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Fredholm and Weyl Theory

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Forward

The present lecture note is based on a graduate course "Topics in Operator Theory" delivered by the author at the Seoul National University, in the spring semester of 2010.

In 1903, Erik I. Fredholm considered integral equations and then gave a complete description of integral equations via the spectral theory of operators, so-called the Fredholm operators. This theory is named in honor of E.I. Fredholm. Since then, the Fredholm theory has been extensively generalized and applied to many branches of mathematics. In 1909, writing about differential equations, Hermann Weyl noticed something about the essential spectrum of a self adjoint operator on a Hilbert space: when you take it away from the spectrum, you are left with the isolated eigenvalues of finite multiplicity. This was soon generalized to normal operators, and then to more general operators, bounded and unbounded, on Hilbert and on Banach spaces. Nowadays, this is called the Weyl's theorem.

In this lecture note we attempt to set forth some of the recent developments that had taken place in the theory of Fredholm operators. In particular, we focus on the Weyl's theorem.

1 Fredholm theory

If $k(x,y)$ is a continuous complex-valued function on $[a,b]\times[a,b]$, then $K: C[a,b]\to C[a,b]$ defined by

$$(Kf)(x) = \int_a^b k(x,y)f(y)dy$$

is a compact operator. The classical Fredholm integral equations is

$$\lambda f(x) - \int_a^b k(x,y)f(y)dy = g(x), \quad a \le x \le b,$$

where $g \in C[a,b]$, λ is a parameter and f is the unknown. Using I to be the identity operator on $C[a,b]$, we can recast this equation into the form $(\lambda I - K)f = g$. Thus we are naturally led to study of operators of the form $T = \lambda I - K$ on any Banach space X. The Riesz-Schauder theory concentrates attention on these operators of the form $T = \lambda I - K$, $\lambda \neq 0$, K compact. The Fredholm theory concentrates attention on operators called Fredholm operators, whose special cases are the operators $\lambda I - K$. After we develop the "Fredholm Theory", we can obtain the following result. Suppose $k(x,y) \in C[a,b]\times C[a,b]$ (or $L^2[a,b]\times L^2[a,b]$). The equation

$$\lambda f(x) - \int_a^b k(x,y)f(y)dy = g(x), \quad \lambda \neq 0 \tag{1}$$

has a unique solution in $C[a,b]$ for each $g \in C[a,b]$ if and only if the homogeneous equation

$$\lambda f(x) - \int_a^b k(x,y)f(y)dy = 0, \quad \lambda \neq 0 \tag{2}$$

has only the trivial solution in $C[a,b]$. Except for a countable set of λ, which has zero as the only possible limit point, the equation (1) has a unique solution for every $g \in C[a,b]$. For $\lambda \neq 0$, the equation (2) has at most a finite number of linear independent solutions.

In this section we explore the Fredholm theory, in the viewpoint of Robin Harte's approach [Har4], which is basically presented by the incomplete space techniques even though we consider most materials in the setting of Banach spaces.

1.1 Preliminaries

Let X and Y be complex Banach spaces. Let $B(X,Y)$ denote the set of bounded linear operators from X to Y and abbreviate $B(X,X)$ to $B(X)$. If $T \in B(X)$, we write $\rho(T)$ for the resolvent set of T ; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the set of eigenvalues of T. For a set $\mathbf{K}$ in $\mathbb{C}$, we write $\operatorname{cl}\mathbf{K}$, $\partial\mathbf{K}$, $\operatorname{iso}\mathbf{K}$, $\operatorname{acc}\mathbf{K}$, and $\operatorname{int}\mathbf{K}$ for the closure, the boundary points, the isolated points, the accumulation points, and the interior points of $\mathbf{K}$. respectively.

We begin with:

Definition 1.1. Let X be a normed space and let X^* be the dual space of X. If Y is a subset of X, then

$$Y^\perp = \{f \in X^* : \ f(x) = 0 \text{ for all } x \in Y\} = \{f \in X^* : Y \subset f^{-1}(0)\}$$

is called the *annihilator* of Y. If Z is a subset of X^* then

$$^\perp Z = \{x \in X : \ f(x) = 0 \text{ for all } f \in Z\} = \bigcap_{f \in Z} f^{-1}(0)$$

is called the *back annihilator* of Z.

Even if Y and Z are not subspaces, $Y^\perp$ and $^\perp Z$ are closed subspaces.

Lemma 1.2. *Let $Y, Y' \subset X$ and $Z, Z' \subset X^*$. Then*
(a) $Y \subset {}^\perp(Y^\perp), \quad Z \subset ({}^\perp Z)^\perp$;
(b) $Y \subset Y' \Longrightarrow (Y')^\perp \subset Y^\perp; \quad Z \subset Z' \Longrightarrow {}^\perp(Z') \subset {}^\perp Z$;
(c) $({}^\perp(Y^\perp))^\perp = Y^\perp, \quad {}^\perp(({}^\perp Z)^\perp) = {}^\perp Z$;
(d) $\{0\}^\perp = X^*, \quad X^\perp = \{0\}, \quad {}^\perp\{0\} = X$.

Proof. This is straightforward. □

Theorem 1.3. *Let M be a subspace of X. Then*
(a) $X^*/M^\perp \cong M^*$;
(b) *If M is closed then* $(X/M)^* \cong M^\perp$;
(c) ${}^\perp(M^\perp) = \operatorname{cl} M$.

Proof. See [Go, p.25]. □

If $T \in B(X,Y)$, we write T^* for the adjoint of T. If X and Y are Hilbert spaces and $T \in B(X,Y)$, T^* denotes the (Hilbert space) adjoint of T.

Theorem 1.4. *If $T \in B(X,Y)$, then*
(a) $T(X)^\perp = (T^*)^{-1}(0)$;
(b) $\operatorname{cl} T(X) = {}^\perp(T^{*-1}(0))$;
(c) $T^{-1}(0) \subset {}^\perp T^*(Y^*)$;
(d) $\operatorname{cl} T^*(Y^*) \subset T^{-1}(0)^\perp$.

Proof. See [Go, p.59]. □

Theorem 1.5. *Let X and Y be Banach spaces and $T \in B(X,Y)$. Then the following are equivalent:*
(a) *T has closed range;*
(b) *T^* has closed range;*
(c) $T^*(Y^*) = T^{-1}(0)^\perp$;
(d) $T(X) = {}^\perp(T^{*-1}(0))$.

Proof. (a) $\Leftrightarrow$ (d): From Theorem 1.4 (b).

(a) $\Rightarrow$ (c): Observe that the operator $T^\wedge : X/T^{-1}(0) \to TX$ defined by

$$x + T^{-1}(0) \mapsto Tx$$

is invertible by the open mapping theorem. Thus we have

$$T^{-1}(0)^\perp \cong \big(X/T^{-1}(0)\big)^* \cong (TX)^* \cong T^*(Y^*).$$

(c) $\Rightarrow$ (b): This is clear because $T^{-1}(0)^\perp$ is closed.

(b) $\Rightarrow$ (a): Observe that if $T_1 : X \to \mathrm{cl}\,(TX)$ then $T_1^* : (\mathrm{cl}\,TX)^* \to X^*$ is one-one. Since $T^*(Y^*) = \mathrm{ran}\,T_1^*$, T_1^* has closed range. Therefore T_1^* is bounded below, so that T_1 is open; therefore TX is closed. □

We introduce:

Definition 1.6. If $T \in B(X, Y)$, write

$$\alpha(T) := \dim T^{-1}(0) \quad \text{and} \quad \beta(T) := \dim Y/\mathrm{cl}\,(TX).$$

We then have:

Theorem 1.7. *If $T \in B(X, Y)$ has a closed range then*

$$\alpha(T^*) = \beta(T) \quad \textit{and} \quad \alpha(T) = \beta(T^*).$$

Proof. This follows form the following observation:

$$T^{*-1}(0) = (TX)^\perp \cong (Y/TX)^* \cong Y/T(X)$$

and

$$T^{-1}(0) \cong (T^{-1}(0))^* \cong X^*/T^{-1}(0)^\perp \cong X^*/T^*(Y^*).$$

□

1.2 Definitions and examples

In the sequel we assume that X and Y are complex Banach spaces and that H and K are complex Hilbert spaces.

Definition 1.8. An operator $T \in B(X, Y)$ is called a *Fredholm operator* if $T(X)$ is closed, $\alpha(T) < \infty$ and $\beta(T) < \infty$. In this case we define the *index* of T by the equality

$$\mathrm{index}\,(T) := \alpha(T) - \beta(T).$$

In the below we shall see that the condition " $T(X)$ is closed " is automatically fulfilled if $\beta(T) < \infty$.

Example 1.9. If X and Y are both finite dimensional then any operator $T \in B(X,Y)$ is Fredholm and

$$\operatorname{index}(T) = \dim X - \dim Y :$$

indeed recall the "rank theorem"

$$\dim X = \dim T^{-1}(0) + \dim T(X),$$

which implies

$$\begin{aligned} \operatorname{index}(T) &= \dim T^{-1}(0) - \dim Y/T(X) \\ &= \dim X - \dim T(X) - (\dim Y - \dim T(X)) \\ &= \dim X - \dim Y. \end{aligned}$$

Thus in particular, if $T \in B(X)$ with $\dim X < \infty$ then T is Fredholm of index zero.

Example 1.10. If $K \in B(X)$ is a compact operator then $T = I - K$ is Fredholm of index 0. This follows from the spectral theory of compact operators (cf. [Con1]).

Example 1.11. If U is the unilateral shift operator on ℓ^2, i.e.,

$$U(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots) \quad \text{for each } (x_1, x_2, x_3, \cdots) \in \ell^2,$$

then a straightforward calculation shows that

$$\operatorname{index} U = -1 \quad \text{and} \quad \operatorname{index} U^* = -1.$$

With U and U^*, we can build a Fredholm operator whose index is equal to an arbitrary prescribed integer. Indeed if

$$T = \begin{pmatrix} U^p & 0 \\ 0 & U^{*q} \end{pmatrix} : \ell^2 \oplus \ell^2 \ \to \ell^2 \oplus \ell^2,$$

then T is Fredholm, $\alpha(T) = q$, $\beta(T) = p$, and hence $\operatorname{index} T = q - p$.

1.3 Operators with closed ranges

If $T \in B(X,Y)$, write

$$\operatorname{dist}\big(x,\ T^{-1}(0)\big) = \inf\big\{||x-y|| :\ Ty = 0\big\} \quad \text{for each } x \in X. \tag{3}$$

If $T \in B(X,Y)$, we define

$$\gamma(T) = \inf\left\{c > 0 :\ ||Tx|| \ge c \operatorname{dist}(x,\ T^{-1}(0)) \ \text{ for each } x \in X\right\} :$$

we call $\gamma(T)$ the *reduced minimum modulus* of T.

We then have:

Theorem 1.12. *If $T \in B(X,Y)$, then*

$$T(X) \text{ is closed} \iff \gamma(T) > 0.$$

Proof. Consider $\widehat{X} = X/T^{-1}(0)$ and thus $\widehat{X}$ is a Banach space with norm

$$||\widehat{x}|| = \operatorname{dist}(x,\ T^{-1}(0)).$$

Define $\widehat{T} : \widehat{X} \to Y$ by $\widehat{T}\widehat{x} = Tx$. Then $\widehat{T}$ is one-one and $\widehat{T}(\widehat{X}) = T(X)$.

($\Rightarrow$) Suppose TX is closed and thus $\widehat{T} : \widehat{X} \to TX$ is bijective. By the open mapping theorem, $\widehat{T}$ is invertible with inverse $\widehat{T}^{-1}$. Thus

$$||Tx|| = ||\widehat{T}\widehat{x}|| \geq \frac{1}{||\widehat{T}^{-1}||}||\widehat{x}|| = \frac{1}{||\widehat{T}^{-1}||}\operatorname{dist}(x,\ T^{-1}(0)),$$

which implies that $\gamma(T) = \frac{1}{||\widehat{T}^{-1}||} > 0$.

($\Leftarrow$) Suppose $\gamma(T) > 0$. Let $Tx_n \to y$. Then by the assumption $||Tx_n|| \geq \gamma(T)\,||\widehat{x_n}||$, and hence, $||Tx_n - Tx_m|| \geq \gamma(T)\,||\widehat{x_n} - \widehat{x_m}||$, which implies that $(\widehat{x_n})$ is a Cauchy sequence in $\widehat{X}$. Thus $\widehat{x_n} \to \widehat{x} \in \widehat{X}$ because $\widehat{X}$ is complete. Hence $Tx_n = \widehat{T}\widehat{x_n} \to \widehat{T}\widehat{x} = Tx$; therefore $y = Tx$. □

Theorem 1.13. *If there is a closed subspace Y_0 of Y for which $T(X) \oplus Y_0$ is closed then T has closed range.*

Proof. Define $T_0 : X \times Y_0 \to Y$ by

$$T_0(x, y_0) = Tx + y_0.$$

The space $X \times Y_0$ is a Banach space with the norm defined by

$$||(x, y_0)|| = ||x|| + ||y_0||.$$

Clearly, T_0 is a bounded linear operator and $T_0(X \times Y_0) = T(X) \oplus Y_0$, which is closed by hypothesis. Moreover, $T_0^{-1}(0) = T^{-1}(0) \times \{0\}$. Theorem 1.12 asserts that there exists a $c > 0$ such that

$$||Tx|| = ||T_0(x,0)|| \geq c \operatorname{dist}\Big((x,0),\ T_0^{-1}(0)\Big) = c \operatorname{dist}\big(x, T^{-1}(0)\big),$$

which implies that $T(X)$ is closed. □

Corollary 1.14. *If $T \in B(X,Y)$ then*

$$T(X) \text{ is complemented} \implies T(X) \text{ is closed.}$$

In particular, if $\beta(T) < \infty$ then $T(X)$ is closed.

Proof. If $T(X)$ is complemented then we can find a closed subspace Y_0 for which $T(X) \oplus Y_0 = Y$. Theorem 1.13 says that $T(X)$ is closed. □

To see the importance of Corollary 1.14, note that for a subspace M of a Banach space Y,

$$Y = M \oplus Y_0 \text{ does not imply that } M \text{ is closed.}$$

Take a non-continuous linear functional f on Y and put $M = \ker f$. Then there exists a one-dimensional subspace Y_0 such that $Y = M \oplus Y_0$ (recall that $Y/f^{-1}(0)$ is one-dimensional). But $M = f^{-1}(0)$ cannot be closed because f is continuous if and only if $f^{-1}(0)$ is closed.

