# An answer to a question of A. Lubin: The lifting problem for commuting subnormals

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**Abstract.** In this paper we give an answer to a long-standing open question on the lifting problem for commuting subnormals (due to A. Lubin): The subnormality for the sum of commuting subnormal operators does not guarantee the existence of commuting normal extensions.

**Keywords.** The lifting problem for commuting subnormal operators, subnormal pairs, jointly subnormal, 2-variable weighted shifts, Berger's Theorem, Agler's criterion, Lambert's Theorem, disintegration of measures, Chu-Vandermonde identity.

## 1. Introduction

§1. **A historical background**. The *Lifting Problem for Commuting Subnormals* (LPCS) asks for necessary and sufficient conditions for a pair of commuting subnormal operators on a Hilbert space to admit commuting normal extensions. This is an old problem in operator theory. The aim of this paper is to answer a long-standing open problem about the LPCS.

To begin with, let $\mathcal{H}$ denote a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, $T^*$ denotes the adjoint of $T$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if its self-commutator $[T^*, T] \equiv T^*T - TT^*$ is positive semi-definite, and *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N$ on $\mathcal{K}$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $T = N|_{\mathcal{H}}$, a restriction of $N$ to $\mathcal{H}$. In this case, $N$ is called a *normal extension* of $T$. In 1950, P.R. Halmos [18] introduced the notion of a subnormal operator for the purpose of the study of dilations and extensions of operators on a Hilbert space. Nowadays, the theory of subnormal operators has become an extensive and highly developed area, which has made significant contributions to a number of problems in functional analysis, operator theory, mathematical physics, and several other fields.

We recall that if $\mathfrak{A}$ is a subset of $\mathcal{B}(\mathcal{H})$ then the *commutant* of $\mathfrak{A}$, denoted $\mathfrak{A}'$, is the set of operators in $\mathcal{B}(\mathcal{H})$ which commute with every operator in $\mathfrak{A}$. If $T \in \mathcal{B}(\mathcal{H})$ is a subnormal operator and $N$ is a normal extension of $T$, then we say that an operator $A$ in $\{T\}'$ *lifts* to $\{N\}'$ if there exists an operator $B$ in $\{N\}'$ such that $B(\mathcal{H}) \subseteq \mathcal{H}$ and $A = B|_{\mathcal{H}}$. In 1971, J.A. Deddens [14] provided an example that not every operator in $\{T\}'$ lifts to $\{N\}'$. As an interesting inquiry in the commutant lifting problem, an old problem (LPCS) in operator theory has been brought up: for

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two commuting subnormal operators $T_1$ and $T_2$, find necessary and sufficient conditions for a pair of $T_1$ and $T_2$ to admit commuting normal extensions. The LPCS has been studied by many authors including [1], [2], [3], [6], [8], [10], [11], [13], [15], [19], [22], [23], [24], [25], [26], [27], [29], [30], etc. There are many known examples of commuting pairs of subnormal operators which admit no lifting (cf. M.B. Abrahamse [1] and A.R. Lubin [22]). Also, many sufficient conditions for the existence of a lifting have been found. For instance, a commuting pair of subnormal operators $T_1$ and $T_2$ admits a lifting if either $T_1$ or $T_2$ is normal (J. Bram [6]), if either $T_1$ or $T_2$ is cyclic (T. Yoshino [30]), if either $T_1$ or $T_2$ is an isometry (M. Slocinski [27]), or if the spectrum of either $T_1$ or $T_2$ is finitely connected and the spectrum of its minimal normal extension is contained in the boundary of its spectrum. On the other hand, in all of the known examples of the absence of lifting, the key property missing is the subnormality of $T_1 + T_2$. Indeed, in 1978, A.R. Lubin [23] addressed a concrete problem about the LPCS: if $T_1$ and $T_2$ are commuting subnormal operators, do they admit commuting normal extensions when $p(T_1, T_2)$ is subnormal for every 2-variable polynomial $p$, or more weakly, when $T_1 + T_2$ is subnormal? In 1994, E. Franks [15] showed that the first condition gives an affirmative answer; indeed, commuting subnormal operators $T_1$ and $T_2$ admit commuting normal extensions if $p(T_1, T_2)$ is subnormal for each 2-variable polynomial $p$ of degree at most 5. However, the second condition still remains open: that is, if $T_1$ and $T_2$ are commuting subnormal operators,

$$\text{does the subnormality of } T_1 + T_2 \text{ guarantee commuting normal extensions of } T_1 \text{ and } T_2\,? \tag{1.1}$$

What is the reason why 36 years passed while question (1.1) remained unanswered ? The difficulty of determining the subnormality of $T_1 + T_2$ is one explanation for failing to answer question (1.1). Probably, the most effective way to determine the subnormality of $T_1 + T_2$ is Agler's criterion for subnormality in [4]. However, in view of Lambert's Theorem [21], a main ingredient to examine the subnormality is weighted shifts and Agler's criterion for the weighted shifts involves quite intricately combinatorial expressions, which are hard problems to solve. Thus, we had to develop the theory of 2-variable weighted shifts before the time is ripe for answering question (1.1). In this paper, we give a negative answer to question (1.1), by using 2-variable weighted shifts together with the disintegration-of-measure technique and ingenious combinatorial computations.

§2. **Joint subnormality**. The notion of joint hyponormality for the general case of $n$-tuples of operators was first formally introduced by A. Athavale [5]. Joint hyponormality originated from the LPCS, and it has also been considered with an aim at understanding the gap between hyponormality and subnormality for single operators. In some sense, the birth of joint hyponormality occurred with the Bram-Halmos theorem for subnormality of an operator. The Bram-Halmos criterion for subnormality (cf. [6], [7]) states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$. Given an $n$-tuple $\mathbf{T} \equiv (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$, we let $[\mathbf{T}^*, \mathbf{T}] \in \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ denote the *self-commutator* of $\mathbf{T}$, defined by

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \ldots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \ldots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \ldots & [T_n^*, T_n] \end{pmatrix},$$

where $[S, T] := ST - TS$ for $S, T \in \mathcal{B}(\mathcal{H})$. By analogy with the case $n = 1$, we shall say ([5], [12]) that $\mathbf{T}$ is *jointly hyponormal* (or simply, *hyponormal*) if $[\mathbf{T}^*, \mathbf{T}]$ is a positive operator on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Thus, the Bram-Halmos criterion can be restated as: $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $(T, T^2, \cdots, T^k)$ is hyponormal for every $k \in \mathbb{Z}_+$. The $n$-tuple $\mathbf{T} \equiv (T_1, \ldots, T_n)$ is said to be (*jointly*) *normal* if $\mathbf{T}$ is commuting and every $T_i$ is a normal operator and is said to be (*jointly*) *subnormal* if $\mathbf{T}$ is the restriction of a normal $n$-tuple to a common invariant subspace, i.e., $\mathbf{T}$ admits commuting normal extensions. Thus the LPCS can be restated as:

**LPCS**: *Find necessary and sufficient conditions for a commuting pair of subnormal operators to be subnormal.*

§3. **A main ingredient of the paper - two variable weighted shifts**. To answer question (1.1), we exploit 2-variable weighted shifts as a main tool. It is well known that the subnormality of an arbitrary operator can be ascertained by examining the subnormality of an associated family of weighted shifts [21]. Thus, single and multivariable weighted shifts have played an important role in the study of the LPCS. They have also played a significant role in the study of cyclicity and reflexivity, in the study of $C^*$-algebras generated by multiplication operators on Bergman spaces, as fertile ground to test new hypotheses, and as canonical models for theories of dilation and positivity. We review the definition and basic properties of 2-variable weighted shifts.

Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called weights or a weight sequence), the (unilateral) weighted shift $W_\alpha$ associated with the sequence $\alpha$ is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$. We shall often write shift$(\alpha_0, \alpha_1, \cdots)$ to denote the weighted shift $W_\alpha$ with a weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$. The *moments* of $\alpha$ are defined by

$$\gamma_k \equiv \gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 \quad (k \geq 1)$$

and $\gamma_0 := 1$. There is a well-known criterion of subnormality of weighted shifts, due to C. Berger (cf. [7, III.8.16]) and independently established by R. Gellar and L.J. Wallen [16]: $W_\alpha$ is subnormal if and only if there exists a probability measure $\xi_\alpha$ supported in $[0, ||W_\alpha||^2]$ (called the *Berger measure* of $W_\alpha$) such that $\gamma_k(\alpha) = \int s^k d\xi_\alpha(s)$ $(k \geq 1)$. If $W_\alpha$ is subnormal with Berger measure $\xi_\alpha$ and $i \geq 1$, and if we let $\mathcal{L}_i := \bigvee\{e_n : n \geq i\}$ denote the invariant subspace obtained by removing the first $i$ vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then

$$\text{the Berger measure of } W_\alpha|_{\mathcal{L}_i} \text{ is } \frac{s^i}{\gamma_i(\alpha)} d\xi_\alpha(s), \tag{1.2}$$

where $W_\alpha|_{\mathcal{L}_i}$ denotes the restriction of $W_\alpha$ to $\mathcal{L}_i$.

We now consider two bounded double-indexed sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}, \beta \equiv \{\beta_{\mathbf{k}}\} \in \ell^\infty(\mathbb{Z}_+^2)$, $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $\ell^2(\mathbb{Z}_+^2)$ be the Hilbert space of square-summable complex sequences indexed by $\mathbb{Z}_+^2$. (Note that $\ell^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $\ell^2(\mathbb{Z}_+) \bigotimes \ell^2(\mathbb{Z}_+)$.) We define a 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$, a pair of $T_1$ and $T_2$ on $\ell^2(\mathbb{Z}_+^2)$, by $T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_1}$ and $T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\varepsilon_2}$, where $\varepsilon_1 := (1, 0)$, $\varepsilon_2 := (0, 1)$, and $\{e_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}_+^2}$ denotes the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+^2)$ (see Figure 1(i)). Clearly,

$$T_1 T_2 = T_2 T_1 \Longleftrightarrow \beta_{\mathbf{k}+\varepsilon_1} \alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\varepsilon_2} \beta_{\mathbf{k}} \text{ (all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{1.3}$$

In the sequel, we assume that all 2-variable weighted shifts $W_{(\alpha,\beta)}$ are commuting, i.e., it satisfies the condition (1.3). Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moment of order $\mathbf{k}$ for a pair $(\alpha, \beta)$ satisfying (1.3) is defined by

$$\gamma_{\mathbf{k}} \equiv \gamma_{\mathbf{k}}(\alpha, \beta) := \begin{cases} 1 & \text{if } k_1 = 0 \text{ and } k_2 = 0; \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0; \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1; \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases}$$

We note that, due to the commutativity condition (1.3), $\gamma_{\mathbf{k}}$ can be computed using any nondecreasing path from $(0, 0)$ to $\mathbf{k}$. We recall that there is a 2-variable Berger's Theorem, due to N. Jewell and A.R. Lubin [20]: a 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ is subnormal if and only if there exists a probability measure $\mu$ (called *Berger measure* of $W_{(\alpha,\beta)}$) defined on the 2-dimensional rectangle $R = [0, ||T_1||^2] \times [0, ||T_2||^2]$ such that

$$\gamma_{\mathbf{k}}(\alpha, \beta) = \iint_R s^{k_1} t^{k_2} d\mu(s, t) \quad \text{for all } \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2 \text{ (called Berger's Theorem).}$$

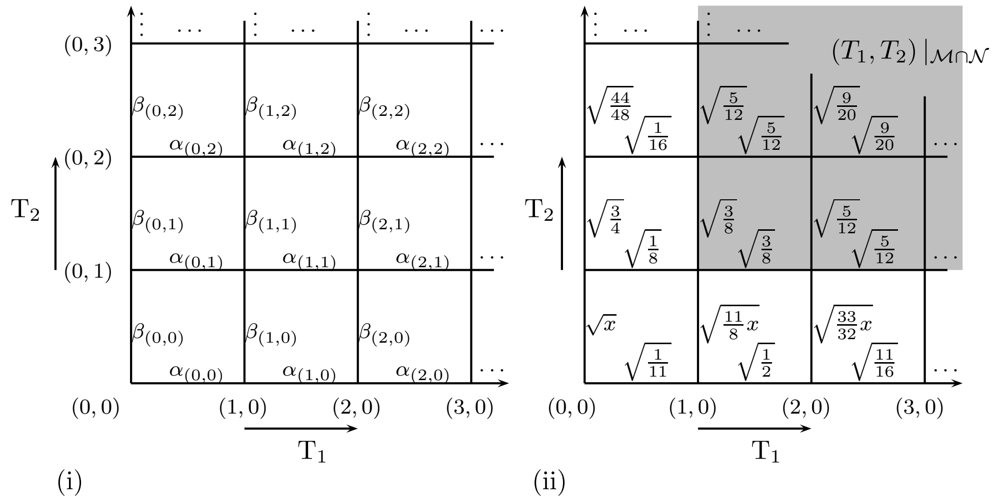


Figure 1. (i) The weight diagram of a generic 2-variable weighted shift;
(ii) The weight diagram of the 2-variable weighted shift $(T_1, T_2)$ given in Theorem 1.1.

§4. **A description of the main theorem**. For an arbitrary commuting 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$, let $(T_1, T_2)|_{\mathcal{R}}$ denote the restriction of $W_{(\alpha,\beta)}$ to $\mathcal{R}$, where $\mathcal{R}$ is a common invariant subspace of $\ell^2(\mathbb{Z}_+^2)$ for $T_1$ and $T_2$. Throughout the paper, we write

$$\begin{aligned}\mathcal{M} &:= \bigvee\{e_{(k_1,k_2)} \in \ell^2(\mathbb{Z}_+^2) : k_1 \geq 0, k_2 \geq 1\};\\ \mathcal{N} &:= \bigvee\{e_{(k_1,k_2)} \in \ell^2(\mathbb{Z}_+^2) : k_1 \geq 1, k_2 \geq 0\}.\end{aligned}$$

To answer question (1.1), we use the 2-variable weighted shift $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ with the weight diagram given by Figure 1(ii), where $\alpha_{(0,1)} := \sqrt{\frac{1}{8}}$, the 0-th horizontal slice of $T_1$ is a weighted shift $W_a := \text{shift}\left(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots\right)$ whose weight sequence $a \equiv \{a_n\}_{n=0}^{\infty}$ is given by

$$a_n := \begin{cases} \sqrt{\frac{1}{11}} & \text{if } n = 0 \\ \sqrt{\frac{4^n+2^n+2}{4^n+2^{n+1}+8}} & \text{if } n \geq 1, \end{cases}$$

and the 0-th vertical slice of $T_2$ is a weighted shift $W_b := \text{shift}\left(\beta_{(0,0)}, \beta_{(0,1)}, \cdots\right)$ whose weight sequence $b \equiv \{b_n\}_{n=0}^{\infty}$ is given by

$$b_n := \begin{cases} \sqrt{x} \quad (x > 0) & \text{if } n = 0 \\ \sqrt{\frac{10 \cdot 2^{2n}+2^n+1}{10 \cdot 2^{2n}+2^{n+1}+4}} & \text{if } n \geq 1. \end{cases}$$

Now, we define $(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}}$. For both of 0-th horizontal and vertical slices of $(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}}$, we put a weighted shift $W_c$ whose weight sequence $c \equiv \{c_n\}_{n=0}^{\infty}$ is given by

$$c_n := \sqrt{\frac{2^{n+1}+1}{2^{n+2}+4}} \quad (n \geq 0):$$

in other words,

$$W_c := \text{shift}\left(\alpha_{(1,1)}, \alpha_{(2,1)}, \cdots\right) = \text{shift}\left(\beta_{(1,1)}, \beta_{(1,2)}, \cdots\right) = \text{shift}\left(\sqrt{\tfrac{3}{8}}, \sqrt{\tfrac{5}{12}}, \cdots\right).$$

