CPD operators is closely related to Stochel's problem. In this talk, we discuss semispectral and dilation type representations for CPD operators. On the basis of Agler's hereditary functional calculus, we build an $L^{\infty}(M)$-functional calculus for CPD operators, where M is an associated semispectral measure. In addition, we derive new necessary and sufficient conditions for a CPD operator to be a subnormal contraction. Finally, we discuss a few natural questions that are usually asked when considering new classes of operators.

(This is a joint work with Z. Jablonski and J. Stochel.)