Consequently, we don't guarantee that

$$\dim(Y/M) < \infty \implies M \text{ is closed.} \tag{4}$$

However Corollary 1.14 asserts that if M is a range of a bounded linear operator then (4) is true. Of course, it is true that

$$M \text{ is closed}, \quad \dim(Y/M) < \infty \implies M \text{ is complemented.}$$

Theorem 1.15. *Let $T \in B(X,Y)$. If T maps bounded closed sets onto closed sets then T has closed range.*

Proof. Suppose $T(X)$ is not closed. Then by Theorem 1.12 there exists a sequence $\{x_n\}$ such that

$$Tx_n \to 0 \quad \text{and} \quad \text{dis}\big(x_n,\ T^{-1}(0)\big) = 1.$$

For each n choose $z_n \in T^{-1}(0)$ such that $||x_n - z_n|| < 2$. Let $V := \text{cl}\,\{x_n - z_n : \ n = 1, 2, \ldots\}$. Since V is closed and bounded in X, $T(V)$ is closed in Y by assumption. Note that $Tx_n = T(x_n - z_n) \in T(V)$. So, $0 \in T(V)$ ($Tx_n \to 0 \in T(V)$) and thus there exists $u \in V \cap T^{-1}(0)$. From the definition of V it follows that

$$||u - (x_{n_0} - z_{n_0})|| < \frac{1}{2} \quad \text{for some } n_0,$$

which implies that

$$\text{dis}\big(x_{n_0},\ T^{-1}(0)\big) < \frac{1}{2}.$$

This contradicts the fact that dist $\big(x_n,\ T^{-1}(0)\big) = 1$ for all n. Therefore $T(X)$ is closed. □

Theorem 1.16. *Let $K \in B(X)$. If K is compact then $T = I - K$ has closed range.*

Proof. Let V be a closed bounded set in X and let

$$y = \lim_{n\to\infty} (I - K)x_n, \quad \text{where } x_n \in V. \tag{5}$$

We have to prove that $y = (I - K)x_0$ for some $x_0 \in V$. Since V is bounded and K is compact the sequence $\{Kx_n\}$ has a convergent subsequence $\{Kx_{n_i}\}$. By (5), we see that

$$x_0 := \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} \big((I - K)x_{n_i} + Kx_{n_i}\big) \text{ exists.}$$

But then $y = (I - K)x_0 \in (I - K)V$; thus $(I - K)V$ is closed. Therefore by Theorem 1.15, $I - K$ has closed range. □

Corollary 1.17. *If $K \in B(X)$ is compact then $I - K$ is Fredholm.*

Proof. From Theorem 1.16 we see that $(I-K)(X)$ is closed. Since $x \in (I-K)^{-1}(0)$ implies $x = Kx$, the identity operator acts as a compact operator on $(I-K)^{-1}(0)$; thus $\alpha(I-K) < \infty$. To prove that $\beta(I-K) < \infty$, recall that $K^* : X^* \to X^*$ is also compact. Since $(I-K)(X)$ is closed it follows from Theorem 1.7 that

$$\beta(I-K) = \alpha(I-K^*) < \infty.$$

□

1.4 The Product of Fredholm operators

Let $T \in B(X,Y)$. Suppose $T^{-1}(0)$ and $T(X)$ are complemented by subspaces X_0 and Y_0; i.e.,

$$X = T^{-1}(0) \oplus X_0 \quad \text{and} \quad Y = T(X) \oplus Y_0.$$

Define $\widetilde{T} : X_0 \times Y_0 \to Y$ by

$$\widetilde{T}(x_0, y_0) = Tx_0 + y_0.$$

The space $X_0 \times Y_0$ is a Banach space with the norm defined by $||(x,y)|| = ||x|| + ||y||$ and $\widetilde{T}$ is a bijective bounded linear operator. We call $\widetilde{T}$ the *bijection associated with* T (cf. [GGK]). If T is Fredholm then such a bijection always exists and Y_0 is finite dimensional. If we identify $X_0 \cong X_0 \times \{0\}$ then the operator $T_0 : X_0 \to Y$ defined by

$$T_0 x = Tx$$

is a common restriction of T and $\widetilde{T}$ to X_0 $(= X_0 \times \{0\})$.

Note that
(a) $\frac{1}{||\widetilde{T}^{-1}||} = \gamma(T)$;
(b) If $\widehat{T} : X/T^{-1}(0) \to TX$ then $\widehat{T} \cong \widetilde{T}$.

Lemma 1.18. *Let $T \in B(X,Y)$ and $M \subset X$ with* $\operatorname{codim} M = n < \infty$. *Then*

$$T \text{ is Fredholm} \iff T_0 := T|_M \text{ is Fredholm},$$

in which case, $\operatorname{index} T = \operatorname{index} T_0 + n$.

Proof. It suffices to prove the lemma for $n = 1$. Put $X := M \oplus \operatorname{span}\{x_1\}$. We consider two cases:

(Case 1) Assume $Tx_1 \notin T_0(M)$. Then $T(X) = T_0(M) \oplus \operatorname{span}\{Tx_1\}$ and $T^{-1}(0) = T_0^{-1}(0)$. Hence

$$\beta(T_0) = \beta(T) + 1 \quad \text{and} \quad \alpha(T_0) = \alpha(T). \tag{6}$$

(Case 2) Assume $Tx_1 \in T_0(M)$. Then $T(X) = T_0(M)$, and hence there exists $u \in M$ such that $Tx_1 = T_0 u$. Thus $T^{-1}(0) = T_0^{-1}(0) \oplus \operatorname{span}\{x_1 - u\}$. Thus

$$\beta(T_0) = \beta(T) \quad \text{and} \quad \alpha(T_0) = \alpha(T) - 1. \tag{7}$$

From (6) and (7) we have the result. □

Theorem 1.19. (Index Product Theorem) *If $T \in B(X,Y)$ and $S \in B(Y,Z)$ then*

$$S \text{ and } T \text{ are Fredholm} \implies ST \text{ is Fredholm with}$$
$$\operatorname{index}(ST) = \operatorname{index} S + \operatorname{index} T.$$

Proof. Let $\widetilde{T}$ be a bijection associated with T, X_0, and Y_0: i.e., $X = T^{-1}(0) \oplus X_0$ and $Y = T(X) \oplus Y_0$. Suppose $T_0 := T|_{X_0}$. Since $\widetilde{T}$ is invertible, $S\widetilde{T}$ is invertible and $\operatorname{index}(S\widetilde{T}) = \operatorname{index} S$. By identifying X_0 and $X_0 \times \{0\}$, we see that ST_0 is a common restriction of $S\widetilde{T}$ and ST to X_0. By Lemma 1.18, ST is Fredholm and

$$\begin{aligned}
\operatorname{index}(ST) &= \operatorname{index}(ST_0) + \dim X/X_0 \\
&= \operatorname{index}(S\widetilde{T}) - \dim\Big(X_0 \times Y_0 / X_0 \times \{0\}\Big) + \alpha(T) \\
&= \operatorname{index} S - \dim Y_0 + \alpha(T) \\
&= \operatorname{index} S - \beta(T) + \alpha(T) \\
&= \operatorname{index} S + \operatorname{index} T.
\end{aligned}$$

□

The converse of Theorem 1.19 is not true in general. To see this, consider the following operators on ℓ^2:

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots)$$
$$S(x_1, x_2, x_3, \ldots) = (x_2, x_4, x_6, \ldots).$$

Then T ad S are not Fredholm, but $ST = I$. However, if $ST = TS$ then we have

$$ST \text{ is Fredholm} \implies S \text{ and } T \text{ are both Fredholm}$$

because $T^{-1}(0) \subset (ST)^{-1}(0)$ and $(ST)(X) = TS(X) \subset T(X)$.

Remark 1.20. For a time being, a Fredholm operator of index 0 will be called a Weyl operator. Then we have the following question: *Is there implication that if $ST = TS$ then*

$$S, T \text{ are Weyl} \iff ST \text{ is Weyl}?$$

Here is an answer. The forward implication comes from the index product theorem without commutativity condition. However the backward implication may fail even with commutativity condition. To see this, let

$$T = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} I & 0 \\ 0 & U^* \end{pmatrix},$$

where U is the unilateral shift on ℓ_2. Evidently,

$$\begin{aligned}
\operatorname{index}(ST) &= \operatorname{index}\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \\
&= \operatorname{index} U + \operatorname{index} U^* \\
&= 0,
\end{aligned}$$

but S and T are not Weyl.

1.5 Perturbation theorems

We begin with:

Theorem 1.21. *Suppose $T \in B(X,Y)$ is Fredholm. If $S \in B(X,Y)$ with $||S|| < \gamma(T)$ then $T+S$ is Fredholm and*
(a) $\alpha(T+S) \leq \alpha(T)$;
(b) $\beta(T+S) \leq \beta(T)$;
(c) $\operatorname{index}(T+S) = \operatorname{index} T$.

Proof. Let $X = T^{-1}(0) \oplus X_0$ and $Y = T(X) \oplus Y_0$. Suppose $\widetilde{T}$ is the bijection with T, X_0 and Y_0. Put $R = T+S$ and define

$$\widetilde{R} : X_0 \times Y_0 \to Y \quad \text{by } \widetilde{R}(x_0, y_0) = Rx_0 + y_0.$$

By definition, $\widetilde{T}(x_0, y_0) = Tx_0 + y_0$. Since $\widetilde{T}$ is invertible and

$$||\widetilde{T} - \widetilde{R}|| \leq ||T - R|| = ||S|| < \gamma(T) = \frac{1}{||\widetilde{T}^{-1}||},$$

we have that $\widetilde{R}$ is also invertible. Note that $R_0 : X_0 \to Y$ defined by

$$R_0 x = Rx$$

is a common restriction of R and $\widetilde{R}$ to X_0. By Lemma 1.18, R is Fredholm and

$$\begin{aligned} \operatorname{index} R &= \operatorname{index} R_0 + \alpha(T) \\ &= \operatorname{index} \widetilde{R} - \beta(T) + \alpha(T) \\ &= \operatorname{index} T \end{aligned}$$

which proves (c). The invertibility of $\widetilde{R}$ implies that $X_0 \cap R^{-1}(0) = \{0\}$. Thus we have

$$\alpha(R) \leq \dim X / X_0 = \alpha(T),$$

which proves (a). Note that (b) is an immediate consequence of (a) and (c). □

The first part of Theorem 1.21 asserts that the set of Fredholm operators forms an open set.

Theorem 1.22. *Let $T, K \in B(X,Y)$. Then*

$$T \textit{ is Fredholm, } K \textit{ is compact} \implies T+K \textit{ is Fredholm with } \operatorname{index}(T+K) = \operatorname{index} T.$$

Proof. Let $X = T^{-1}(0) \oplus X_0$ and $Y = T(X) \oplus Y_0$. Define $\widetilde{T},\ \widetilde{K} : X_0 \times Y_0 \to Y$ by

$$\widetilde{T}(x_0, y_0) = Tx_0 + y_0, \quad \widetilde{K}(x_0, y_0) = Kx_0 + y_0.$$

Therefore $\widetilde{K}$ is compact since K is compact and $\dim Y_0 < \infty$. From $(\widetilde{T} + \widetilde{K})(x_0, 0) = (T+K)x_0$ and Lemma 1.18 it follows that

$$T + K \text{ is Fredholm} \iff \widetilde{T} + \widetilde{K} \text{ is Fredholm}.$$

But $\widetilde{T}$ is invertible. So
$$\widetilde{T}+\widetilde{K}=\widetilde{T}\left(I+\widetilde{T}^{-1}\widetilde{K}\right).$$
Observe that $\widetilde{T}^{-1}\widetilde{K}$ is compact. Thus by Corollary 1.17, $I+\widetilde{T}^{-1}\widetilde{K}$ is Fredholm. Hence $T+K$ is Fredholm.

To prove the statement about the index consider the integer valued function $F(\lambda) := \operatorname{index}(T+\lambda K)$. Applying Theorem 1.21 to $T+\lambda K$ in place of T shows that f is continuous on $[0,1]$. Consequently, f is constant. In particular,

$$\operatorname{index} T = f(0) = f(1) = \operatorname{index}(T+K).$$

□

Corollary 1.23. *If $K \in B(X)$ then*

$$K \text{ is compact} \implies I-K \text{ is Fredholm with } \operatorname{index}(I-K)=0.$$

Proof. Apply the preceding theorem with $T=I$ and note that $\operatorname{index} I = 0$. □

1.6 The Calkin algebra

We begin with:

Theorem 1.24. *If $T \in B(X,Y)$ then*

$$T \text{ is Fredholm} \iff \exists\, S \in B(Y,X) \text{ such that } I-ST \text{ and } I-TS \text{ are finite rank.}$$

Proof. ($\Rightarrow$) Suppose T Fredholm and let

$$X = T^{-1}(0) \oplus X_0 \quad \text{and} \quad Y = T(X) \oplus Y_0.$$

Define $T_0 := T|_{X_0}$. Since T_0 is one-one and $T_0(X_0) = T(X)$ is closed

$$T_0^{-1} : T(X) \to X_0 \text{ is invertible.}$$

Put $S := T_0^{-1}Q$, where $Q : Y \to T(X)$ is a projection. Evidently, $S(Y) = X_0$ and $S^{-1}(0) = Y_0$. Furthermore,
$$I-ST \text{ is the projection of } X \text{ onto } T^{-1}(0)$$
$$I-TS \text{ is the projection of } Y \text{ onto } Y_0.$$
In particular, $I-ST$ and $I-TS$ are of finite rank.

($\Leftarrow$) Assume $ST = I-K_1$ and $TS = I-K_2$, where K_1, K_2 are finite rank. Since

$$T^{-1}(0) \subset (ST)^{-1}(0) \quad \text{and} \quad (TS)X \subset T(X),$$

we have
$$\alpha(T) \le \alpha(ST) = \alpha(I-K_1) < \infty$$
$$\beta(T) \le \beta(TS) = \beta(I-K_2) < \infty,$$
which implies that T is Fredholm. □

Theorem 1.24 remains true if the statement "$I-ST$ and $I-TS$ are of finite rank" is replaced by "$I-ST$ and $I-TS$ are compact operators." In other words,

$$T \text{ is Fredholm} \iff T \text{ is invertible modulo compact operators.}$$

Let $K(X)$ be the space of all compact operators on X. Note that $K(X)$ is a closed ideal of $B(X)$. On the quotient space $B(X)/K(X)$, define the product

$$[S][T] = [ST], \quad \text{where } [S] \text{ is the coset } S + K(X).$$

The space $B(X)/K(X)$ with this additional operation is an algebra, which is called the *Calkin algebra*, with identity $[I]$.

Theorem 1.25. (Atkinson's Theorem) *Let $T \in B(X)$. Then*

$$T \textit{ is Fredholm} \iff [T] \textit{ is invertible in } B(X)/K(X).$$

Proof. ($\Rightarrow$) If T is Fredholm then

$$\exists\ S \in B(X) \text{ such that } ST - I \text{ and } TS - I \text{ are compact.}$$

Hence $[S][T] = [T][S] = [I]$, so that $[S]$ is the inverse of $[T]$ in the Calkin algebra.

($\Leftarrow$) If $[S][T] = [T][S] = [I]$ then

$$ST = I - K_1 \quad \text{and} \quad TS = I - K_2,$$

where K_1, K_2 are compact operators. Thus T is Fredholm. $\square$

Let $T \in B(X)$. The *essential spectrum* $\sigma_e(T)$ of T is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredholm}\}$$

We thus have

$$\sigma_e(T) = \sigma_{B(X)/K(X)}(T + K(X)).$$

Evidently $\sigma_e(T)$ is compact. If $\dim X = \infty$, then

$$\sigma_e(T) \neq \emptyset \quad (\text{because } B(X)/K(X) \neq \emptyset).$$

In particular, Theorem 1.22 implies that

$$\sigma_e(T) = \sigma_e(T + K) \quad \text{for every } K \in K(X).$$

Theorem 1.26. *If $T \in B(X, Y)$ then*

$$T \textit{ is Weyl} \iff \exists \textit{ a finite rank operator } F \textit{ such that } T + F \textit{ is invertible.}$$

Proof. (⇒) Let T be Weyl and put

$$X = T^{-1}(0) \oplus X_0 \quad \text{and} \quad Y = T(X) \oplus Y_0.$$

Since $\operatorname{index} T = 0$, it follows that

$$\dim T^{-1}(0) = \dim Y_0.$$

Thus there exists an invertible operator $F_0 : T^{-1}(0) \to Y_0$. Define $F := F_0(I - P)$, where P is the projection of X onto X_0. Obviously, $T + F$ is invertible.