In turn, both of the $i$-th horizontal and vertical slices of $(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}}$ are defined by a restriction of $W_c$ to the subspace $\mathcal{L}_i := \bigvee\{e_n : n \geq i\}$, that is,

$$\begin{aligned} W_c|_{\mathcal{L}_i} &= \text{shift}\left(\alpha_{(i,i)}, \alpha_{(i+1,i)}, \cdots\right) = \text{shift}\left(\beta_{(i,i)}, \beta_{(i,i+1)}, \cdots\right) \\ &= \text{shift}\left(\sqrt{\tfrac{2^{i+1}+1}{2^{i+2}+4}}, \sqrt{\tfrac{2^{i+2}+1}{2^{i+3}+4}}, \cdots\right). \end{aligned}$$

Then, the remaining weights of $T_1$ and $T_2$ are automatically determined by the commutativity of $T_1$ and $T_2$. Via Berger's Theorem, we can show that

(a) $W_a$ is subnormal with the 4-atomic Berger measure $\xi_a := \frac{3}{4}\delta_0 + \frac{2}{11}\delta_{\frac{1}{4}} + \frac{1}{22}\delta_{\frac{1}{2}} + \frac{1}{44}\delta_1$;
(b) $W_b$ is subnormal with the 4-atomic Berger measure $\xi_b := \left(1 - \frac{15x}{8}\right)\delta_0 + x\left(\delta_{\frac{1}{4}} + \frac{1}{4}\delta_{\frac{1}{2}} + \frac{5}{8}\delta_1\right)$;
(c) $W_c$ is subnormal with the 2-atomic Berger measure $\xi_c := \frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{2}\delta_{\frac{1}{2}}$,

where $\delta_p$ denotes Dirac measure at $p$.

{*Proof*: For $\ell \geq 1$,

$$\begin{aligned}
&\int s^\ell d\xi_a(s) = \gamma_\ell\left(W_a\right) = a_0^2 a_1^2 \cdots a_{\ell-2}^2 a_{\ell-1}^2 \\
&= \tfrac{1}{11} \cdot \tfrac{4+2+2}{4+2^2+8} \cdot \tfrac{4^2+2^2+2}{4^2+2^3+8} \cdots \tfrac{4^{\ell-2}+2^{\ell-2}+2}{4^{\ell-2}+2^{\ell-1}+8} \cdot \tfrac{4^{\ell-1}+2^{\ell-1}+2}{4^{\ell-1}+2^{\ell}+8} \\
&= \tfrac{1}{11} \cdot \tfrac{8\left(\frac{1}{4}\right)^2+2\left(\frac{1}{2}\right)^2+1}{8\left(\frac{1}{4}\right)+2\left(\frac{1}{2}\right)+1} \cdot \tfrac{8\left(\frac{1}{4}\right)^3+2\left(\frac{1}{2}\right)^3+1}{8\left(\frac{1}{4}\right)^2+2\left(\frac{1}{2}\right)^2+1} \cdots \tfrac{8\left(\frac{1}{4}\right)^{\ell-1}+2\left(\frac{1}{2}\right)^{\ell-1}+1}{8\left(\frac{1}{4}\right)^{\ell-2}+2\left(\frac{1}{2}\right)^{\ell-2}+1} \cdot \tfrac{8\left(\frac{1}{4}\right)^{\ell}+2\left(\frac{1}{2}\right)^{\ell}+1}{8\left(\frac{1}{4}\right)^{\ell-1}+2\left(\frac{1}{2}\right)^{\ell-1}+1} \\
&= \tfrac{1}{44} \cdot \left(8\left(\tfrac{1}{4}\right)^\ell + 2\left(\tfrac{1}{2}\right)^\ell + 1\right) = \tfrac{2}{11} \cdot \left(\tfrac{1}{4}\right)^\ell + \tfrac{1}{22} \cdot \left(\tfrac{1}{2}\right)^\ell + \tfrac{1}{44},
\end{aligned} \tag{1.4}$$

giving (a). The assertions (b) and (c) follow from the same argument as (a).}

Then, our main theorem follows:

**Theorem 1.1.** *Let* $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ *be given by Figure* 1*(ii). Then, we have:*

(i) $T_1$ *and* $T_2$ *are both subnormal if and only if* $0 < x \leq \frac{8}{33}$*;*
(ii) $(T_1, T_2)$ *is subnormal if and only if* $0 < x \leq \frac{2}{11}$*;*
(iii) $T_1 + T_2$ *is subnormal if* $0 < x \leq \frac{2}{11} + \varepsilon$ *for some* $\varepsilon > 0$*.*

Consequently, Theorem 1.1 proves that there exists a commuting pair $(T_1, T_2)$ of subnormal operators such that $T_1 + T_2$ is subnormal, but the pair $(T_1, T_2)$ is not subnormal; that is, the pair $(T_1, T_2)$ does not admit commuting normal extensions. This answers Lubin's question (1.1) in the negative.

In Section 2, we give a proof of Theorem 1.1.

## 2. Proof of Theorem 1.1

To examine the subnormality of 2-variable weighted shifts, we need some definitions.

(i) Let $\mu$ and $\nu$ be two positive measures on a set $X \equiv \mathbb{R}_+$. We say that $\mu \leq \nu$ on $X$ if $\mu(E) \leq \nu(E)$ for each Borel subset $E \subseteq X$; equivalently, $\mu \leq \nu$ if and only if $\int f d\mu \leq \int f d\nu$ for all $f \in C(X)$ such that $f \geq 0$ on $X$, where $C(X)$ denotes the set of all continuous functions on $X$.
(ii) Let $\mu$ be a probability measure on $X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$ and assume that $\frac{1}{t} \in L^1(\mu)$, i.e., $\iint \frac{1}{t}\, d\mu(s,t) < \infty$. The *extremal measure* $\mu_{ext}$ (which is also a probability measure) on $X \times Y$ is given by

$$d\mu_{ext}(s,t) := \frac{1}{t\left\|\frac{1}{t}\right\|_{L^1(\mu)}} d\mu(s,t).$$

(iii) Given a measure $\mu$ on $X \times Y$, the *marginal measure* $\mu^X$ is given by $\mu^X := \mu \circ \pi_X^{-1}$, where $\pi_X : X \times Y \to X$ is the canonical projection onto $X$. Thus $\mu^X(E) = \mu(E \times Y)$ for every $E \subseteq X$.

We provide several auxiliary lemmas which are needed for the proof of Theorem 1.1. Recall the subnormal backward extension of 1-variable weighted shifts (cf. [9]): If shift$(\alpha_1, \alpha_2, \cdots)$ is subnormal with Berger measure $\xi$, then shift$(\alpha_0, \alpha_1, \alpha_2, \cdots)$ is subnormal if and only if

$$\frac{1}{s} \in L^1(\xi) \ \text{ and } \ \alpha_0^2 \le \left( \left\| \frac{1}{s} \right\|_{L^1(\xi)} \right)^{-1}. \tag{2.1}$$

The following lemma is the 2-variable version of (2.1).

**Lemma 2.1.** *([13, Proposition 3.10]) (Subnormal backward extension of 2-variable weighted shifts) Assume that $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ is a commuting pair of subnormal operators and $(T_1, T_2)|_{\mathcal{M}}$ is subnormal with associated Berger measure $\mu_{\mathcal{M}}$. Then, $W_{(\alpha,\beta)}$ is subnormal if and only if the following conditions hold:*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\beta_{(0,0)}^2 \le (\|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})})^{-1}$;

(iii) $\beta_{(0,0)}^2 \|\frac{1}{t}\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \le \xi_0$, *where $\xi_0$ is the Berger measure of* shift$(\alpha_{(0,0)}, \alpha_{(1,0)}, \cdots)$.