(⇐) Assume $S = T + F$ is invertible, where F is of finite rank. By Theorem 1.22, T is Fredholm and $\operatorname{index} T = \operatorname{index} S = 0$. □

The *Weyl spectrum*, $\omega(T)$, of $T \in B(X)$ is defined by

$$\omega(T) = \Big\{ \lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl} \Big\}$$

Evidently, $\omega(T)$ is compact and in particular,

$$\omega(T) = \bigcap_{K \text{ compact}} \sigma(T + K).$$

Definition 1.27. An operator $T \in B(X, Y)$ is said to be *regular* if there is $T' \in B(Y, X)$ for which

$$T = TT'T; \tag{8}$$

then T' is called a *generalized inverse* of T. We can always arrange

$$T' = T'TT' : \tag{9}$$

indeed if (8) holds then

$$T'' = T'TT' \implies TT''T \quad \text{and} \quad T'' = T''TT''.$$

If T' satisfies (8) and (9) then it will be called a *generalized inverse of T in the strong sense.* Also $T \in B(X, Y)$ is said to be *decomposably regular* if there exists $T' \in B(Y, X)$ such that

$$T = TT'T \quad \text{and} \quad T' \text{ is invertible.}$$

The operator $S := T_0^{-1}Q$, which was defined in the proof of Theorem 1.24, is a generalized inverse of in the strong sense. Thus we have

$$T \text{ is Fredholm} \iff I - T'T \text{ and } I - TT' \text{ are finite rank.}$$

Generalized inverses are useful in solving linear equations. Suppose T' is a generalized inverse of T. If $Tx = y$ is solvable for a given $y \in Y$, then $T'y$ is a solution (not necessary the only one). Indeed,

$$\begin{aligned} Tx = y \text{ is solvable} &\implies \exists\, x_0 \text{ such that } Tx_0 = y \\ &\implies TT'y = TT'Tx_0 = Tx_0 = y. \end{aligned}$$

Theorem 1.28. *If $T \in B(X,Y)$, then*

$$T \text{ is regular} \iff T^{-1}(0) \text{ and } T(X) \text{ are complemented.}$$

Proof. ($\Leftarrow$) If $X = X_0 \oplus T^{-1}(0)$ and $Y = Y_0 \oplus T(X)$ then $T' : Y \to X$ defined by

$$T'(Tx_0 + y_0) = x_0, \quad \text{where } x_0 \in X_0 \text{ and } y_0 \in Y_0$$

is a generalized inverse of T because for $x_0 \in X_0$ and $z \in T^{-1}(0)$,

$$TT'T(x_0 + z) = TT'(Tx_0) = Tx_0 = T(x_0 + z).$$

($\Rightarrow$) Assume T' is a generalized inverse of T: $TT'T = T$. Obviously, TT' and $T'T$ are both projections. Also,

$$T(X) = TT'T(X) \subset TT'(X) \subset T(X);$$
$$T^{-1}(0) \subset (T'T)^{-1}(0) \subset (TT'T)^{-1}(0) = T^{-1}(0),$$

which gives

$$TT'(X) = T(X) \quad \text{and} \quad (T'T)^{-1}(0) = T^{-1}(0),$$

which implies that $T^{-1}(0)$ and $T(X)$ are complemented. □

Corollary 1.29. *If $T \in B(X,Y)$ then*

$$T \text{ is Fredholm} \implies T \text{ is regular.}$$

Proof. Immediate from Theorem 1.28. □

Theorem 1.30. *If $T \in B(X,Y)$ is Fredholm with $T = TT'T$, then T' is also Fredholm with*

$$\text{index}\,(T') = -\text{index}\,(T).$$

Proof. We first claim that

$$ST \text{ is Fredholm} \implies (S \text{ Fredholm} \iff T \text{ Fredholm}) : \tag{10}$$

indeed,

$$ST \text{ is Fredholm} \implies I - (ST)'(ST) \in K_0 \quad \text{and} \quad I - (ST)(ST)' \in K_0,$$

which implies

$$T \text{ is Fredholm} \iff I - T(ST)'S \in K_0 \iff S \text{ is Fredholm}.$$

Thus by (10), T' is Fredholm and by the index product theorem,

$$\text{index}\,(T) = \text{index}\,(TT'T) = \text{index}\,(T) + \text{index}\,(T') + \text{index}\,(T).$$

□

Theorem 1.31. *If $T \in B(X,Y)$ is Fredholm with generalized inverse $T' \in B(Y,X)$ in the strong sense then*

$$\text{index}\,(T) = \dim T^{-1}(0) - \dim\,(T')^{-1}(0).$$

Proof. Observe that

$$(T')^{-1}(0) = (TT')^{-1}(0) \cong X/TT'(X) \cong X/T(X),$$

which gives that $\beta(T) = \alpha(T')$. □

Theorem 1.32. *If $T \in B(X,Y)$ is Fredholm with generalized inverse $T' \in B(Y,X)$, then*

$$\text{index}\,(T) = \text{trace}\,(TT' - T'T).$$

Proof. If $T = TT'T$ is Fredholm then

$$I - T'T \text{ and } I - TT' \text{ are both finite rank.}$$

Observe that

$$\begin{aligned} &\dim\,(I - T'T)(X) = \dim\,(T'T)^{-1}(0) = \dim T^{-1}(0) = \alpha(T); \\ &\dim\,(I - TT')(Y) = \dim\,(TT')^{-1}(0) = \dim X/TT'(Y) = \dim X/T(X) = \beta(T). \end{aligned}$$

Thus we have

$$\begin{aligned} \text{trace}\,(TT' - T'T) &= \text{trace}\,\big((I - T'T) - (I - TT')\big) \\ &= \text{trace}\,(I - T'T) - \text{trace}\,(I - TT') \\ &= \text{rank}\,(I - T'T)(X) - \dim\,(I - TT')(X) \\ &= \alpha(T) - \beta(T) \\ &= \text{index}\,(T). \end{aligned}$$

□

1.7 The punctured neighborhood theorem

If $T \in B(X,Y)$ then
(a) T is said to be *upper semi-Fredholm* if $T(X)$ is closed and $\alpha(T) < \infty$;
(b) T is said to be *lower semi-Fredholm* if $T(X)$ is closed and $\beta(T) < \infty$.
(c) T is said to be *semi-Fredholm* if it is upper or lower semi-Fredholm.

Theorem 1.21 remains true for semi-Fredholm operators. Thus we have:

Lemma 1.33. *Suppose $T \in B(X,Y)$ is semi-Fredholm. If $||S|| < \gamma(T)$ then*
(i) *$T + S$ has a closed range;*
(ii) *$\alpha(T + S) \le \alpha(T)$, $\beta(T + S) \le \beta(T)$;*
(iii) *$\text{index}\,(T + S) = \text{index}\,T$.*

Proof. This follows from a slight change of the argument for Theorem 1.21. □

We are ready for the punctured neighborhood theorem; this proof is due to Harte and Lee [HaL1].

Theorem 1.34. (Punctured Neighborhood Theorem) *If $T \in B(X)$ is semi-Fredholm then there exists $\rho > 0$ such that $\alpha(T-\lambda I)$ and $\beta(T-\lambda I)$ are constant in the annulus $0 < |\lambda| < \rho$.*

Proof. Assume that T is upper semi-Fredholm and $\alpha(T) < \infty$. First we argue

$$(T-\lambda I)^{-1}(0) \subset \bigcap_{n=1}^{\infty} T^n(X) =: T^{\infty}(X). \tag{11}$$

Indeed,

$$\begin{aligned} x \in (T-\lambda I)^{-1}(0) &\implies Tx = \lambda x, \text{ and hence } x \in T(X) \\ &\implies \text{Note that } \lambda x = Tx \in T(TX) = T^2(X) \\ &\implies \text{By induction}, x \in T^n(X) \text{ for all } n. \end{aligned}$$

Next we claim that

$$T^{\infty}(X) \text{ is closed:}$$

indeed, since T^n is upper semi-Fredholm for all n, $T^n(X)$ is closed and hence $T^{\infty}(X)$ is closed.

If S commutes with T, so that also $S(T^{\infty}(X)) \subset T^{\infty}(X)$, we shall write $\widetilde{S} : T^{\infty}(X) \to T^{\infty}(X)$. We claim that

$$\widetilde{T} : T^{\infty}(X) \to T^{\infty}(X) \text{ is onto.} \tag{12}$$

To see this, let $y \in T^{\infty}(X)$ and thus

$$\exists\ x_n \in T^n(X) \text{ such that } Tx_n = y \quad (n = 1, 2, \ldots).$$

Since $T^{-1}(0)$ is finite dimensional and $T^n(X) \supset T^{n+1}(X)$,

$$\exists n_0 \in \mathbb{N} \text{ such that } T^{-1}(0) \cap T^{n_0}(X) = T^{-1}(0) \cap T^n(X) \quad \text{for } n \geq n_0.$$

From the fact that $T^n(X) \subset T^{n_0}(X)$, we have

$$x_n - x_{n_0} \in T^{-1}(0) \cap T^{n_0}(X) = T^{-1}(0) \cap T^n(X) \subset T^n(X).$$

Hence

$$x_{n_0} \in \bigcap_{n \geq n_0} T^n(X) = T^{\infty}(X) \quad \text{and} \quad Tx_{n_0} = y,$$

which says that $\widetilde{T}$ is onto. This proves (12). Now observe

$$\dim (T-\lambda I)^{-1}(0) = \dim \widetilde{T-\lambda I}^{-1}(0) = \operatorname{index} \widetilde{T-\lambda I} = \operatorname{index} \widetilde{T} : \tag{13}$$

the first equality comes from (11), the second equality follows from the fact that $\beta(\widetilde{T-\lambda I}) \leq \beta(\widetilde{T}) = 0$ by Lemma 1.33, and the third equality follows the observation that $\widetilde{T}$ is semi-Fredholm. Since the right-hand side of (13) is independent of λ, $\alpha(T-\lambda I)$ is constant and hence also is $\beta(T-\lambda I)$.

If instead $\beta(T) < \infty$, apply the above argument with T^*. $\square$

Theorem 1.35. *Define*

$$U := \left\{ \lambda \in \mathbb{C} : \ T - \lambda I \ \textit{is semi-Fredholm} \right\}.$$

Then we have:
(a) U *is an open set;*
(b) *If* C *is a component of* U *then on* C*, with the possible exception of isolated points,*

$$\alpha(T - \lambda I) \ \textit{and} \ \beta(T - \lambda I) \ \textit{have constant values} \ n_1 \ \textit{and} \ n_2, \ \textit{respectively.}$$

At the isolated points,

$$\alpha(T - \lambda I) > n_1 \quad \textit{and} \quad \beta(T - \lambda I) > n_2.$$

Proof. (a) For $\lambda \in U$ apply Lemma 1.33 to $T - \lambda I$ in place of T.

(b) The component C is open since any component of an open set in $\mathbb{C}$ is open. Let $\alpha(\lambda_0) = n_1$ be the smallest integer which is attained by

$$\alpha(\lambda) = \alpha(T - \lambda I) \quad \text{on } C.$$

Suppose $\alpha(\lambda') \neq n_1$. Since C is connected there exists an arc Γ lying in C with endpoints λ_0 and λ'. It follows from Theorem 1.34 and the fact that C is open that for each $\mu \in \Gamma$, there exists an open ball $S(\mu)$ in C such that

$$\alpha(\lambda) \text{ is constant on the set } S(\mu) \text{ with the point } \mu \text{ deleted.}$$

Since Γ is compact and connected there exist points $\lambda_1, \lambda_2, \cdots, \lambda_n = \lambda'$ on Γ such that

$$S(\lambda_0), S(\lambda_1), \ldots, S(\lambda_n) \text{ cover } \Gamma \quad \text{and} \quad S(\lambda_i) \cap S(\lambda_{i+1}) \neq \emptyset \ (0 \leq i \leq n-1) \tag{14}$$

We claim that $\alpha(\lambda) = \alpha(\lambda_0)$ on all of $S(\lambda_0)$. Indeed it follows from the Lemma 1.33 that

$$\alpha(\lambda) \leq \alpha(\lambda_0) \text{ for } \lambda \text{ sufficiently close to } \lambda_0.$$

Therefore, since $\alpha(\lambda_0)$ is the minimum of $\alpha(\lambda)$ on C,

$$\alpha(\lambda) = \alpha(\lambda_0) \text{ for } \lambda \text{ sufficiently close to } \lambda_0.$$

Since $\alpha(\lambda)$ is constant for all $\lambda \neq \lambda_0$ in $S(\lambda_0)$, which is $\alpha(\lambda_0)$. Now $\alpha(\lambda)$ is constant on the set $S(\lambda_i)$ with the point λ_i deleted $(1 \leq i \leq n)$. Hence it follows from (14) and the observation $\alpha(\lambda) = \alpha(\lambda_0)$ for all $\lambda \in S(\lambda_0)$ that $\alpha(\lambda) = \alpha(\lambda_0)$ for all $\lambda \neq \lambda'$ in $S(\lambda')$ and $\alpha(\lambda') > n_1$. The result just obtained can be applied to the adjoint. This completes the proof. □

1.8 The Riesz-Schauder (or Browder) theory

An operator $T \in B(X)$ is said to be *quasinilpotent* if

$$\|T^n\|^{\frac{1}{n}} \longrightarrow 0 \text{ as } n \to \infty$$

and is said to be *nilpotent* if

$$T^n = 0 \quad \text{for some } n \in \mathbb{N}.$$

An example for quasinilpotent but not nilpotent:

$$\begin{aligned} T : \ell^2 &\to \ell^2 \\ T(x_1, x_2, x_3, \dots) &\longmapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots). \end{aligned}$$

An example for quasinilpotent but neither nilpotent nor compact:

$$T = T_1 \oplus T_2 : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2,$$

where

$$\begin{aligned} T_1 &: (x_1, x_2, x_3, \dots) \longmapsto (0, x_1, 0, x_3, 0, x_5, \dots) \\ T_2 &: (x_1, x_2, x_3, \dots) \longmapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots). \end{aligned}$$

Remember that if $T \in B(X)$ we define $L_T, R_T \in B(B(X))$ by

$$L_T(S) := TS \quad \text{and} \quad R_T(S) := ST \quad \text{for } S \in B(X).$$

Lemma 1.36. *We have:*
(a) L_T *is 1-1* $\Longleftrightarrow$ T *is 1-1;*
(b) R_T *is 1-1* $\Longleftrightarrow$ T *is dense;*
(c) L_T *is bounded below* $\Longleftrightarrow$ T *is bounded below;*
(d) R_T *is bounded below* $\Longleftrightarrow$ T *is open.*

Proof. See [Be3]. □

Theorem 1.37. *If $T \in B(X)$, then*
(a) T *is nilpotent* $\Longrightarrow$ T *is neither 1-1 nor dense;*
(b) T *is quasinilpotent* $\Longrightarrow$ T *is neither bounded below nor open.*

Proof. By Lemma 1.36,

(a) T is nilpotent $\Longrightarrow T^{n+1} = 0 \neq T^n$
$\Longrightarrow L_T(T^n) = R_T(T^n) = 0 \neq T^n$
$\Longrightarrow L_T$ and R_T are not 1-1
$\Longrightarrow T$ is not 1-1 and not dense.

(b) T is quasinilpotent $\Longrightarrow \forall\, \varepsilon > 0$, $\exists\, n \in \mathbb{N}$ such that $\|T^n\|^{\frac{1}{n}} \geq \varepsilon > \|T^{n+1}\|^{\frac{1}{n+1}}$
$\Longrightarrow \|L_T(T^n)\| = \|R_T(T^n)\| < \varepsilon \|T^n\|$
$\Longrightarrow L_T$ and R_T are not bounded below
$\Longrightarrow T$ is not bounded below and not open.

□

We would remark that

$$\{\text{quasinilpotents}\} \subseteq \partial B^{-1}(X).$$

Observe that quasinilpotents of finite rank or cofinite rank are nilpotents.

Definition 1.38. An operator $T \in B(X)$ is said to be *quasipolar* [*polar*, resp.] if there is a projection P commuting with T for which T has a matrix representation

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : \begin{pmatrix} P(X) \\ P^{-1}(0) \end{pmatrix} \to \begin{pmatrix} P(X) \\ P^{-1}(0) \end{pmatrix},$$

where T_1 is invertible and T_2 is quasinilpotent [nilpotent, resp.]

Definition 1.39. An operator $T \in B(X)$ is said to be *simply polar* if there is $T' \in B(X)$ for which

$$T = TT'T \quad \text{with} \quad TT' = T'T$$

Proposition 1.40. *Simply polar operators are decomposably regular.*

Proof. Assume $T = TT'T$ with $TT' = T'T$. Then

$$T'' = T' + (1 - T'T) \implies \begin{cases} T = TT''T \\ (T'')^{-1} = T + (1 - T'T) \end{cases}.$$

□

Theorem 1.41. *If $T \in B(X)$ then*

$$T \textit{ is quasipolar but not invertible} \iff 0 \in \operatorname{iso}\sigma(T)$$

Proof. ($\Rightarrow$) If T is quasipolar we may write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : \begin{pmatrix} P(X) \\ P^{-1}(0) \end{pmatrix} \to \begin{pmatrix} P(X) \\ P^{-1}(0) \end{pmatrix},$$

where T_1 is invertible and T_2 is quasinilpotent. Thus for sufficiently small $\lambda \neq 0$, $T_1 - \lambda I$ and $T_2 - \lambda I$ are both invertible, which implies that $0 \in \operatorname{iso}\sigma(T)$

($\Leftarrow$) If $0 \in \operatorname{iso}\sigma(T)$, construct open discs D_1 and D_2 such that D_1 contains 0, D_2 contains the spectrum $\sigma(T)$ and $D_1 \cap D_2 = \emptyset$. If we define $f : D_1 \cup D_2 \longrightarrow \mathbb{C}$ by setting

$$f(\lambda) = \begin{cases} 0 & \text{on} \quad D_1 \\ 1 & \text{on} \quad D_2 \end{cases}$$

then f is analytic on $D_1 \cup D_2$ and $f(\lambda)^2 = f(\lambda)$. Observe that

$$P = P_{D_2} = f(T) = \frac{1}{2\pi i} \int_{\partial D_2} (\lambda - T)^{-1} d\lambda$$

and $PT = TP$. Thus we may write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : P(X) \oplus P^{-1}(0) \longrightarrow P(X) \oplus P^{-1}(0),$$

where $\sigma(T_1) = \sigma(T)\backslash\{0\}$ and $\sigma(T_2) = \{0\}$. Therefore T_1 is invertible and T_2 is quasinilpotent; so that T is quasipolar.

□

Theorem 1.42. *If $T \in B(X)$ then*

$$T \textit{ is simply polar } \iff T(X) = T^2(X), \quad T^{-1}(0) = T^{-2}(0)$$

Proof. ($\Rightarrow$) Observe

$$\begin{aligned} T(X) &= TT'T(X) = T^2T'(X) \subseteq T^2(X) \subseteq T(X); \\ T^{-1}(0) &= (TT'T)^{-1}(0) = (T'T^2)^{-1}(0) \supseteq T^{-2}(0) \supseteq T^{-1}(0). \end{aligned}$$

($\Leftarrow$) (i) $x \in T(X) \cap T^{-1}(0) \;\Rightarrow\; x = Ty$ for some $y \in X$ and $Tx = 0$
$\Rightarrow\; T^2y = 0 \;\Rightarrow\; y \in T^{-2}(0) = T^{-1}(0)$
$\Rightarrow\; Ty = 0 \;\Rightarrow\; x = 0,$
which gives $T(X) \cap T^{-1}(0) = \{0\}$.

(ii) By assumption, $T(T(X)) = T(X)$. Let $T_1 := T|_{T(X)}$, so that $T_1(X) = T^2(X) = T(X)$. Thus for all $x \in X$,

$$\exists\, y \in T(X) \text{ such that } Tx = T_1y = Ty.$$

Define $z = x - y$, and hence $z \in T^{-1}(0)$. Thus $X = T(X) + T^{-1}(0)$. In particular, $T(X)$ is closed by Theorem 1.13, so that

$$X = T(X) \oplus T^{-1}(0).$$

Therefore we can find a projection $P \in B(X)$ for which

$$P(X) = T(X) \quad \text{and} \quad P^{-1}(0) = T^{-1}(0).$$

We thus write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} P(X) \\ P^{-1}(0) \end{pmatrix} \to \begin{pmatrix} P(X) \\ P^{-1}(0) \end{pmatrix},$$

where T_1 is invertible because $T_1 := T|_{TX}$ is 1-1 and onto since $T(X) = T^2(X)$. If we put

$$T' = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

then $TT'T = T$ and

$$TT' = T'T = \begin{pmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = P,$$

which says that T is simply polar.

□

Theorem 1.43. *If $T \in B(X)$ then*

$$T \text{ is polar} \iff T^n \text{ is simply polar for some } n \in \mathbb{N}$$

Proof. ($\Rightarrow$) If T is polar then we can write $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ with T_1 invertible and T_2 nilpotent. So $T^n = \begin{pmatrix} T_1^n & 0 \\ 0 & 0 \end{pmatrix}$, where n is the nilpotency of T_2. If we put $S = \begin{pmatrix} T_1^{-n} & 0 \\ 0 & I \end{pmatrix}$, then $T^n S T^n = T^n$ and $ST^n = T^n S$.

($\Leftarrow$) If T^n is simply polar then $X = T^n(X) \oplus T^{-n}(0)$. Observe that since T^n is simply polar we have

$$T(T^n X) = T^{n+1}(X) \supseteq T^{2n}(X) = T^n(X)$$
$$T(T^{-n}(0)) \subseteq T^{-n+1}(0) \subseteq T^{-n}(0)$$

Thus we see that $T|_{T^n(X)}$ is 1-1 and onto, so that invertible. Thus we may write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : T^n(X) \oplus T^{-n}(0) \longrightarrow T^n(X) \oplus T^{-n}(0),$$

where $T_1 = T|_{T^n(X)}$ is invertible and $T_2 = T|_{T^{-n}(0)}$ is nilpotent with nilpotency n. Therefore T is polar. $\square$

If $T \in B(X, Y)$, then we have

$$\{0\} \subseteq T^{-1}(0) \subseteq T^{-2}(0) \subseteq \cdots \subseteq T^{-n}(0) \subseteq T^{-n-1}(0) \subseteq \cdots$$

and

$$X \supseteq T(X) \supseteq T^2(X) \supseteq \cdots \supseteq T^n(X) \supseteq T^{n+1}(X) \supseteq \cdots :$$

We shall say that T has *ascent* k if k is the smallest number such that

$$T^{-k}(0) = T^{-\infty}(0) \equiv \bigcup_{n=1}^{\infty} T^{-n}(0) :$$

in this case, we write ascent $(T) = k$. Also we say that T has *descent* k such that if k is the smallest number such that

$$T^k(X) = T^{\infty}(X) \equiv \bigcap_{n=1}^{\infty} T^k(X) :$$

in this case, we write descent $(T) = k$.

The following is an immediate result of Theorem 1.43 :

Corollary 1.44. *If $T \in B(X)$ then*

$$T \text{ is polar} \iff \text{ascent}\,(T) = \text{descent}\,(T) < \infty$$

Corollary 1.45. *If $S, T \in B(X)$ with $ST = TS$, then*

$$S \text{ and } T \text{ are polar} \implies ST \text{ is polar.}$$

Proof. Suppose $S^n(X) = S^{n+1}(X)$ and $T^n(X) = T^{n+1}(X)$. Then

$$\begin{aligned}(ST)^{mn+1}(X) &= S^{mn+1}T^{mn+1}(X) = S^{mn+1}T^{mn}(X) = T^{mn}S^{mn+1}(X) \\ &= T^{mn}S^{mn}(X) = (ST)^{mn}(X)\end{aligned}$$

Similarly,

$$(ST)^{-p-1}(0) = (ST)^{-p}(0).$$

□

Definition 1.46. An operator $T \in B(X)$ is called a *Browder* (or *Riesz-Schauder*) *operator* if T is Fredholm and quasipolar.

If T is Fredholm then by the remark above Definition 1.38,

$$T \text{ is quasipolar} \iff T \text{ is polar.}$$

Thus we have

$$T \text{ is Browder} \iff T \text{ is Fredholm and polar.}$$

Theorem 1.47. *If $T \in B(X)$, the following are equivalent:*
(a) *T is Browder, but not invertible;*
(b) *T is Fredholm and $0 \in \operatorname{iso}\sigma(T)$;*
(c) *T is Weyl and $0 \in \operatorname{iso}\sigma(T)$;*
(d) *T is Fredholm and $\operatorname{ascent}(T) = \operatorname{descent}(T) < \infty$.*

Proof. (a) ⇔ (b) : Theorem 1.41
(b) ⇔ (c) : From the continuity of the index
(b) ⇔ (d) : From Corollary 1.44. □

Theorem 1.48. *If $K \in B(X)$ then*

$$K \text{ is compact} \implies I + K \text{ is Browder.}$$

Proof. From the spectral theory of the compact operators,

$$-1 \in \operatorname{iso}\sigma(K) \quad (\text{in fact, } \lambda \neq 0 \Rightarrow \lambda \notin \operatorname{acc}\sigma(K)),$$

which gives

$$0 \in \operatorname{iso}\sigma(I + K).$$

From Corollary 1.17, $I + K$ is Fredholm. Now Theorem 1.47 says that $I + K$ is Browder.

□

Theorem 1.49. (Riesz-Schauder Theorem) *If $T \in B(X)$ then*

$$T \text{ is Browder} \iff T = S + K, \text{ where } S \text{ is invertible and } K \text{ is compact with } SK = KS.$$

Proof. (⇒) If T is Browder then it is polar, so that we can write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where T_1 is invertible and T_2 is nilpotent. Since T is Fredholm, T_2 is also Fredholm. If we put

$$S = \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 \\ 0 & T_1 - I \end{pmatrix},$$

then evidently $T_2 - I$ is of finite rank. Thus S is invertible and K is of finite rank. Further,

$$T = S + K \quad \text{and} \quad SK = KS.$$

(⇐) Suppose $T = S + K$ and $SK = KS$. Since, by Theorem 1.48, $I + S^{-1}K$ is Browder, so that $I + S^{-1}K$ is Fredholm and polar. Therefore, by Theorem 1.19 and Corollary 1.45, $T = S(I + S^{-1}K)$ is Fredholm and polar, and hence Browder. Here, note that S and $I + S^{-1}K$ commutes. □

Remark 1.50. If $S, T \in B(X)$ and $ST = TS$ then
(a) S, T are Browder $\iff$ ST is Browder;
(b) S is Browder and T is compact $\implies$ $S + T$ is Browder.

Example 1.51. There exists a Weyl operator which is not Browder.

Proof. Put $T = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$, where U is the unilateral shift. Evidently, T is Fredholm and $\operatorname{index} T = \operatorname{index} U + \operatorname{index} U^* = 0$, which says that T is Weyl. However, $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$; so that $0 \notin \operatorname{iso} \sigma(T)$, which implies that T is not Browder. □

1.9 Essential spectra

If $T \in B(X)$ we define:
(a) The *essential spectrum* of $T := \sigma_e(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Fredolm}\}$
(b) The *Weyl spectrum* of $T := \omega(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Weyl}\}$
(c) The *Browder spectrum* of $T := \sigma_b(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not Browder}\}$

Evidently, $\sigma_e(T)$, $\omega(T)$ and $\sigma_b(T)$ are all compact;

$$\sigma_e(T) \subset \omega(T) \subset \sigma_b(T);$$

these are nonempty if $\dim X = \infty$.

Theorem 1.52. *If $T \in B(X)$, then*
(a) $\sigma(T) = \sigma_e(T) \cup \sigma_p(T) \cup \sigma_{\text{com}}(T)$*;*
(b) $\sigma(T) = \omega(T) \cup \big(\sigma_p(T) \cap \sigma_{\text{com}}(T)\big)$*;*
(c) $\sigma_b(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T)$*,*
where $\sigma_{\text{com}}(T) := \{\lambda \in \mathbb{C} : T - \lambda I$ *does not have dense range*$\}$.

Proof. Immediate follow from definition. □

Definition 1.53. We shall write

$$P_{00}(T) = \text{iso}\,\sigma(T)\backslash\sigma_e(T)$$

for the *Riesz points* of $\sigma(T)$. Evidently, $\lambda \in P_{00}(T)$ means that $T - \lambda I$ is Browder, but not invertible.

Lemma 1.54. *If Ω is locally connected and $H,\ K \subset \Omega$, then*

$$\partial K \subseteq H \cup \text{iso}\,K \implies K \subset \eta H \cup \text{iso}\,K,$$

where $\eta(\cdot)$ denotes the polynomially convex hull.

Proof. See [Har4]. □

Theorem 1.55. *If $T \in B(X)$, then*
(a) $\partial\sigma(T)\backslash\sigma_e(T) \subseteq \text{iso}\,\sigma(T)$*;*
(b) $\sigma(T) \subseteq \eta\sigma_e(T) \cup P_{00}(T)$

Proof. (a) This is an immediate consequence of the punctured neighborhood theorem.

(b) From (a) and Lemma 1.54,

$$\begin{aligned}\sigma(T) &\subseteq \eta\sigma_e(T) \cup \text{iso}\,\sigma(T)\\ &= \eta\sigma_e(T) \cup P_{00}(T)\end{aligned}$$

by the fact that if $\lambda \notin \eta\sigma_e(T)$ and $\lambda \in \text{iso}\,\sigma(T)$, then $T - \lambda I$ is Fredholm and $\lambda \in \text{iso}\,\sigma(T)$ thus $T - \lambda I$ is Browder.

□

1.10 Spectral mapping theorems

Recall the Calkin algebra $B(X)/K(X)$. The *Calkin homomorphism* π is defined by

$$\begin{aligned}\pi : B(X) &\longrightarrow B(X)/K(X)\\ \pi(T) &= T + K(X).\end{aligned}$$

Evidently, by the Atkinson's theorem,

$$T \text{ is Fredholm} \iff \pi(T) \text{ is invertible.}$$

If $\mathbf{K}$ is a compact set in $\mathbb{C}$, write $\text{Hol}\,(\mathbf{K})$ for the set of all analytic (holomorphic) functions defined on an open set containing $\mathbf{K}$.