*In the case when $W_{(\alpha,\beta)}$ is subnormal, the Berger measure $\mu$ of $W_{(\alpha,\beta)}$ is given by*

$$d\mu(s,t) = \beta_{(0,0)}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}(s,t) + \Big( d\xi_0(s) - \beta_{(0,0)}^2 \left\| \frac{1}{t} \right\|_{L^1(\mu_{\mathcal{M}})} d(\mu_{\mathcal{M}})_{ext}^X(s) \Big) d\delta_0(t).$$

On the other hand, we also employ disintegration-of-measure techniques. To do so, we need to review some basic properties on disintegration of measures; most of the discussion is taken from [7, VII.2, pp. 317-319]. Let $X$ and $Z$ be compact metric spaces and let $\mu$ be a positive regular Borel measure on $Z$. Let $\mathcal{L}^1(\mu)$ denote the set of all Borel functions $f$ on $Z$ such that $\int |f| d\mu < \infty$ and let $L^1(\mu)$ be the corresponding Lebesgue space of the equivalence classes of those functions. For a Borel mapping $\phi : Z \to X$, let $\nu$ be the Borel measure $\mu \circ \phi^{-1}$ on $X$; that is,

$$\nu(\Delta) := \mu(\phi^{-1}(\Delta)) \tag{2.2}$$

for every Borel set $\Delta \subseteq X$. If $f \in \mathcal{L}^1(\mu)$ then the map $\psi \mapsto \int_Z (\psi \circ \phi) f \, d\mu$ defines a bounded linear functional on $L^\infty(\nu)$. When restricted to characteristic functions $\chi_\Delta$ in $L^\infty(\nu)$, $\Delta \mapsto \int_Z (\chi_\Delta \circ \phi) f \, d\mu = \int_{\phi^{-1}(\Delta)} f \, d\mu$ is a Borel measure on $X$ which is absolutely continuous with respect to $\nu$. Then, there exists a unique element $E(f)$ in $L^1(\nu)$ such that $\int_Z (\chi_\Delta \circ \phi) f \, d\mu = \int_X \chi_\Delta E(f) d\nu$ for every Borel set $\Delta$ of $X$. Via convergence theorems, one can show that

$$\int_Z (\psi \circ \phi) \, f \, d\mu = \int_X \psi \, E(f) d\nu \tag{2.3}$$

for all $\psi \in L^\infty(\nu)$. This defines a map $E : \mathcal{L}^1(\mu) \to L^1(\nu)$ called the *expectation operator*. We write $M(Z)$ for the set of all regular Borel measures on $Z$. A *disintegration of the measure* $\mu$ with respect to $\phi$ is a function $x \mapsto \lambda_x$ from $X$ to $M(Z)$ such that $\lambda_x$ is a probability measure for each $x \in X$ and $E(f)(x) = \int_Z f \, d\lambda_x$ a.e. $[\nu]$ for each $f \in \mathcal{L}^1(\mu)$. Then we have the existence and uniqueness of the disintegration of a measure (cf. [7, Theorem VII.2.11]): (i) given a regular Borel measure $\mu$ on a compact metric space $Z$, and a Borel function $\phi$ from $Z$ into a compact metric space $X$, there is a disintegration $x \mapsto \lambda_x$ of $\mu$ with respect to $\phi$; (ii) if $x \mapsto \lambda_x^{'}$ is another disintegration of $\mu$ with respect to $\phi$, then $\lambda_x = \lambda_x^{'}$ a.e. $[\nu]$.

The following lemma is useful in the sequel.

**Lemma 2.2.** *If $\mu$ is a positive regular Borel measure defined on $Z := X \times Y \equiv \mathbb{R}_+ \times \mathbb{R}_+$ and $\frac{1}{t} \in L^1(\mu)$, then*

$$\left\| \frac{1}{t} \right\|_{L^1(\mu)} = \left\| \frac{1}{t} \right\|_{L^1(\mu^Y)},$$

*where $\mu^Y := \mu \circ \pi_Y^{-1}$ and $\pi_Y : Z \to Y$ is the canonical projection onto $Y$.*

*Proof.* Put $\phi = \pi_Y$ in the preceding argument. Then, for the disintegration $t \mapsto \lambda_t$ of the measure $\mu$ with respect to $\phi$, we know (cf. [7, Proposition VII.2.10]) that $\operatorname{supp}(\lambda_t) = \phi^{-1}(t) = X \times \{t\} \subseteq Z$. Thus, we may regard $\lambda_t$ as a measure on $X$ for each $t \in Y$ and write $d\lambda_t(s)$ for $d\lambda_t(s,t)$. Note that

$$E(f)(t) = \iint_{X\times Y} f \, d\lambda_t(s,t) = \iint_{X\times\{t\}} f \, d\lambda_t(s,t).$$

We thus have

$$\begin{aligned}
\left\|\frac{1}{t}\right\|_{L^1(\mu)} &= \iint \frac{1}{t} d\mu(s,t) \\
&= \int_Y E\left(\frac{1}{t}\right) d\mu^Y(t) \quad (\text{by (2.3) with } \psi \equiv 1) \\
&= \int_Y \left(\iint_{X\times\{t\}} \frac{1}{t} d\lambda_t(s,t)\right) d\mu^Y(t) \\
&= \int_Y \left(\int_X \frac{1}{t} d\lambda_t(s)\right) d\mu^Y(t) = \int_Y \frac{1}{t} d\mu^Y(t) \\
&= \left\|\frac{1}{t}\right\|_{L^1(\mu^Y)},
\end{aligned}$$

which proves the lemma. □

The following is a well-known combinatoric identity, where the first equality is called the Chu-Vandermonde identity.

**Lemma 2.3.**

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} = \frac{1}{2\pi i} \int_{|z|=1} \frac{(1+z)^{2n}}{z^{n+1}} dz = \frac{1}{\pi} \int_0^4 \frac{s^n}{\sqrt{4s - s^2}} ds.$$

*Proof.* The first equality comes from [17, (3.66)], the second equality follows from the Cauchy integral formula, and the last equality follows from a direct calculation. □

**Lemma 2.4.** *If $W_{(\alpha,\beta)} \equiv (T_1, T_2)$ is a 2-variable weighted shift given by Figure 1(ii), then $(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}}$ is subnormal with Berger measure*

$$\mu_{\mathcal{M}\cap\mathcal{N}} \equiv \frac{1}{2}\delta_{(\frac{1}{4},\frac{1}{4})} + \frac{1}{2}\delta_{(\frac{1}{2},\frac{1}{2})}. \tag{2.4}$$

*Proof.* For each $t \in [0,1]$, define

$$\delta_t(s) := \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the weight diagram of $(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}}$ given in Figure 1(ii), we can see that for all $k_1, k_2 \geq 0$,

$$\begin{aligned}
\iint_{[0,1]^2} s^{k_1} t^{k_2} d\mu_{\mathcal{M}\cap\mathcal{N}}(s,t) &= \int_0^1 t^{k_1+k_2} d\left(\mu_{\mathcal{M}\cap\mathcal{N}}\right)^Y(t) \\
&= \int_0^1 t^{k_2} \left[\int_0^1 s^{k_1} d\delta_t(s)\right] d\left(\mu_{\mathcal{M}\cap\mathcal{N}}\right)^Y(t) \\
&= \iint_{[0,1]^2} s^{k_1} t^{k_2} d\delta_t(s)\, d\xi_c(t) \\
&= \iint_{[0,1]^2} s^{k_1} t^{k_2} d\delta_t(s)\, d\left(\frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{2}\delta_{\frac{1}{2}}\right)(t) \\
&= \iint_{[0,1]^2} s^{k_1} t^{k_2} d\left(\frac{1}{2}\delta_{(\frac{1}{4},\frac{1}{4})} + \frac{1}{2}\delta_{(\frac{1}{2},\frac{1}{2})}\right)(s,t),
\end{aligned}$$

which gives (2.4). $\square$

We are now ready for:

**Proof of Theorem** 1.1.

(i) For $m \geq 0$, let $W_c|_{\mathcal{L}_m}$ denote the restriction of $W_c$ to $\mathcal{L}_m \equiv \bigvee\{e_{(k_1,0)} : k_1 \geq m\}$. Since $\xi_c = \frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{2}\delta_{\frac{1}{2}}$, it follows that for each $m = 1, 2, \cdots$, $W_c|_{\mathcal{L}_m}$ is also subnormal with Berger measure

$$d(\xi_c)_{\mathcal{L}_m}(s) := \frac{s^m}{\gamma_m(W_c)} d\xi_c(s) = \frac{1}{\gamma_m(W_c)}\left(\frac{1}{2}\left(\frac{1}{4}\right)^m d\delta_{\frac{1}{4}}(s) + \frac{1}{2}\left(\frac{1}{2}\right)^m d\delta_{\frac{1}{2}}(s)\right).$$

Note

$$\left\|\frac{1}{s}\right\|_{L^1((\xi_c)_{\mathcal{L}_m})} = \frac{2\left(\frac{1}{4}\right)^m + \left(\frac{1}{2}\right)^m}{\gamma_m(W_c)} \quad \text{and} \quad \alpha^2_{(0,m+1)} = \frac{\gamma_m(W_c)}{8 \cdot \gamma_m(W_b|_{\mathcal{L}_1})},$$

where the Berger measure of $W_b|_{\mathcal{L}_1}$ is $(\xi_b)_{\mathcal{L}_1} := \frac{1}{4}\delta_{\frac{1}{4}} + \frac{1}{8}\delta_{\frac{1}{2}} + \frac{5}{8}\delta_1$. Since the $m$-th horizontal slice of $(T_1, T_2)|_{\mathcal{M}\cap\mathcal{N}}$ is a restriction of $W_c$ to $\mathcal{L}_m$, it follows from (2.1) that $T_1$ is subnormal if and only if $\alpha^2_{(0,m+1)} \leq \left\|\frac{1}{s}\right\|^{-1}_{L^1((\xi_c)_{\mathcal{L}_m})}$ (all $m \geq 0$). Since

$$2\left(\frac{1}{4}\right)^m + \left(\frac{1}{2}\right)^m \leq 8 \cdot \gamma_m(W_b|_{\mathcal{L}_1}) = 8\int s^m d(\xi_b)_{\mathcal{L}_1}(s) = 2\left(\frac{1}{4}\right)^m + \left(\frac{1}{2}\right)^m + 5 \quad (\text{all } m \geq 0),$$

it follows at once that $T_1$ is subnormal.

Similarly, if $n \geq 0$, then

$$\left\|\frac{1}{t}\right\|_{L^1((\xi_c)_{\mathcal{L}_n})} = \frac{2\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n}{\gamma_n(W_c)} \quad \text{and} \quad \beta^2_{(n+1,0)} = \frac{11x \cdot \gamma_n(W_c)}{8 \cdot \gamma_n(W_a|_{\mathcal{L}_1})}, \tag{2.5}$$

where the Berger measure of $W_a|_{\mathcal{L}_1}$ is $(\xi_a)_{\mathcal{L}_1} := \frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{4}\delta_{\frac{1}{2}} + \frac{1}{4}\delta_1$. Since $T_2$ is subnormal if and only if $\beta^2_{(n+1,0)} \leq \left\|\frac{1}{t}\right\|^{-1}_{L^1((\xi_c)_{\mathcal{L}_n})}$ (all $n \geq 0$), a direct calculation together with (2.1) and (2.5) shows that

$$T_2 \text{ is subnormal} \iff x \leq \frac{8\left(\frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{1}{4}\left(\frac{1}{2}\right)^n + \frac{1}{4}\right)}{11\left(2\left(\frac{1}{4}\right)^n + \left(\frac{1}{2}\right)^n\right)} \ (\text{all } n \geq 0) \iff x \leq \frac{8}{33},$$

where the second implication follows from the observation that the fractional function of the second term is increasing on $n \geq 0$. This proves (i).

(ii) We first claim that

$$(T_1, T_2)|_{\mathcal{M}} \text{ is subnormal with Berger measure } \mu_{\mathcal{M}} \equiv \frac{1}{4}\delta_{\left(\frac{1}{4},\frac{1}{4}\right)} + \frac{1}{8}\delta_{\left(\frac{1}{2},\frac{1}{2}\right)} + \frac{5}{8}\delta_{(0,1)}. \tag{2.6}$$

For (2.6), we first observe that by Lemma 2.2, $\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} = \left\|\frac{1}{s}\right\|_{L^1\left((\mu_{\mathcal{M}\cap\mathcal{N}})^X\right)} = 3$ since $(\mu_{\mathcal{M}\cap\mathcal{N}})^X = \frac{1}{2}\delta_{\frac{1}{4}} + \frac{1}{2}\delta_{\frac{1}{2}}$ (by Lemma 2.4). We thus have

$$(\mu_{\mathcal{M}\cap\mathcal{N}})^Y_{ext} = \left(\left\|\frac{1}{s}\right\|^{-1}_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} \frac{\mu_{\mathcal{M}\cap\mathcal{N}}}{s}\right)^Y = \frac{2}{3}\delta_{\frac{1}{4}} + \frac{1}{3}\delta_{\frac{1}{2}}. \tag{2.7}$$

Hence, by Lemma 2.1(iii), $(T_1, T_2)|_{\mathcal{M}}$ is subnormal if and only if

$$\alpha^2_{(0,1)}\left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} (\mu_{\mathcal{M}\cap\mathcal{N}})^Y_{ext} \leq (\xi_b)_{\mathcal{L}_1} \iff \frac{1}{4}\delta_{\frac{1}{4}} + \frac{1}{8}\delta_{\frac{1}{2}} \leq \frac{1}{4}\delta_{\frac{1}{4}} + \frac{1}{8}\delta_{\frac{1}{2}} + \frac{5}{8}\delta_1,$$

which is always true. Therefore, $(T_1,T_2)\,|_{\mathcal{M}}$ is always subnormal. By Lemma 2.1 and (2.7), we get the desired Berger measure of $(T_1,T_2)\,|_{\mathcal{M}}$:

$$\begin{aligned} d\mu_{\mathcal{M}}(s,t) &= \alpha_{(0,1)}^2 \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} d(\mu_{\mathcal{M}\cap\mathcal{N}})_{ext}(s,t) \\ &\quad + \left(d(\xi_b)_{\mathcal{L}_1}(t) - \alpha_{(0,1)}^2 \left\|\frac{1}{s}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} d(\mu_{\mathcal{M}\cap\mathcal{N}})_{ext}^Y(t)\right) d\delta_0(s) \\ &= \frac{3}{8}\left(\frac{2}{3} d\delta_{(\frac14,\frac14)}(s,t) + \frac{1}{3} d\delta_{(\frac12,\frac12)}(s,t)\right) + \frac{5}{8} d\delta_1(t) d\delta_0(s) \\ &= \frac{1}{4} d\delta_{(\frac14,\frac14)}(s,t) + \frac{1}{8} d\delta_{(\frac12,\frac12)}(s,t) + \frac{5}{8} d\delta_{(0,1)}(s,t), \end{aligned}$$

which gives

$$\mu_{\mathcal{M}} = \frac{1}{4}\delta_{(\frac14,\frac14)} + \frac{1}{8}\delta_{(\frac12,\frac12)} + \frac{5}{8}\delta_{(0,1)}.$$

We next claim that

$$(T_1,T_2)\,|_{\mathcal{N}} \text{ is subnormal } \iff 0 < x \leq \frac{2}{11}.$$

By Lemma 2.2, we note that $\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} = \left\|\frac{1}{t}\right\|_{L^1\left((\mu_{\mathcal{M}\cap\mathcal{N}})^Y\right)} = 3$ and $(\mu_{\mathcal{M}\cap\mathcal{N}})_{ext}^X(s) = \frac{2}{3}\delta_{\frac14} + \frac{1}{3}\delta_{\frac12}$. Thus, by Lemma 2.1(iii), $(T_1,T_2)\,|_{\mathcal{N}}$ is subnormal if and only if

$$\begin{aligned} &\beta_{(1,0)}^2 \left\|\tfrac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}\cap\mathcal{N}})} (\mu_{\mathcal{M}\cap\mathcal{N}})_{ext}^X \leq (\xi_a)_{\mathcal{L}_1} \\ &\iff \tfrac{11x}{8} \cdot 3 \cdot \left(\tfrac{2}{3}\delta_{\frac14} + \tfrac{1}{3}\delta_{\frac12}\right) \leq \tfrac{1}{2}\delta_{\frac14} + \tfrac{1}{4}\delta_{\frac12} + \tfrac{1}{4}\delta_1 \iff x \leq \tfrac{2}{11}. \end{aligned}$$