Theorem 1.56. *If $T \in B(X)$ and $f \in \mathrm{Hol}\,(\sigma(T))$, then*

$$f(\sigma_e(T)) = \sigma_e(f(T))$$

Proof. Since $f(\pi(T)) = f(T + K(X)) = f(T) + K(X) = \pi(f(T))$ it follows that

$$f(\sigma_e(T)) = f(\sigma(\pi(T))) = \sigma(f(\pi(T))) = \sigma(\pi(f(T))) = \sigma_e(f(T)).$$

□

Theorem 1.57. *If $T \in B(X)$ and $f \in \mathrm{Hol}\,(\sigma(T))$, then*

$$f(\sigma_b(T)) = \sigma_b(f(T))$$

Proof. Since by the analyticity of f, $f(\mathrm{acc}\, K) = \mathrm{acc}\, f(K)$, it follows that

$$\begin{aligned} f(\sigma_b(T)) &= f(\sigma_e(T) \cup \mathrm{acc}\,\sigma(T)) \\ &= f(\sigma_e(T)) \cup f(\mathrm{acc}\,\sigma(T)) \\ &= \sigma_e(f(T)) \cup \mathrm{acc}\,\sigma(f(T)) \\ &= \sigma_b(f(T)). \end{aligned}$$

□

Theorem 1.58. *If $T \in B(X)$ and p is a polynomial then*

$$\omega(p(T)) \subseteq p(\omega(T)).$$

Proof. Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$; thus $p(z) = c_0(z - \alpha_1) \cdots (z - \alpha_n)$. Then

$$p(T) = c_0(T - \alpha_1 I) \cdots (T - \alpha_n I).$$

We now claim that

$$\begin{aligned} 0 \notin p(\omega(T)) &\implies c_0(z - \alpha_1) \cdots (z - \alpha_n) \neq 0 \quad \text{for each } \lambda \in \omega(T) \\ &\implies \lambda \neq \alpha_i \quad \text{for each } \lambda \in \omega(T) \\ &\implies T - \alpha_i I \text{ is Weyl for each } i = 1, 2, \ldots n \\ &\implies c_0(T - \alpha_1 I) \cdots (T - \alpha_n I) \text{ is Weyl} \\ &\implies 0 \notin \omega(p(T)) \end{aligned}$$

□

In fact, we can show that $\omega(f(T)) \subseteq f(\omega(T))$ for any $f \in \mathrm{Hol}\,(\sigma(T))$.

The inclusion of Theorem 1.58 may be proper. For example, if U is the unilateral shift, consider

$$T = \begin{pmatrix} U + I & 0 \\ 0 & U^* - I \end{pmatrix} : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2.$$

Then

$$\omega(T) = \sigma(T) = \{z \in \mathbb{C} : |1 + z| \leq 1\} \cup \{z \in \mathbb{C} : |1 - z| \leq 1\}.$$

Let $p(z) = (z + 1)(z - 1)$. Then

$$p(\omega(T)) \text{ is a Cardioid containing } 0.$$

Therefore $0 \in p(\omega(T))$. However

$$p(T) = (T+I)(T-I) = \begin{pmatrix} U+2I & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U^*-2I \end{pmatrix},$$

so that $\operatorname{index}(p(T)) = \operatorname{index} U^* + \operatorname{index} U = 0$, which implies $0 \notin \omega(p(T))$. Therefore

$$p(\omega(T)) \nsubseteq \omega(p(T)).$$

1.11 The continuity of spectra

Let σ_n be a sequence of compact subsets of $\mathbb{C}$.

(a) The *limit inferior*, $\liminf \sigma_n$, is the set of all $\lambda \in \mathbb{C}$ such that every neighborhood of λ has a nonempty intersection with all but finitely many σ_n.

(b) The *limit superior*, $\limsup \sigma_n$, is the set of all $\lambda \in \mathbb{C}$ such that every neighborhood of λ intersects infinitely many σ_n.

(c) If $\liminf \sigma_n = \limsup \sigma_n$ then $\lim \sigma_n$ is said to be exist and is the common limit.

A mapping $\mathcal{T}$ on $B(X)$ whose values are compact subsets of $\mathbb{C}$ is said to be *upper semi-continuous* at T when

$$T_n \longrightarrow T \implies \limsup \mathcal{T}(T_n) \subset \mathcal{T}(T)$$

and to be *lower semi-continuous* at T when

$$T_n \longrightarrow T \implies \mathcal{T}(T) \subset \liminf \mathcal{T}(T_n).$$

If $\mathcal{T}$ is both upper and lower semi-continuous, then it is said to be continuous.

Example 1.59. The spectrum $\sigma : T \longmapsto \sigma(T)$ is not continuous in general: for example, if

$$T_n := \begin{pmatrix} U & \frac{1}{n}(I-UU^*) \\ 0 & U^* \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}$$

then $\sigma(T_n) = \partial\mathbb{D}$, $\sigma(T) = \mathbb{D}$, and $T_n \longrightarrow T$.

Proposition 1.60. *σ is upper semi-continuous.*

Proof. Suppose $T^n \to T$ and $\lambda \in \limsup \sigma(T_n)$. Then there exists $\lambda_n \in \limsup \sigma(T_n)$ so that $\lambda_{n_k} \to \lambda$. Since $T_{n_k} - \lambda_{n_k} I$ is singular and $T_{n_k} - \lambda_{n_k} I \longrightarrow T - \lambda I$, it follows that $T - \lambda I$ is singular; therefore $\lambda \in \sigma(T)$. □

Let H be a complex Hilbert space. Then $T \in B(H)$ is called a *hyponormal* operator if $T^*T \geq TT^*$, i.e., the self-commutator $[T^*, T] \equiv T^*T - TT^*$ is positive semi-definite.

We then have:

Theorem 1.61. *σ is continuous on the set of all hyponormal operators.*

Proof. Let $T_n,\ T$ be hyponormal operators such that $T^n \to T$ in norm. We want to prove that

$$\sigma(T) \subset \liminf \sigma(T_n).$$

Assume $\lambda \notin \liminf \sigma(T_n)$. Then there exists a neighborhood $N(\lambda)$ of λ such that it does not intersect infinitely many $\sigma(T_n)$. Thus we can choose a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that for some $\varepsilon > 0$,

$$\operatorname{dist}\Big(\lambda, \sigma(T_{n_k})\Big) > \varepsilon.$$

Since T_{n_k} is hyponormal, it follows that

$$\operatorname{dist}(\lambda, \sigma(T_{n_k})) = \min_{\mu\in\sigma(T_{n_k}-\lambda)} |\mu| = \frac{1}{\max\limits_{\mu\in\sigma((T_{n_k}-\lambda)^{-1})} |\mu|} = \frac{1}{\|(T_{n_k}-\lambda)^{-1}\|},$$

where the second equality follows from the observation

$$\sigma(T^{-1}) = \left\{\frac{1}{z} : z \in \sigma(T)\right\}$$

because if $f(z) = \frac{1}{z}$ then $\sigma(T^{-1}) = \sigma(f(T)) = f(\sigma(T)) = \{\frac{1}{z} : z \in \sigma(T)\}$ and the last equality uses the fact that $(T_{n_k} - \lambda I)^{-1}$ is normaloid. So $\|(T_{n_k} - \lambda I)^{-1}\| < \frac{1}{\varepsilon}$. We thus have

$$\begin{aligned}
&\|(T_{n_k} - \lambda I)^{-1} - (T_{n_l} - \lambda I)^{-1}\| \\
&= \|(T_{n_k} - \lambda I)^{-1}\Big\{(T_{n_k} - \lambda I) - (T_{n_l} - \lambda I)\Big\} - (T_{n_l} - \lambda I)^{-1}\| \\
&\le \|(T_{n_k} - \lambda I)^{-1}\| \cdot \|T_{n_l} - T_{n_k}\| \cdot \|(T_{n_l} - \lambda I)^{-1}\| \\
&< \frac{1}{\varepsilon^2}\|T_{n_l} - T_{n_k}\|.
\end{aligned}$$

Since $T_{n_k} \to T$, it follows that $\{(T_{n_k} - \lambda I)^{-1}\}$ converges, to some operator B, say. Therefore

$$\begin{aligned}
(T - \lambda I)B &= \lim(T_{n_k} - \lambda I) \cdot \lim(T_{n_k} - \lambda I)^{-1} \\
&= \lim(T_{n_k} - \lambda I)(T_{n_k} - \lambda I)^{-1} = 1.
\end{aligned}$$

Similarly, $B(T - \lambda I) = 1$ and hence $\lambda \notin \sigma(T)$. □

Lemma 1.62. *Let A be a commutative Banach algebra. If $x \in A$ is not invertible and $\|y - x\| < \varepsilon$, then there exists λ such that $y - \lambda$ is not invertible and $|\lambda| < \varepsilon$.*

Proof. Since x is not invertible, it generates an ideal $\neq A$. Thus there exists a maximal ideal M containing x. So $z \in M \implies z$ is not invertible. Since $A/M \cong \mathbb{F},\ \lambda \cdot 1 \in y + M$ for some y. Thus $y - \lambda \cdot 1 \in M$. Since $x \in M$ we have $y - x - \lambda \cdot 1 \in M$, so that $\lambda \in \sigma(y - x)$. Finally, $|\lambda| \le \|y - x\| < \varepsilon$. □

Theorem 1.63. *If in a Banach algebra A, $x_i \to x$ and $x_i x = x x_i$ for all i, then $\lim \sigma(x_i) = \sigma(x)$.*

Proof. Let B be the algebra generated by $1,\ x$, and x_i. Then $(x-\mu)^{-1}$ and $(x_i-\mu)^{-1}$ are commutative whenever they exist. Let $\lambda \in \sigma(x)$, i.e., $x-\lambda$ is not invertible. By Lemma 1.62, there exists N such that

$$i > N \implies \sigma(x_i - \lambda) \cap N_\varepsilon(0) \neq \emptyset.$$

So $0 \in \liminf \sigma(x_i - \lambda)$, or $\lambda \in \liminf \sigma(x_i)$, so that

$$\sigma(x) \subseteq \liminf \sigma(x_i) \subseteq \limsup \sigma(x_i) \subseteq \sigma(x).$$

□

Theorem 1.64. *ω is upper semi-continuous.*

Proof. We want to prove that

$$\limsup \omega(T_n) \subset \omega(T) \quad \text{if } T_n \to T.$$

Let $\lambda \notin \omega(T)$, so $T-\lambda I$ is Weyl. Since the set of Weyl operators forms an open set,

$$\exists\ \eta > 0 \text{ such that } \|T - \lambda I - S\| < \eta \implies S \text{ is Weyl.}$$

Let N be such that

$$\|(T-\lambda I) - (T_n - \lambda I)\| < \frac{\eta}{2} \quad \text{for } n \geq N.$$

Let $V = B(\lambda\,; \frac{\eta}{2})$. Then for $\mu \in V,\ n \geq N$,

$$\|(T-\lambda I) - (T_n - \mu I)\| < \eta,$$

so that $T_n - \mu I$ is Weyl, which implies that $\lambda \notin \limsup \omega(T_n)$. □

Theorem 1.65. *Let $T_n \to T$. If $T_nT = TT_n$ for all n, then $\lim \omega(T_n) = \omega(T)$.*

Proof. In view of Theorem 1.64, it suffices to show that

$$\omega(T) \subseteq \liminf \omega(T_n) \tag{15}$$

Observe that $\pi(T_n)\pi(T) = \pi(T)\pi(T_n)$ and hence by Theorem 1.63, $\lim \sigma_e(T_n) = \sigma_e(T)$. Towards (15), suppose $\lambda \notin \liminf \omega(T_n)$. So there exists a neighborhood $V(x)$ which does not intersect infinitely many $\omega(T_n)$. Since $\sigma_e(T_n) \subset \omega(T_n)$, V does not intersect infinitely many $\sigma_e(T_n)$, i.e., $\lambda \notin \lim \sigma_e(T_n) = \sigma_e(T)$. This shows that $T - \lambda I$ is Fredholm. By the continuity of index, $T - \lambda I$ is Weyl, i.e., $\lambda \notin \omega(T)$. □

We now have:

Theorem 1.66. *If S and T are commuting hyponormal operators then*

$$S,\ T \text{ are Weyl} \iff ST \text{ is Weyl.}$$

Hence if f is analytic in a neighborhood of $\sigma(T)$, then

$$\omega(f(T)) = f(\omega(T)).$$

Proof. See [LeL1]. □

1.12 Concluding remarks and open problems

Let H be an infinite dimensional separable Hilbert space. An operator $T \in B(H)$ is called *a Riesz operator* if $\sigma_e(T) = 0$. If $T \in B(H)$ then the *West decomposition theorem* [Wes] says that

$$T \text{ is Riesz} \iff T = K + Q \text{ with compact } K \text{ and quasinilpotent } Q:$$

this is equivalent to the following: if $Q_{B(H)}$ and $Q_{C(H)}$ denote the sets of quasinilpotents of $B(H)$ and $C(H)$, respectively, then

$$\pi\left(Q_{B(H)}\right) = Q_{C(H)}, \tag{16}$$

where $C(H) = B(H)/K(H)$ is the Calkin algebra and π denotes the Calkin homomorphism. It remains still open whether the West decomposition theorem survives in the Banach space setting.

Problem 1.67. *Is the equality (16) true if H is a Banach space ?*

Suppose A is a Banach algebra with identity 1: we shall write A^{-1} for the invertible group of A and A_0^{-1} for the connected components of the identity in A^{-1}. It was [Har3] known that

$$A_0^{-1} := \text{Exp}(A) = \{e^{c_1} e^{c_2} \cdots e^{c_k} : \; k \in \mathbb{N}, \; c_i \in A\}.$$

Evidently, $\text{Exp}\,(A)$ is open, relatively closed in A^{-1}, connected and a normal subgroup. Write

$$\kappa(A) := A^{-1}/\text{Exp}\,(A)$$

for the *abstract index group*. The exponential spectrum $\epsilon(a)$ of $a \in A$ is defined by

$$\epsilon(a) := \{\lambda \in \mathbb{C} : \; a - \lambda \notin \text{Exp}\,(A)\}.$$

Clearly,

$$\partial\epsilon(a) \subset \sigma(a) \subset \epsilon(a).$$

If $A = B(H)$ then $\epsilon(a) = \sigma(a)$. We have known that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$: indeed (cf. [GGK, p.38])

$$\begin{pmatrix} ab-1 & 0 \\ 0 & 1 \end{pmatrix} = F \begin{pmatrix} ba-1 & 0 \\ 0 & 1 \end{pmatrix} E,$$

where

$$E := \begin{pmatrix} b & 1 \\ ab-1 & a \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} a & 1-ab \\ -1 & b \end{pmatrix}$$

are both invertible.

However we were not able to answer to the following:

Problem 1.68. *If A is a Banach algebra and $a, b \in A$, does it follow that*

$$\epsilon(ab) \setminus \{0\} = \epsilon(ba) \setminus \{0\}\,?$$

2 Weyl theory

In 1909, writing about differential equations, Hermann Weyl noticed something about the essential spectrum of a self adjoint operator on a Hilbert space: when you take it away from the spectrum, you are left with the isolated eigenvalues of finite multiplicity. This was soon generalized to normal operators, and then to more general operators, bounded and unbounded, on Hilbert and on Banach spaces. Nowadays, this theorem is called the Weyl's theorem. In this section, we explore a recent development of the study on the Weyl's theorem.

2.1 Weyl's theorem

If $T \in B(X)$ write $\pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0i}(T)$ for the eigenvalues of infinite multiplicity. We often write $\ker T$ and $\operatorname{ran} T$ for the null space and the range of T, respectively. Define

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \dim \ker(T - \lambda I) < \infty\} \tag{17}$$

for the isolated eigenvalues of finite multiplicity, and ([Har1], [Har2], [Har3], [Har4])

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) \tag{18}$$

for the *Riesz points* of $\sigma(T)$, then by the punctured neighborhood theorem, i.e., $\partial\,\sigma(T) \setminus \sigma_e(T) \subseteq \operatorname{iso}\sigma(T)$ (cf. [HaL1]),

$$\operatorname{iso}\sigma(T) \setminus \sigma_e(T) = \operatorname{iso}\sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T). \tag{19}$$

H. Weyl [We] examined the spectra of all compact perturbations $T + K$ of a single hermitian operator T and discovered that $\lambda \in \sigma(T + K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(T)$. Today this result is known as Weyl's theorem: that is, we say that *Weyl's theorem holds for* $T \in B(X)$ if there is equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T). \tag{20}$$

In this section we explore the class of operators satisfying Weyl's theorem.

If $T \in B(X)$, write $r(T)$ for the spectral radius of T. It is familiar that $r(T) \leq ||T||$. An operator T is called *normaloid* if $r(T) = ||T||$ and *isoloid* if $\operatorname{iso}\sigma(T) \subseteq \pi_0(T)$. If X is a Hilbert space, an operator $T \in B(X)$ is called *reduction-isoloid* if the restriction of T to any reducing subspace is isoloid.

Let H be a Hilbert space and suppose that $T \in B(H)$ is reduced by each of its finite-dimensional eigenspaces. If

$$\mathfrak{M} := \bigvee \{\ker(T - \lambda I) : \ \lambda \in \pi_{0f}(T)\},$$

then $\mathfrak{M}$ reduces T. Let $T_1 := T|\mathfrak{M}$ and $T_2 := T|\mathfrak{M}^\perp$. Then we have ([Be2, Proposition 4.1]) that

(a) T_1 is a normal operator with pure point spectrum;
(b) $\pi_0(T_1) = \pi_{0f}(T)$;
(c) $\sigma(T_1) = \mathrm{cl}\, \pi_0(T_1)$;
(d) $\pi_0(T_2) = \pi_0(T) \setminus \pi_{0f}(T) = \pi_{0i}(T)$.

In this case, S. Berberian ([Be2, Definition 5.4]) defined

$$\tau(T) := \sigma(T_2) \cup \mathrm{acc}\, \pi_{0f}(T). \tag{21}$$

We shall call $\tau(T)$ the *Berberian spectrum* of T. S. Berberian has also shown that $\tau(T)$ is a nonempty compact subset of $\sigma(T)$. We can, however, show that Weyl spectra, Browder spectra, and Berberian spectra all coincide for operators reduced by each of its finite-dimensional eigenspaces:

Theorem 2.1. *If H is a Hilbert space and $T \in B(H)$ is reduced by each of its finite-dimensional eigenspaces then*

$$\tau(T) = \omega(T) = \sigma_b(T). \tag{22}$$

Proof. Let $\mathfrak{M}$ be the closed linear span of the eigenspaces $\ker(T - \lambda I)$ $(\lambda \in \pi_{0f}(T))$ and write

$$T_1 := T|\mathfrak{M} \quad \text{and} \quad T_2 := T|\mathfrak{M}^\perp.$$

From the preceding arguments it follows that T_1 is normal, $\pi_0(T_1) = \pi_{0f}(T)$ and $\pi_{0f}(T_2) = \emptyset$. For (22) it will be shown that

$$\omega(T) \subseteq \tau(T) \subseteq \sigma_b(T) \tag{23}$$

and

$$\sigma_b(T) \subseteq \omega(T). \tag{24}$$

For the first inclusion of (23) suppose $\lambda \in \sigma(T) \setminus \tau(T)$. Then $T_2 - \lambda I$ is invertible and $\lambda \in \mathrm{iso}\, \pi_0(T_1)$. Since also $\pi_0(T_1) = \pi_{0f}(T_1)$, we have that $\lambda \in \pi_{00}(T_1)$. But since T_1 is normal, it follows that $T_1 - \lambda I$ is Weyl and hence so is $T - \lambda I$. This proves the first inclusion. For the second inclusion of (23) suppose $\lambda \in \sigma(T) \setminus \sigma_b(T)$. Thus $T - \lambda I$ is Browder but not invertible. Observe that the following equality holds with no other restriction on either R or S:

$$\sigma_b(R \oplus S) = \sigma_b(R) \cup \sigma_b(S) \quad \text{for each } R \in B(X_1) \text{ and } S \in B(X_2). \tag{25}$$

Indeed if $\lambda \in \mathrm{iso}\, \sigma(R \oplus S)$ then λ is either an isolated point of the spectra of direct summands or a resolvent element of direct summands, so that if $R - \lambda I$ and $S - \lambda I$ are Fredholm then by (19), λ is either a Riesz point or a resolvent element of direct summands, which implies that $\sigma_b(R) \cup \sigma_b(S) \subseteq \sigma_b(R \oplus S)$, and the reverse inclusion is evident. From this we can see that $T_1 - \lambda I$ and $T_2 - \lambda I$ are both Browder. But since $\pi_{0f}(T_2) = \emptyset$, it follows that $T_2 - \lambda I$ is one-one and hence invertible. Therefore $\lambda \in \pi_{00}(T_1) \setminus \sigma(T_2)$, which implies that $\lambda \notin \tau(T)$. This proves the second inclusion of (23). For (24) suppose $\lambda \in \sigma(T) \setminus \omega(T)$ and hence $T - \lambda I$ is Weyl but not invertible. Observe that if H_1 is a Hilbert space and if an operator $R \in B(H_1)$ satisfies the equality $\omega(R) = \sigma_e(R)$, then

$$\omega(R \oplus S) = \omega(R) \cup \omega(S) \quad \text{for each Hilbert space } H_2 \text{ and } S \in B(H_2): \tag{26}$$

this follows from the fact that the index of a direct sum is the sum of the indices

$$\text{index}\,(R \oplus S - \lambda(I \oplus I)) = \text{index}\,(R - \lambda I) + \text{index}\,(S - \lambda I)$$

whenever $\lambda \notin \sigma_e(R \oplus S) = \sigma_e(R) \cup \sigma_e(S)$. Since T_1 is normal, applying the equality (26) to T_1 in place of R gives that $T_1 - \lambda I$ and $T_2 - \lambda I$ are both Weyl. But since $\pi_{0f}(T_2) = \emptyset$, we must have that $T_2 - \lambda I$ is invertible and therefore $\lambda \in \sigma(T_1) \setminus \omega(T_1)$. Thus from Weyl's theorem for normal operators we can see that $\lambda \in \pi_{00}(T_1)$ and hence $\lambda \in \text{iso}\,\sigma(T_1) \cap \rho(T_2)$, which by (19), implies that $\lambda \notin \sigma_b(T)$. This proves (24) and completes the proof. □

As applications of Theorem 2.1 we will give several corollaries below.

Corollary 2.2. *If H is a Hilbert space and $T \in B(H)$ is reduced by each of its finite-dimensional eigenspaces then $\sigma(T) \setminus \omega(T) \subseteq \pi_{00}(T)$.*

Proof. This follows at once from Theorem 2.1. □

Weyl's theorem is not transmitted to dual operators: for example if $T : \ell^2 \to \ell^2$ is the unilateral weighted shift defined by

$$Te_n = \frac{1}{n+1} e_{n+1} \quad (n \geq 0), \tag{27}$$

then $\sigma(T) = \omega(T) = \{0\}$ and $\pi_{00}(T) = \emptyset$, and therefore Weyl's theorem holds for T, but fails for its adjoint T^*. We however have:

Corollary 2.3. *Let H be a Hilbert space. If $T \in B(H)$ is reduced by each of its finite-dimensional eigenspaces and $\text{iso}\,\sigma(T) = \emptyset$, then Weyl's theorem holds for T and T^*. In this case, $\sigma(T) = \omega(T)$.*

Proof. If $\text{iso}\,\sigma(T) = \emptyset$, then it follows from Corollary 2.2 that $\sigma(T) = \omega(T)$, which says that Weyl's theorem holds for T. The assertion that Weyl's theorem holds for T^* follows from noting that $\sigma(T)^* = \big(\sigma(T)\big)^-$, $\omega(T^*) = \big(\omega(T)\big)^-$ and $\pi_{00}(T^*) = \big(\pi_{00}(T)\big)^- = \emptyset$. □

In Corollary 2.3, the condition "$\text{iso}\,\sigma(T) = \emptyset$" cannot be replaced by the condition "$\pi_{00}(T) = \emptyset$": for example consider the operator T defined by (27).

Corollary 2.4. ([Be1, Theorem]) *If H is a Hilbert space and $T \in B(H)$ is reduction-isoloid and is reduced by each of its finite-dimensional eigenspaces then Weyl's theorem holds for T.*

Proof. In view of Corollary 2.2, it suffices to show that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Suppose $\lambda \in \pi_{00}(T)$. Then with the preceding notations, $\lambda \in \pi_{00}(T_1) \cap \big[\text{iso}\,\sigma(T_2) \cup \rho(T_2)\big]$. If $\lambda \in \text{iso}\,\sigma(T_2)$, then since by assumption T_2 is isoloid we have that $\lambda \in \pi_0(T_2)$ and hence $\lambda \in \pi_{0f}(T_2)$. But since $\pi_{0f}(T_2) = \emptyset$, we should have that $\lambda \notin \text{iso}\,\sigma(T_2)$. Thus $\lambda \in \pi_{00}(T_1) \cap \rho(T_2)$. Since T_1 is normal it follows that $T_1 - \lambda I$ is Weyl and so is $T - \lambda I$; therefore $\lambda \in \sigma(T) \setminus \omega(T)$. □

Since hyponormal operators are isoloid and are reduced by each of its eigenspaces, it follows from Corollary 2.4 that Weyl's theorem holds for hyponormal operators.

If the condition "reduction-isoloid" is replaced by "isoloid" then Corollary 2.4 may fail: for example, consider the operator $T = T_1 \oplus T_2$, where T_1 is the one-dimensional zero operator and T_2 is an injective quasinilpotent compact operator.

If H is a Hilbert space, an operator $T \in B(H)$ is said to be *p-hyponormal* if $(T^*T)^p - (TT^*)^p \geq 0$ (cf. [Al],[Ch3]). If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal.

Corollary 2.5. [CIO] *Weyl's theorem holds for every p-hyponormal operator.*

Proof. This follows from the fact that every p-hyponormal operator is isoloid and is reduced by each of its eigenspaces ([Ch3]). □

L. Coburn [Co, Corollary 3.2] has shown that if $T \in B(H)$ is hyponormal and $\pi_{00}(T) = \emptyset$, then T is *extremally noncompact*, in the sense that

$$||T|| = ||\pi(T)||,$$

where π is the canonical map of $B(H)$ onto the Calkin algebra $B(H)/K(H)$. His proof relies upon the fact that Weyl's theorem holds for hyponormal operators, and hence $\sigma(T) = \omega(T)$ since $\pi_{00}(T) = \emptyset$. Now we can strengthen the Coburn's argument slightly:

Corollary 2.6. *If $T \in B(H)$ is normaloid and $\pi_{00}(T) = \emptyset$, then T is extremally noncompact.*

Proof. Since $\sigma(T) \subseteq \eta\,\omega(T) \cup p_{00}(T)$ for any $T \in B(H)$, we have that $\eta\,\sigma(T) \setminus \eta\,\omega(T) \subseteq \pi_{00}(T)$. Thus by our assumption, $\eta\,\sigma(T) = \eta\,\omega(T)$. Therefore we can argue that for each compact operator $K \in B(H)$,

$$||T|| = r(T) = r_\omega(T) = r_\omega(T + K) \leq r(T + K) \leq ||T + K||,$$

where $r_\omega(T)$ denotes the "Weyl spectral radius". This completes the proof. □

Note that if $T \in B(H)$ is normaloid and $\pi_{00}(T) = \emptyset$, then Weyl's theorem may fail for T; for example take $X = \ell_2 \oplus \ell_2$ and $T = U \oplus U^*$, where U is the unilateral shift.

We next consider Weyl's theorem for Toeplitz operators.

The Hilbert space $L^2 \equiv L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and the Hardy space $H^2 \equiv H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \dots\}$. An element $f \in L^2$ is referred to as analytic if $f \in H^2$ and coanalytic if $f \in L^2 \ominus H^2$. If P denotes the projection operator $L^2 \to H^2$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operator T_φ on H^2 defined by

$$T_\varphi g = P(\varphi g) \quad \text{for all } g \in H^2 \tag{28}$$

is called the *Toeplitz operator with symbol* φ.

Theorem 2.7. [Co] *Weyl's theorem holds for every Toeplitz operator T_φ.*

Proof. It was known [Wi] that $\sigma(T_\varphi)$ is always connected. Since there are no quasinilpotent Toeplitz operators except 0, $\sigma(T_\varphi)$ can have no isolated eigenvalues of finite multiplicity. Thus Weyl's theorem is equivalent to the fact that

$$\sigma(T_\varphi) = \omega(T_\varphi). \tag{29}$$

Since $T_\varphi - \lambda I = T_{\varphi-\lambda}$, it suffices to show that if T_φ is Weyl then T_φ is invertible. If T_φ is not invertible, but is Weyl then it is easy to see that both T_φ and $T_\varphi^* = T_{\overline{\varphi}}$ must have nontrivial kernels. Thus we want to show that this can not happen, unless $\varphi = 0$ and hence T_φ is the non-Weyl operator.

Suppose that there exist nonzero functions φ, f, and g ($\varphi \in L^\infty$ and $f, g \in H^2$) such that $T_\varphi f = 0$ and $T_{\overline{\varphi}} g = 0$. Then $P(\varphi f) = 0$ and $P(\overline{\varphi} g) = 0$, so that there exist functions $h, k \in H^2$ such that

$$\int h \, d\theta = \int k \, d\theta = 0 \quad \text{and} \quad \varphi f = \overline{h}, \;\; \overline{\varphi} g = \overline{k}.$$

Thus by the F. and M. Riesz's theorem, φ, f, g, h, k are all nonzero except on a set of measure zero. We thus have that $\overline{f}/g = h/\overline{k}$ pointwise a.e., so that $\overline{fk} = gh$ a.e., which implies $gh = 0$ a.e. Again by the F. and M. Riesz's theorem, we can conclude that either $g = 0$ a.e. or $h = 0$ a.e. This contradiction completes the proof. □

We review here a few essential facts concerning Toeplitz operators with continuous symbols, using [Do1] as a general reference. The sets $C(\mathbb{T})$ of all continuous complex-valued functions on the unit circle $\mathbb{T}$ and $H^\infty(\mathbb{T}) = L^\infty \cap H^2$ are Banach algebras, and it is well-known that every Toeplitz operator with symbol $\varphi \in H^\infty$ is subnormal. The C^*-algebra $\mathfrak{A}$ generated by all Toeplitz operators T_φ with $\varphi \in C(\mathbb{T})$ has an important property which is very useful for spectral theory: the commutator ideal of $\mathfrak{A}$ is the ideal $K(H^2)$ of compact operators on H^2. As $C(\mathbb{T})$ and $\mathfrak{A}/K(H^2)$ are $*$-isomorphic C^*-algebras, then for every $\varphi \in C(\mathbb{T})$,

$$T_\varphi \text{ is a Fredholm operator if and only if } \varphi \text{ is invertible} \tag{30}$$

$$\operatorname{index} T_\varphi = -wn(\varphi)\ , \tag{31}$$

$$\sigma_e(T_\varphi) = \varphi(\mathbb{T})\ , \tag{32}$$

where $wn(\varphi)$ denotes the winding number of φ with respect to the origin. Finally, we make note that if $\varphi \in C(\mathbb{T})$ and if $f \in \operatorname{Hol}(\sigma(T_\varphi))$, then $f \circ \varphi \in C(\mathbb{T})$ and $f(T_\varphi)$ is well-defined by the analytic functional calculus.