We now claim that

$$(T_1,T_2) \text{ is subnormal } \iff 0 < x \leq \frac{2}{11}. \tag{2.8}$$

Towards (2.8), observe that the commutativity of $T_1$ and $T_2$ comes directly from Figure 1(ii). By the proof of (i) just given above, we know that $T_1$ is always subnormal and

$$T_2 \text{ is subnormal } \iff 0 < x \leq \frac{8}{33}.$$

By Lemma 2.2, we have $\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = \left\|\frac{1}{t}\right\|_{L^1\left((\mu_{\mathcal{M}})^Y\right)} = \frac{15}{8}$ (since $\mu_{\mathcal{M}}^Y = \frac14\delta_{\frac14} + \frac18\delta_{\frac12} + \frac58\delta_1$) and

$$(\mu_{\mathcal{M}})_{ext}^X = \left(\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}^{-1} \frac{\mu_{\mathcal{M}}}{t}\right)^X = \frac{1}{3}\delta_0 + \frac{8}{15}\delta_{\frac14} + \frac{2}{15}\delta_{\frac12}.$$

Hence, by Lemma 2.1(iii), $(T_1,T_2)$ is subnormal if and only if

$$\begin{aligned} &\beta_{00}^2 \left\|\tfrac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} (\mu_{\mathcal{M}})_{ext}^X \leq \xi_a \\ &\iff x\left(\tfrac{5}{8}\delta_0 + \delta_{\frac14} + \tfrac{1}{4}\delta_{\frac12}\right) \leq \tfrac{3}{4}\delta_0 + \tfrac{2}{11}\delta_{\frac14} + \tfrac{1}{22}\delta_{\frac12} + \tfrac{1}{44}\delta_1 \iff x \leq \tfrac{2}{11}, \end{aligned}$$

which proves (ii).

(iii) For the subnormality of $T_1+T_2$, we shall use Agler's criterion for subnormality in [4], which states that a contraction $S \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{\ell=0}^{n}(-1)^\ell\binom{n}{l}\left\|S^\ell x\right\|^2 \geq 0$ for all $n \geq 1$ and all $x \in \mathcal{H}$. Since $(T_1,T_2)|_{\mathcal{M}}$ is subnormal, it is enough to consider Agler's criterion at $\left\{e_{(k,0)}\right\}_{k=0}^{\infty}$: indeed, if $x = \sum_k a_k e_{(k,0)}$, then

$$\left\|\left(\frac{T_1+T_2}{2}\right)^\ell x\right\|^2 = \sum_k |a_k|^2 \left\|\left(\frac{T_1+T_2}{2}\right)^\ell e_{(k,0)}\right\|^2,$$

and hence

$$\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{l}\left\|\left(\frac{T_1+T_2}{2}\right)^{\ell}x\right\|^2=\sum_k|a_k|^2\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{l}\left\|\left(\frac{T_1+T_2}{2}\right)^{\ell}e_{(k,0)}\right\|^2,$$

which gives

$$\frac{T_1+T_2}{2}\text{ is subnormal}\iff P_n\left(k,0\right):=\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}\left\|\left(\frac{T_1+T_2}{2}\right)^{\ell}e_{(k,0)}\right\|^2\geq 0\quad(\text{all } n\geq 1)\,.$$

Hence, we see that $T_1+T_2$ is subnormal if and only if $\inf\left\{P_n\left(k,0\right):\ n\in\mathbb{Z}_+\right\}\geq 0$ for all $k\geq 0$. For $\ell\geq 1$, we observe

$$\left(\frac{T_1+T_2}{2}\right)^{\ell}=2^{-\ell}\left(T_1^{\ell}+T_2^{\ell}+\sum_{i=1}^{\ell-1}\binom{\ell}{i}T_1^{\ell-i}T_2^{i}\right).$$

First of all, we suppose $k\geq 1$. We then have

$$\begin{aligned}P_n\left(k,0\right)&=\sum_{\ell=0}^{n}(-1)^{\ell}\tbinom{n}{\ell}\left\|\left(\tfrac{T_1+T_2}{2}\right)^{\ell}e_{(k,0)}\right\|^2\\&=1+\sum_{\ell=1}^{n}\left(-1\right)^{\ell}\tbinom{n}{\ell}2^{-2\ell}\left(\tfrac{\gamma_{k+\ell}(\xi_a)}{\gamma_k(\xi_a)}+\tfrac{x}{8}\tfrac{\gamma_{k+\ell-2}\left((\mu_{\mathcal{M}\cap\mathcal{N}})^X\right)}{\gamma_k(\xi_a)}+\sum_{i=1}^{\ell-1}\tbinom{\ell}{i}^2\tfrac{x}{8}\tfrac{\gamma_{k+\ell-2}\left((\mu_{\mathcal{M}\cap\mathcal{N}})^X\right)}{\gamma_k(\xi_a)}\right)\\&=1+\sum_{\ell=1}^{n}\left(-1\right)^{\ell}\tbinom{n}{\ell}2^{-2\ell}\left(\tfrac{\gamma_{k+\ell}(\xi_a)}{\gamma_k(\xi_a)}+\tfrac{x}{8}\tfrac{\gamma_{k+\ell-2}\left((\mu_{\mathcal{M}\cap\mathcal{N}})^X\right)}{\gamma_k(\xi_a)}\left(\sum_{i=1}^{\ell-1}\tbinom{\ell}{i}^2+1\right)\right),\end{aligned}$$

where $\gamma_{\ell}(\xi_a)$ and $\gamma_{\ell}\left((\mu_{\mathcal{M}\cap\mathcal{N}})^X\right)$ denote the $\ell$-th moments of shift $\left(\alpha_{(0,0)},\alpha_{(1,0)},\cdots\right)$ and the 0-th horizontal slice of $(T_1,T_2)|_{\mathcal{M}\cap\mathcal{N}}$, respectively. Note that

$$\begin{cases}\gamma_{\ell}(\xi_a)=\frac{2}{11}\left(\frac{1}{4}\right)^{\ell}+\frac{1}{22}\left(\frac{1}{2}\right)^{\ell}+\frac{1}{44}\\ \gamma_{\ell}\left((\mu_{\mathcal{M}\cap\mathcal{N}})^X\right)=\frac{1}{2}\left(\frac{1}{4}\right)^{\ell}+\frac{1}{2}\left(\frac{1}{2}\right)^{\ell}.\end{cases}$$

We thus have

$$\begin{aligned}P_n(k,0)=1+\tfrac{1}{\gamma_k(\xi_a)}&\left(\sum_{\ell=1}^{n}\left(-1\right)^{\ell}\tbinom{n}{\ell}2^{-2\ell}\left(\tfrac{2}{11}\left(\tfrac{1}{4}\right)^{k+\ell}+\tfrac{1}{22}\left(\tfrac{1}{2}\right)^{k+\ell}+\tfrac{1}{44}\right)\right.\\&\left.+\tfrac{x}{8}\sum_{\ell=1}^{n}\left(-1\right)^{\ell}\tbinom{n}{\ell}2^{-2\ell}\left(\tfrac{1}{2}\left(\tfrac{1}{4}\right)^{k+\ell-2}+\tfrac{1}{2}\left(\tfrac{1}{2}\right)^{k+\ell-2}\right)\left(\sum_{i=1}^{\ell-1}\tbinom{\ell}{i}^2+1\right)\right).\end{aligned}$$