We require the use of certain closed subspaces and subalgebras of $L^\infty(\mathbb{T})$, which are described in further detail in [Do2] and Appendix 4 of [Ni]. Recall that the subspace $H^\infty(\mathbb{T}) + C(\mathbb{T})$ is a closed subalgebra of L^∞. The elements of the closed selfadjoint subalgebra QC, which is defined to be

$$QC = \big(H^\infty(\mathbb{T}) + C(\mathbb{T})\big) \cap \overline{\big(H^\infty(\mathbb{T}) + C(\mathbb{T})\big)}\,,$$

are called quasicontinuous functions. The subspace PC is the closure in $L^\infty(\mathbb{T})$ of the set of all piecewise continuous functions on $\mathbb{T}$. Thus $\varphi \in PC$ if and only if it is right continuous and has both a left- and right-hand limit at every point. There are certain algebraic relations among Toeplitz operators whose symbols come from these classes, including

$$T_\psi T_\varphi - T_{\psi\varphi} \in K(H^2) \;\text{ for every } \varphi \in H^\infty(\mathbb{T}) + C(\mathbb{T}) \text{ and } \psi \in L^\infty(\mathbb{T})\,, \tag{33}$$

and

$$\text{the commutator } [T_\varphi, T_\psi] \text{ is compact for every } \varphi, \psi \in PC\,. \tag{34}$$

We add to these relations the following one.

Lemma 2.8. *If T_φ is a Toeplitz operator with quasicontinuous symbol φ, and if $f \in \mathrm{Hol}\,(\sigma(T_\varphi))$, then $T_{f\circ\varphi} - f(T_\varphi)$ is a compact operator.*

Proof. Assume that $\varphi \in QC$. Recall from [Do1, p.188] that if $\psi \in H^\infty + C(\mathbb{T})$, then T_ψ is Fredholm if and only if ψ is invertible in $H^\infty + C(\mathbb{T})$. Therefore for every $\lambda \notin \sigma(T_\varphi)$, both $\varphi - \lambda$ and $\overline{\varphi - \lambda}$ are invertible in $H^\infty + C(\mathbb{T})$; hence, $(\varphi - \lambda)^{-1} \in QC$. Using this fact together with (33) we have that, for $\psi \in L^\infty$ and $\lambda, \mu \in \mathbb{C}$,

$$T_{\varphi-\mu} T_\psi T_{(\varphi-\lambda)^{-1}} - T_{(\varphi-\mu)\psi(\varphi-\lambda)^{-1}} \in K(H^2) \quad \text{whenever } \lambda \notin \sigma(T_\varphi)\,.$$

The arguments above extend to rational functions to yield: if r is any rational function with all of its poles outside of $\sigma(T_\varphi)$, then $r(T_\varphi) - T_{r\circ\varphi} \in K(H^2)$. Suppose that f is an analytic function on an open set containing $\sigma(T_\varphi)$. By Runge's theorem there exists a sequence of rational functions r_n such that the poles of each r_n lie outside of $\sigma(T_\varphi)$ and $r_n \to f$ uniformly on $\sigma(T_\varphi)$. Thus $r_n(T_\varphi) \to f(T_\varphi)$ in the norm-topology of $L(H^2)$. Furthermore, because $r_n \circ \varphi \to f \circ \varphi$ uniformly, we have $T_{r_n\circ\varphi} \to T_{f\circ\varphi}$ in the norm-topology. Hence, $T_{f\circ\varphi} - f(T_\varphi) = \lim \big(T_{r_n\circ\varphi} - r_n(T_\varphi)\big)$, which is compact. □

Lemma 2.8 does not extend to piecewise continuous symbols $\varphi \in PC$, as we cannot guarantee that $T_\varphi^n - T_{\varphi^n} \in K(H^2)$ for each $n \in \mathbb{Z}^+$. For example, if $\varphi(e^{i\theta}) = \chi_{\mathbb{T}_+} - \chi_{\mathbb{T}_-}$, where $\chi_{\mathbb{T}_+}$ and $\chi_{\mathbb{T}_-}$ are characteristic functions of, respectively, the upper semicircle and the lower semicircle, then $T_\varphi^2 - I$ is not compact.

Corollary 2.9. *If T_φ is a Toeplitz operator with quasicontinuous symbol φ, then for every $f \in \mathrm{Hol}\,(\sigma(T_\varphi))$,*

(a) *$\omega(f(T_\varphi)) = \sigma(T_{f\circ\varphi})$, and*

(b) *$f(T_\varphi)$ is essentially normal and is unitarily equivalent to a compact perturbation of $f(T_\varphi) \oplus M_{f\circ\varphi}$, where $M_{f\circ\varphi}$ is the operator of multiplication by $f \circ \varphi$ on $L^2(\mathbb{T})$.*

Proof. Because the Weyl spectrum is stable under the compact perturbations, it follows from Lemma 2.8 that $\omega(f(T_\varphi)) = \omega(T_{f\circ\varphi}) = \sigma(T_{f\circ\varphi})$, which proves statement (a). To prove (b), observe that because QC is a closed algebra, the composition of the analytic function f with $\varphi \in QC$ produces a quasicontinuous function $f\circ\varphi \in QC$. Moreover, by (33), every Toeplitz operator with quasicontinuous symbol is essentially normal. The (normal) Laurent operator $M_{f\circ\varphi}$ on $L^2(\mathbb{T})$ has its spectrum contained within the spectrum of the (essentially normal) Toeplitz operator $T_{f\circ\varphi}$. Thus there is the following relationship involving the essentially normal operators $f(T_\varphi)$ and $M_{f\circ\varphi} \oplus f(T_\varphi)$:

$$\sigma_e\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big) = \sigma_e(f(T_\varphi)) \quad \text{and} \quad \mathcal{SP}(f(T_\varphi)) = \mathcal{SP}\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big),$$

where $\mathcal{SP}(T)$ denotes the spectral picture of an operator T. (The spectral picture $\mathcal{SP}(T)$ is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the Fredholm indices associated with these holes and pseudoholes.) Thus it follows from the Brown-Douglas-Fillmore theorem [Pe] that $f(T_\varphi)$ is compalent to $f(T_\varphi) \oplus M_{f\circ\varphi}$, in the sense that there exists a unitary operator W and a compact operator K such that $W\big(f(T_\varphi) \oplus M_{f\circ\varphi}\big)W^* + K = f(T_\varphi)$. □

Corollary 2.9 (a) can be viewed as saying that $\sigma(f(T_\varphi)) \setminus \sigma(T_{f\circ\varphi})$ consists of holes with winding number zero.

We consider the following question ([Ob2]):

$$\textit{if } T_\varphi \textit{ is a Toeplitz operator, then does Weyl's theorem hold for } T_\varphi^2\,? \tag{35}$$

To answer the above question, we need a spectral property of Toeplitz operators with continuous symbols.

Lemma 2.10. *Suppose that φ is continuous and that $f \in \mathrm{Hol}\,(\sigma(T_\varphi))$. Then*

$$\sigma(T_{f\circ\varphi}) \subseteq f(\sigma(T_\varphi))\,, \tag{36}$$

and equality occurs if and only if Weyl's theorem holds for $f(T_\varphi)$.

Proof. By Corollary 2.9, $\sigma(T_{f\circ\varphi}) = \omega(f(T_\varphi)) \subseteq \sigma(f(T_\varphi)) = f(\sigma(T_\varphi))$. Because $\sigma(T_\varphi)$ is connected, so is $f(\sigma(T_\varphi)) = \sigma(f(T_\varphi))$; therefore the set $\pi_{00}(f(T_\varphi))$ is empty. Again by Corollary 2.9, $\omega(f(T_\varphi)) = \sigma(T_{f\circ\varphi})$ and so $\omega(f(T_\varphi)) = \sigma(f(T_\varphi)) \setminus \pi_{00}(f(T_\varphi))$ if and only if $\sigma(T_{f\circ\varphi}) = f(\sigma(T_\varphi))$. □

If φ is not continuous, it is possible for Weyl's theorem to hold for some $f(T_\varphi)$ without $\sigma(T_{f\circ\varphi})$ being equal to $f(\sigma(T_\varphi))$. One example is as follows. Let $\varphi(e^{i\theta}) = e^{\frac{i\theta}{3}}$ $(0 \le \theta < 2\pi)$, a piecewise continuous function. The operator T_φ is invertible but T_{φ^2} is not; hence $0 \in \sigma(T_{\varphi^2}) \setminus \{\sigma(T_\varphi)\}^2$. However $\omega(T_\varphi^2) = \sigma(T_\varphi^2)$, and $\pi_{00}(T_\varphi^2)$ is empty. Therefore Weyl's theorem holds for T_φ^2.

We can now answer the question (35): the answer is negative.

Example 2.11. [FL] *There exists a continuous function $\varphi \in C(\mathbb{T})$ such that $\sigma(T_{\varphi^2}) \neq \{\sigma(T_\varphi)\}^2$.*

Proof. Let φ be defined by

$$\varphi(e^{i\theta}) = \begin{cases} -e^{2i\theta} + 1 & (0 \le \theta \le \pi) \\ e^{-2i\theta} - 1 & (\pi \le \theta \le 2\pi)\,. \end{cases}$$

Then the graph of φ consists of two adjacent circles C_1 (in the right half-plane) and C_2 (in the left half-plane). Evidently, φ is continuous and φ has winding number $+1$ with respect to the hole of C_1; the hole of C_2 has winding number -1. Thus we have $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$ and $\sigma(T_\varphi) = \mathrm{conv}\,\varphi(\mathbb{T})$. On the other hand, a straightforward calculation shows that $\varphi^2(\mathbb{T})$ is the Cardioid $r = 2(1 + \cos\theta)$. In particular, $\varphi^2(\mathbb{T})$ traverses the Cardioid once in a counterclockwise direction and then traverses the Cardioid once in a clockwise direction. Thus $wn(\varphi^2 - \lambda) = 0$ for each λ in the hole of $\varphi^2(\mathbb{T})$. Hence $T_{\varphi^2-\lambda}$ is a Weyl operator and is, therefore, invertible for each λ in the hole of $\varphi^2(\mathbb{T})$. This implies that $\sigma(T_{\varphi^2})$ is the Cardioid $r = 2(1 + \cos\theta)$. But because $\{\sigma(T_\varphi)\}^2 = \{\mathrm{conv}\,\varphi(\mathbb{T})\}^2 = \{(r,\theta) : r \le 2(1 + \cos\theta)\}$, it follows that $\sigma(T_{\varphi^2}) \neq \{\sigma(T_\varphi)\}^2$. □

We next consider Weyl's theorem through the local spectral theory. Local spectral theory is based on the existence of analytic solutions $f : U \to X$ to the equation $(T - \lambda I)f(\lambda) = x$

on an open subset $U \subset \mathbb{C}$, for a given operator $T \in B(X)$ and a given element $x \in X$. We define the spectral subspace as follows: for a closed set $F \subset \mathbb{C}$, let

$$\mathcal{X}_T(F) := \{x \in X : (T - \lambda I)f(\lambda) = x \text{ has an analytic solution } f : \mathbb{C} \setminus F \to X\}.$$

We say that $T \in B(X)$ has the *single valued extension property* (SVEP) at $\lambda_0 \in \mathbb{C}$ if for every neighborhood U of λ_0, $f = 0$ is the only analytic solution $f : U \to X$ satisfying $(T - \lambda I)f(\lambda) = 0$. We also say that T has the SVEP if T has this property at every $\lambda \in \mathbb{C}$. The local spectrum of T at x is defined by

$$\sigma_T(x) := \mathbb{C} \setminus \bigcup \Big\{(T - \lambda I)f(\lambda) = x \text{ has an analytic solution } f : U \to X \text{ on the open subset } U \subset \mathbb{C}\Big\}.$$

If T has the SVEP then $\mathcal{X}_T(F) = \{x \in X : \sigma_T(x) \subset F\}$.

The following lemma gives a connection of the SVEP with a finite ascent property.

Lemma 2.12. [Fin] *If $T \in B(X)$ is semi-Fredholm then*

$$T \text{ has the SVEP at } 0 \iff T \text{ has a finite ascent at } 0.$$

The finite dimensionality of $\mathcal{X}_T(\{\lambda\})$ is necessary ad sufficient for $T - \lambda I$ to be Fredholm whenever λ is an isolated point of the spectrum.