Observe that

$$\sum_{i=1}^{\ell-1}\binom{\ell}{i}^2+1=\binom{2\ell}{\ell}-1\quad(\text{by Lemma 2.3})\tag{2.9}$$

and

$$\sum_{\ell=1}^{n}\left(-1\right)^{\ell}\binom{n}{\ell}c^{\ell}=\left(1-c\right)^{n}-1\ \ (0<c<1).\tag{2.10}$$

By (2.9) and (2.10), $P_n\left(k,0\right)$ can be written as

$$\begin{aligned}P_n\left(k,0\right)=1+\tfrac{1}{\gamma_k(\xi_a)}&\left(\left(\tfrac{2}{11}-x\right)\left(\tfrac{1}{4}\right)^{k}\left(\left(\tfrac{15}{16}\right)^{n}-1\right)+\left(\tfrac{1}{22}-\tfrac{x}{4}\right)\left(\tfrac{1}{2}\right)^{k}\left(\left(\tfrac{7}{8}\right)^{n}-1\right)+\tfrac{1}{44}\left(\left(\tfrac{3}{4}\right)^{n}-1\right)\right.\\&\left.+x\left(\tfrac{1}{4}\right)^{k}\sum_{\ell=1}^{n}(-1)^{\ell}\tbinom{n}{\ell}\tbinom{2\ell}{\ell}\left(\tfrac{1}{16}\right)^{\ell}+\tfrac{x}{4}\left(\tfrac{1}{2}\right)^{k}\sum_{\ell=1}^{n}(-1)^{\ell}\tbinom{n}{\ell}\tbinom{2\ell}{\ell}\left(\tfrac{1}{8}\right)^{\ell}\right).\end{aligned}\tag{2.11}$$

Now, we should resolve the last two terms of (2.11). To do so, we consider the following weighted shift

$$W_S:=\text{shift}\left(\frac{\left\|Se_{(0,0)}\right\|}{\left\|e_{(0,0)}\right\|},\frac{\left\|S^2e_{(0,0)}\right\|}{\left\|Se_{(0,0)}\right\|},\frac{\left\|S^3e_{(0,0)}\right\|}{\left\|S^2e_{(0,0)}\right\|},\cdots\right),$$

where $S := U_+ \otimes I + I \otimes U_+ \in \mathcal{B}(\ell^2(\mathbb{Z}_+^2))$ (where $U_+ \equiv \text{shift}(1,1,\cdots)$ is the unilateral shift), which is subnormal. By Lambert's Theorem in [21] and Berger's Theorem, we can see that $W_S$ is subnormal and

$$\int_0^4 s^\ell d\mu(s) = \gamma_\ell(W_S) = \left\|S^\ell e_{(0,0)}\right\|^2 = \sum_{k=0}^{\ell} \binom{\ell}{k}^2 = \binom{2\ell}{\ell}, \tag{2.12}$$

where $\mu$ is the Berger measure corresponding to the subnormal weighted shift $W_S$. We thus have

$$\begin{aligned} \sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell}\binom{2\ell}{\ell}\left(\frac{1}{16}\right)^\ell &= \sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell}\left(\int_0^4 s^\ell d\mu(s)\right)\left(\frac{1}{16}\right)^\ell \quad \text{(by (2.12))} \\ &= \int_0^4 \left(\sum_{\ell=0}^{n} \binom{n}{\ell}(-1)^\ell \left(\frac{s}{16}\right)^\ell\right) d\mu(s) - 1 \qquad (2.13) \\ &= \int_0^4 \left(1-\frac{s}{16}\right)^n d\mu(s) - 1 \quad \text{(by (2.10))} \end{aligned}$$

and similarly,

$$\sum_{\ell=1}^{n} (-1)^\ell \binom{n}{\ell}\binom{2\ell}{\ell}\left(\frac{1}{8}\right)^\ell = \int_0^4 \left(1-\frac{s}{8}\right)^n d\mu(s) - 1. \tag{2.14}$$

By (2.13) and (2.14), (2.11) can be written as

$$\begin{aligned} P_n(k,0) = 1 + \tfrac{1}{\gamma_k(\xi_a)} &\Big(\left(\tfrac{2}{11}-x\right)\left(\tfrac{1}{4}\right)^k\left(\left(\tfrac{15}{16}\right)^n - 1\right) + \left(\tfrac{1}{22}-\tfrac{x}{4}\right)\left(\tfrac{1}{2}\right)^k\left(\left(\tfrac{7}{8}\right)^n - 1\right) + \tfrac{1}{44}\left(\left(\tfrac{3}{4}\right)^n - 1\right) \\ &+ x\left(\tfrac{1}{4}\right)^k\left(\textstyle\int_0^4 \left(1-\tfrac{s}{16}\right)^n d\mu(s) - 1\right) + \tfrac{x}{4}\left(\tfrac{1}{2}\right)^k\left(\textstyle\int_0^4 \left(1-\tfrac{s}{8}\right)^n d\mu(s) - 1\right)\Big). \end{aligned}$$

Since $\gamma_k(\xi_a) = \frac{2}{11}\left(\frac{1}{4}\right)^k + \frac{1}{22}\left(\frac{1}{2}\right)^k + \frac{1}{44}$, it follows that

$$\begin{aligned} P_n(k,0) = \tfrac{1}{\gamma_k(\xi_a)} &\Big(\left(\tfrac{2}{11}-x\right)\left(\tfrac{1}{4}\right)^k\left(\tfrac{15}{16}\right)^n + \left(\tfrac{1}{22}-\tfrac{x}{4}\right)\left(\tfrac{1}{2}\right)^k\left(\tfrac{7}{8}\right)^n + \tfrac{1}{44}\left(\tfrac{3}{4}\right)^n \\ &+ x\left(\tfrac{1}{4}\right)^k \textstyle\int_0^4 \left(1-\tfrac{s}{16}\right)^n d\mu(s) + \tfrac{x}{4}\left(\tfrac{1}{2}\right)^k \int_0^4 \left(1-\tfrac{s}{8}\right)^n d\mu(s)\Big), \end{aligned} \tag{2.15}$$

which implies that

$$\begin{aligned} P_n(k,0)\,\gamma_k(\xi_a) &= \tfrac{2}{11}\left(\tfrac{1}{4}\right)^k\left(\tfrac{15}{16}\right)^n + \tfrac{1}{22}\left(\tfrac{1}{2}\right)^k\left(\tfrac{7}{8}\right)^n + \tfrac{1}{44}\left(\tfrac{3}{4}\right)^n \\ &+ x\left(\tfrac{1}{4}\right)^k\left(\textstyle\int_0^4 \left(1-\tfrac{s}{16}\right)^n d\mu(s) - \left(\tfrac{15}{16}\right)^n\right) + \tfrac{x}{4}\left(\tfrac{1}{2}\right)^k\left(\textstyle\int_0^4 \left(1-\tfrac{s}{8}\right)^n d\mu(s) - \left(\tfrac{7}{8}\right)^n\right). \end{aligned} \tag{2.16}$$

Observe that by Lemma 2.3,

$$d\mu(s) = \frac{1}{\pi}\frac{ds}{\sqrt{4s-s^2}}.$$

We thus have

$$\begin{aligned} \frac{\int_0^4 \left(1-\frac{s}{16}\right)^n d\mu(s)}{\left(\frac{15}{16}\right)^n} &= \frac{1}{\pi}\int_0^4 \left(\frac{16-s}{15}\right)^n \frac{ds}{\sqrt{4s-s^2}} \\ &\geq \frac{1}{\pi}\int_{\frac{1}{3}}^{\frac{1}{2}} \left(\frac{16-s}{15}\right)^n \frac{ds}{\sqrt{4s-s^2}} \\ &= \frac{1}{6\pi}\left(\frac{16-s_0}{15}\right)^n \frac{1}{\sqrt{4s_0-s_0^2}} \quad \left(\text{for some } s_0 \text{ with } \tfrac{1}{3} < s_0 < \tfrac{1}{2}\right), \end{aligned}$$

which tends to $\infty$ as $n \to \infty$ and similarly,

$$\frac{\int_0^4 \left(1-\frac{s}{8}\right)^n d\mu(s)}{\left(\frac{7}{8}\right)^n} \to \infty \quad \text{as } n \to \infty.$$

This implies that by (2.16), there exists $n_0 \in \mathbb{Z}_+$ such that

$$P_n(k,0) \geq 0 \quad \text{if } n > n_0. \tag{2.17}$$

Now, suppose

$$\varepsilon_1 := \min_{1\le n\le n_0} \int_0^4 \left(1-\frac{s}{16}\right)^n d\mu(s) = \min_{1\le n\le n_0} \frac{1}{\pi}\int_0^4 \left(1-\frac{s}{16}\right)^n \frac{ds}{\sqrt{4s-s^2}};$$
$$\varepsilon_2 := \min_{1\le n\le n_0} \int_0^4 \left(1-\frac{s}{8}\right)^n d\mu(s) = \min_{1\le n\le n_0} \frac{1}{\pi}\int_0^4 \left(1-\frac{s}{8}\right)^n \frac{ds}{\sqrt{4s-s^2}}$$

and put $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Obviously, $\varepsilon > 0$. Thus, by (2.15),

$$P_n(k,0) \ge \tfrac{1}{\gamma_k(\xi_a)} \left( \left(\tfrac{2}{11} - x + \varepsilon\right)\left(\tfrac{1}{4}\right)^k \left(\tfrac{15}{16}\right)^n + \left(\tfrac{1}{22} - \tfrac{x}{4} + \tfrac{\varepsilon}{4}\right)\left(\tfrac{1}{2}\right)^k \left(\tfrac{7}{8}\right)^n + \tfrac{1}{44}\left(\tfrac{3}{4}\right)^n \right),$$

which implies that

$$P_n(k,0) \ge 0 \quad (1 \le n \le n_0) \quad \text{whenever } 0 < x \le \tfrac{2}{11} + \varepsilon. \tag{2.18}$$

By (2.17) and (2.18), we can conclude that for each $k \ge 1$, $P_n(k,0) \ge 0$ for all $n \in \mathbb{Z}_+$ if $0 < x \le \frac{2}{11} + \varepsilon$ (some $\varepsilon > 0$).

If instead $k = 0$ then the same argument shows that

$$\begin{aligned} P_n(0,0) &= \left(\tfrac{3}{4} - \tfrac{5x}{8}\right) + \left(\tfrac{2}{11} - x\right)\left(\tfrac{15}{16}\right)^n + \left(\tfrac{1}{22} - \tfrac{x}{4}\right)\left(\tfrac{7}{8}\right)^n + \left(\tfrac{1}{44} + \tfrac{5x}{8}\right)\left(\tfrac{3}{4}\right)^n \\ &\quad + x\int_0^4 \left(1 - \tfrac{s}{16}\right)^n d\mu(s) + \tfrac{x}{4}\int_0^4 \left(1 - \tfrac{s}{8}\right)^n d\mu(s), \end{aligned}$$

which also implies that

$$P_n(0,0) \ge 0 \quad (\text{all } n \in \mathbb{Z}_+) \quad \text{whenever } 0 < \epsilon \le \tfrac{2}{11} + \varepsilon.$$

Therefore, we can conclude that $T_1 + T_2$ is subnormal if $0 < x \le \frac{2}{11} + \varepsilon$ (some $\varepsilon > 0$). This proves the theorem. □

**Remark 2.5.** Our 2-variable weighted shift in Theorem 1.1 has 4-atomic Berger measures in the 0-th horizontal and vertical slices of $(T_1, T_2)$. However, if we take 3-atomic Berger measures in the 0-th horizontal and vertical slices of $(T_1, T_2)$, then our extensively numerous trials resisted resolution for finding a gap between the subnormality of $T_1 + T_2$ and the subnormality of $(T_1, T_2)$.

## References


[1] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43**(1976), 597–604.

[2] M.B. Abrahamse, *Commuting subnormal operators*, Ill. J. of Math. **22(1)**(1978), 171–176.

[3] M.B. Abrahamse and R.G. Douglas, *A class of subnormal operators related to multiply connected domains*, Adv. Math. **19**(1976), 106–148.

[4] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory **13(2)**(1985), 203–217.

[5] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103(2)**(1988), 417–423.

[6] J. Bram, *Subnormal operators*, Duke Math. J. **22**(1955), 75–94.

[7] J.B. Conway, *The Theory of Subnormal Operators*, Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

[8] R.E. Curto, Joint hyponormality: A bridge between hyponormality and subnormality, *Proc. Symposia Pure Math.* 51(1990), 69–91.

[9] R.E. Curto, *Quadratically hyponormal weighted shifts*, Integral Equations Operator Theory **13(1)**(1990), 49–66.

[10] R.E. Curto, S.H. Lee and J. Yoon, *k-hyponormality of multivariable weighted shifts*, J. Funct. Anal. **229(2)**(2005), 462–480.

[11] R.E. Curto, S.H. Lee and J. Yoon, *Subnormality of 2-variable weighted shifts with diagonal core*, C. R. Acad. Sci. Paris **351**(2013), 203–207.

[12] R.E. Curto, P. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Contributions to operator theory and its applications (Mesa, AZ, 1987), 1–22, Oper. Theory Adv. Appl. 35, Birkhäuser, Basel, 1988.

[13] R.E. Curto and J. Yoon, *Jointly hyponormal pairs of commuting subnormal operators need not be jointly subnormal*, Tran. Amer. Math. Soc. **358(11)**(2006), 5139–5159.

[14] J.A. Deddens, *Intertwining analytic Toeplitz operators*, Mich. Math. J. **18**(1971), 243–246.

[15] E. Franks, *Polynomially subnormal operator tuples*, J. Operator Theory **31(2)**(1994), 219–228.

[16] R. Gellar and L.J. Wallen, *Subnormal weighted shifts and the Halmos-Bram criterion*, Proc. Japan Acad. **46**(1970), 375–378.

[17] H.W. Gould, *Combinatorial identities*, Morgantown Printing and Binding Co, Morgantown, 1972.

[18] P.R. Halmos, *Normal dilations and extensions of operators*, Summa Bras. Math. **2**(1950), 125–134.

[19] T. Ito, *On the commutative family of subnormal operators*, J. Fac. Sci. Hokkaido Univ. Ser. I **14(3)**(1958), 1–15.

[20] N.P. Jewell and A.R. Lubin, *Commuting weighted shifts and analytic function theory in several variables*, J. Operator Theory **1(2)**(1979), 207–223.

[21] A. Lambert, *Subnormality and weighted shifts*, J. London Math. Soc. **14(3)**(1976), 476–480.

[22] A.R. Lubin, *Weighted shifts and products of subnormal operators*, Indiana Univ. Math. J. **26(5)**(1977), 839–845.

[23] A.R. Lubin, *Extensions of commuting subnormal operators*, (Proc. Conf. Calif. State Univ., Long Beach, Calif., 1977), pp.115-120, Lecture Notes in Math. 693, Springer, Berlin, 1978.

[24] A.R. Lubin, *Weighted shifts and commuting normal extension*, J. Austral Math. Soc. Ser. A **27(1)**(1979), 17–26.

[25] W. Mlak, *Commutants of subnormal operators*, Bull. Acad. Pol. Sci. **19**(1971), 837–842.

[26] R.F. Olin and J.E. Thomson, *Lifting the commutant of a subnormal operator*, Canad. J. Math. **31(1)**(1979), 148–156.

[27] M. Slocinski, *Normal extensions of commuting subnormal operators*, Studia Math. **54**(1975), 259–266.

[28] J. Stampfli, *Which weighted shifts are subnormal?*, Pacific J. Math. **17**(1966), 367–379.

[29] W. Szymanski, *Dilations and subnormality*, Proc. Amer. Math. Soc. **101(2)**(1987), 251–259.

[30] T. Yoshino, *Subnormal operator with a cyclic vector*, Tohoku Math. J. **21**(1969), 47–55.

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