Lemma 2.13. [Ai] *Let $T \in B(X)$. If $\lambda \in \operatorname{iso}\sigma(T)$ then*

$$\lambda \notin \sigma_e(T) \iff \mathcal{X}_T(\{\lambda\}) \text{ is finite dimensional.}$$

Theorem 2.14. *If $T \in B(X)$ has the SVEP then the following are equivalent:*
(a) *Weyl's theorem holds for T;*
(b) $\operatorname{ran}(T - \lambda I)$ *is closed for every $\lambda \in \pi_{00}(T)$;*
(c) $\mathcal{X}_T(\{\lambda\})$ *is finite dimensional for every $\lambda \in \pi_{00}(T)$.*

Proof. (a) $\Rightarrow$ (b): Evident.
(b) $\Rightarrow$ (a): If $\lambda \in \sigma(T) \setminus \omega(T)$ then by Lemma 2.12, $T - \lambda I$ has a finite ascent. Thus $T - \lambda I$ is Browder and hence $\lambda \in \pi_{00}(T)$. Conversely, if $\lambda \in \pi_{00}(T)$ then by assumption $T - \lambda I$ is Browder, so $\lambda \in \sigma(T) \setminus \omega(T)$.
(b) $\Leftrightarrow$ (c): Immediate from Lemma 2.13. $\square$

An operator $T \in B(X)$ is called *reguloid* if each isolated point of spectrum is a regular point, in the sense that there is a generalized inverse:

$$\lambda \in \operatorname{iso} \sigma(T) \Longrightarrow T - \lambda I = (T - \lambda I)S_\lambda(T - \lambda I) \text{ with } S_\lambda \in B(X).$$

It was known [Har4] that if T is reguloid then $\operatorname{ran}(T - \lambda I)$ is closed for each $\lambda \in \operatorname{iso}\sigma(T)$. Also an operator $T \in B(X)$ is said to satisfy *the growth condition* (G_1), if for all $\lambda \in \mathbb{C} \setminus \sigma(T)$

$$||(T - \lambda I)^{-1}||\operatorname{dist}(\lambda, \sigma(T)) \leq 1.$$

Lemma 2.15. *If $T \in B(X)$ then*

$$(G_1) \implies \textit{reguloid} \implies \textit{isoloid}. \tag{37}$$

Proof. Recall ([Har4, Theorem 7.3.4]) that if $T - \lambda I$ has a generalized inverse and if $\lambda \in \partial\sigma(T)$ is in the boundary of the spectrum then $T - \lambda I$ has an invertible generalized inverse. If therefore T is reguloid and $\lambda \in \operatorname{iso}\sigma(T)$ then $T - \lambda I$ has an invertible generalized inverse, and hence ([Har4, (3.8.6.1)])

$$\ker(T - \lambda I) \cong X/\operatorname{ran}(T - \lambda I).$$

Thus if $\ker(T - \lambda I) = \{0\}$ then $T - \lambda I$ is invertible, a contradiction. Therefore λ is an eigenvalue of T, which proves the second implication of (37). Towards the first implication we may write T in place of $T - \lambda I$ and hence assume $\lambda = 0$: then using the spectral projection at $0 \in \mathbb{C}$ we can represent T as a 2×2 operator matrix:

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix},$$

where $\sigma(T_0) = \{0\}$ and $\sigma(T_1) = \sigma(T) \setminus \{0\}$. Now we can borrow an argument of J. Stampfli ([Sta, Theorem C]): take $0 < \epsilon \le \frac{1}{2}\operatorname{dist}(0, \sigma(T) \setminus \{0\})$ and argue

$$T_0 = \frac{1}{2\pi i}\int_{|z|=\epsilon} z(T - zI)^{-1} dz,$$

using the growth condition (G_1) to see that

$$||T_0|| \le \frac{1}{2\pi}\int_{|z|=\epsilon} |z|\ ||(T - zI)^{-1}||\ |dz| \le \frac{1}{2\pi}\epsilon\frac{1}{\epsilon}2\pi\epsilon = \epsilon, \tag{38}$$

which tends to 0 with ϵ. It follows that $T_0 = 0$ and hence that

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix} = TST \text{ with } S = \begin{pmatrix} 0 & 0 \\ 0 & T_1^{-1} \end{pmatrix}$$

has a generalized inverse. □

Corollary 2.16. *If $T \in B(X)$ is reguloid and has the SVEP then Weyl's theorem holds for T.*

Proof. Immediate from Theorem 2.14. □

Lemma 2.17. *Let $T \in B(X)$. If for any $\lambda \in \mathbb{C}$, $\mathcal{X}_T(\{\lambda\})$ is closed then T has the SVEP.*

Proof. This follows from [Ai, Theorem 2.31] together with the fact that

$$\mathcal{X}_T(\{\lambda\}) = \{x \in X : \lim_{n\to\infty} ||(T - \lambda I)^n x||^{\frac{1}{n}} = 0\}.$$

□

Corollary 2.18. *If $T \in B(X)$ satisfies*

$$\mathcal{X}_T(\{\lambda\}) = \ker(T - \lambda I) \quad \textit{for every } \lambda \in \mathbb{C}, \tag{39}$$

then T has the SVEP and both T and T^* are reguloid. Thus in particular if T satisfies (39) *then Weyl's theorem holds for T.*

Proof. If T satisfies the condition (39) then by Lemma 2.17, T has the SVEP. The second assertion follows from [Ai, Theorem 3.96]. The last assertion follows at once from Corollary 2.16. □

An operator $T \in B(X)$ is said to be *paranormal* if

$$||Tx||^2 \le ||T^2x||\,||x|| \quad \text{for every } x \in X.$$

It was well known that if $T \in B(X)$ is paranormal then the following hold:

(a) T is normaloid;

(b) T has finite ascent;

(c) if x and y are nonzero eigenvectors corresponding to, respectively, distinct nonzero eigenvalues of T, then $||x|| \le ||x + y||$ ([ChR, Theorem 2,6])

In particular, p-hyponormal operators are paranormal (cf. [FIY]). An operator $T \in B(X)$ is said to be *totally paranormal* if $T - \lambda I$ is paranormal for every $\lambda \in \mathbb{C}$. Evidently, every hyponormal operator is totally paranormal. On the other hand, every totally paranormal operator satisfies (39): indeed, for every $x \in X$ and $\lambda \in \mathbb{C}$,

$$||(T - \lambda I)^n x||^{\frac{1}{n}} \ge ||(T - \lambda I)x|| \quad \text{for every } n \in \mathbb{N}.$$

So if $x \in \mathcal{X}_T(\{\lambda\})$ then $||(T - \lambda I)^n x||^{\frac{1}{n}} \to 0$ as $n \to \infty$, so that $x \in \ker(T - \lambda I)$, which gives $\mathcal{X}_T(\{\lambda\}) \subset \ker(T - \lambda I)$. The reverse inclusion is true for every operator. Therefore by Corollary 2.18 we can conclude that Weyl's theorem holds for totally paranormal operators. We can prove more:

Theorem 2.19. *Weyl's theorem holds for paranormal operators on a separable Banach space.*

Proof. It was known [ChR] that paranormal operators T on a separable Banach space X have the SVEP. So in view of Theorem 2.14 it suffices to show that $\text{ran}(T - \lambda I)$ is closed for each $\lambda \in \pi_{00}(T)$. Suppose $\lambda \in \pi_{00}(T)$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial B}(\lambda I - T)^{-1} d\lambda$, where B is an open disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = T_1 \oplus T_2, \quad \text{where } \sigma(T_1) = \{\lambda\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

If $\lambda = 0$ then T_1 is a quasinilpotent paranormal operator, so that $T_1 = 0$. If $\lambda \ne 0$ write $T_A = \frac{1}{\lambda} T_1$. Then T_A is paranormal and $\sigma(T_A) = \{1\}$. Since T_A is invertible we have that T_A and T_A^{-1} are paranormal, and hence normaloid. So $||T_A|| = ||T_A^{-1}|| = 1$ and hence

$$||x|| = ||T_A^{-1} T_A x|| \le ||T_A x|| \le ||x|| \quad \text{for each } x \in X,$$

which implies that T_A and T_A^{-1} are isometries. Also since $T_A - 1$ is a quasinilpotent operator it follows that $T_A = I$, and hence $T_1 = \lambda I$. Thus we have that $T - \lambda I = 0 \oplus (T_2 - \lambda I)$ has closed range. This completes the proof. □

Does Weyl's theorem hold for paranormal operators on an *arbitrary* Banach space? Paranormal operators on an arbitrary Banach space may not have the SVEP. So the proof of Theorem 2.19 does not work for arbitrary Banach spaces. In spite of it Weyl's theorem holds for paranormal operators on an arbitrary Banach space. To see this recall the *reduced minimum modulus* of T is defined by

$$\gamma(T) := \inf \frac{||Tx||}{\text{dist}\,(x,\,\ker(T))} \quad (x \notin \ker(T)).$$

It was known [Go] that $\gamma(T) > 0$ if and only if T has closed range.

Theorem 2.20. *Weyl's theorem holds for paranormal operators on a Banach space.*

Proof. The proof of Theorem 2.19 shows that with no restriction on X, $\pi_{00}(T) \subset \sigma(T)\setminus\omega(T)$ for every paranormal operator $T \in B(X)$. Thus we must show that $\sigma(T) \setminus \omega(T) \subset \text{iso}\,\sigma(T)$. Suppose $\lambda \in \sigma(T) \setminus \omega(T)$. If $\lambda = 0$ then T is Weyl and has finite ascent. Thus T is Browder, and hence $0 \in \text{iso}\,\sigma(T)$. If $\lambda \neq 0$ and $\lambda \notin \text{iso}\,\sigma(T)$ then we can find a sequence $\{\lambda_n\}$ of nonzero eigenvalues such that $\lambda_n \to \lambda$. By the property (c) above Theorem 2.19,

$$\text{dist}\left(x_{\lambda_n},\,\ker(T-\lambda I)\right) \geq 1 \quad \text{for each unit vector } x_{\lambda_n} \in \ker(T-\lambda_n I).$$

We thus have

$$\frac{||(T-\lambda I)x_n||}{\text{dist}\,(x_{\lambda_n},\,\ker(T-\lambda I))} = \frac{|\lambda_n - \lambda|}{\text{dist}\,(x_{\lambda_n},\,\ker(T-\lambda I))} \to 0,$$

which shows that $\gamma(T-\lambda I) = 0$ and hence $T-\lambda I$ does not have closed range, a contradiction. Therefore $\lambda \in \text{iso}\,\sigma(T)$. This completes the proof. □

2.2 The spectral mapping theorem for the Weyl spectrum

Let $\mathfrak{S}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. If $\mathfrak{A}$ is a unital Banach algebra then the spectrum can be viewed as a function $\sigma : \mathfrak{A} \to \mathfrak{S}$, mapping each $T \in \mathfrak{A}$ to its spectrum $\sigma(T)$. It is well-known that the function σ is upper semicontinuous, i.e., if $T_n \to T$ then $\limsup \sigma(T_n) \subset \sigma(T)$ and that in noncommutative algebras, σ does have points of discontinuity. The work of J. Newburgh [Ne] contains the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [CoM] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^*-algebra of all operators acting on a complex separable Hilbert space. Of interest is the identification of points of spectral continuity, and of classes $\mathfrak{C}$ of operators for which σ becomes continuous when restricted to $\mathfrak{C}$. In [BGS], the continuity of the spectrum was considered when restricted to certain subsets of the entire manifold of Toeplitz operators. The set of normal operators is perhaps the most immediate in the latter direction: σ is continuous on the set of normal operators. As noted in Solution 104 of [Ha3], Newburgh's argument uses the fact that the inverses of normal resolvents are normaloid. This argument can be easily extended to the set of hyponormal operators because the inverses of hyponormal resolvents are also hyponormal and hence normaloid.

Although p-hyponormal operators are normaloid, it was shown [HwL1] that σ is continuous on the set of all p-hyponormal operators.

We now examine the continuity of the Weyl spectrum in pace of the spectrum. In general the Weyl spectrum is not continuous: indeed, it was in [BGS] that the spectrum is discontinuous on the entire manifold of Toeplitz operators. Since the spectra and the Weyl spectra coincide for Toeplitz operators, it follows at once that the weyl spectrum is discontinuous.

However the Weyl spectrum is upper semicontinuous.

Lemma 2.21. *The map $T \to \omega(T)$ is upper semicontinuous.*

Proof. Let $\lambda \in \omega(T)$. Since the set of Weyl operators forms an open set, there exists $\delta > 0$ such that if $S \in B(X)$ and $||T - \lambda I - S|| < \delta$ then S is Weyl. So there exists an integer N such that $||T - \lambda I - (T_n - \lambda I)|| < \frac{\delta}{2}$ for $n \geq N$. Let V be an open $(\delta/2)$-neighborhhod of λ. We have, for $\mu \in V$ and $n \geq N$,

$$||T - \lambda I - (T_n - \mu I)|| < \delta,$$

so that $T_n - \mu I$ is Weyl. This shows that $\lambda \notin \limsup \omega(T_n)$. Thus $\limsup \omega(T_n) \subset \omega(T)$. □

Lemma 2.22. [Ne, Theorem 4] *If $\{T_n\}_n$ is a sequence of operators converging to an operator T and such that $[T_n, T]$ is compact for each n, then $\lim \sigma_e(T_n) = \sigma_e(T)$.*

Proof. Newburgh's theorem is stated as follows: if in a Banach algebra A, $\{a_i\}_i$ is a sequence of elements commuting with $a \in A$ and such that $a_i \to a$, then $\lim \sigma(a_i) = \sigma(a)$. If π denotes the canonical homomorphism of $B(X)$ onto the Calkin algebra $B(X)/K(X)$, then the assumptions give that $\pi(T_n) \to \pi(T)$ and $[\pi(T_n), \pi(T)] = 0$ for each n. Hence, $\lim\, \sigma(\pi(T_n)) = \sigma(\pi(T))$; that is, $\lim \sigma_e(T_n) = \sigma_e(T)$. □

Theorem 2.23. *Suppose that $T, T_n \in B(X)$, for $n \in \mathbb{Z}^+$, are such that T_n converges to T. If $[T_n, T] \in K(X)$ for each n, then*

$$\lim\, \omega(f(T_n)) = \omega(f(T)) \quad \textit{for every } f \in \mathrm{Hol}(\sigma(T)). \tag{40}$$

Remark. Because $T_n \to T$, by the upper-semicontinuity of the spectrum, there is a positive integer N such that $\sigma(T_n) \subseteq V$ whenever $n > N$. Thus, in the left-hand side of (40) it is to be understood that $n > N$.

Proof. If T_n and T commute modulo the compact operators then, whenever T_n^{-1} and T^{-1} exist, T_n, T, T_n^{-1} and T^{-1} all commute modulo the compact operators. Therefore $r(T_n)$ and $r(T)$ also commute modulo $K(X)$ whenever r is a rational function with no poles in $\sigma(T)$ and n is sufficiently large. Using Runge's theorem we can approximate f on compact subsets of V by rational functions r who poles lie off of V. So there exists a sequence of rational functions r_i whose poles lie outside of V and $r_i \to f$ uniformly on compact subsets of V. If $n > N$, then by the functional calculus,

$$f(T_n)f(T) - f(T)f(T_n) = \lim_i \big(r_i(T_n)r_i(T) - r_i(T)r_i(T_n)\big),$$

which is compact for each n. Furthermore,

$$\begin{aligned}||f(T_n)-f(T)|| &= ||\frac{1}{2\pi i}\int_\Gamma f(\lambda)\big((\lambda-T_n)^{-1}-(\lambda-T)^{-1}\big)\,d\lambda|| \\ &\le \frac{1}{2\pi i}\,\ell(\Gamma)\,\max_{\lambda\in\Gamma}|f(\lambda)|\cdot\max_{\lambda\in\Gamma}||(\lambda-T_n)^{-1}-(\lambda-T)^{-1}||\,,\end{aligned}$$

where Γ is the boundary of V and $\ell(\Gamma)$ denotes the arc length of Γ. The right-hand side of the above inequality converges to 0, and so $f(T_n)\to f(T)$. By Lemma 2.22, $\lim\,\sigma_e(f(T_n))=\sigma_e(f(T))$. The arguments used by J.B. Conway and B.B. Morrel in Proposition 3.11 of [CoM] can now be used here to obtain the conclusion $\lim\,\omega(f(T_n))=\omega(f(T))$. □

In general there is only inclusion for the Weyl spectrum:

Theorem 2.24. *If $T\in B(X)$ then*

$$\omega(p(T))\subseteq p(\omega(T))\quad \textit{for every polynomial } p.$$

Proof. We can suppose p is nonconstant. Suppose $\lambda\notin p\omega(T)$. Writing $p(\mu)-\lambda=a(\mu-\mu_1)(\mu-\mu_2)\cdots(\mu-\mu_n)$, we have

$$p(T)-\lambda I=a(T-\mu_1 I)\cdots(T-\mu_n I). \tag{41}$$

For each i, $p(\mu_i)=\lambda\notin p\omega(T)$, so that $\mu_i\notin\omega(T)$, i.e., $T-\mu_i I$ is weyl. It thus follows from (41) that $p(T)-\lambda I$ is Weyl since the product of Weyl operators is Weyl. □

In general the spectral mapping theorem is liable to fail for the Weyl spectrum:

Example 2.25. *Let $T=U\oplus(U^*+2I)$, where U is the unilateral shift on ℓ_2, and let $p(\lambda):=\lambda(\lambda-2)$. Then $0\in p(\omega(T))$ but $0\notin\omega(p(T))$.*

Proof. Observe $p(T)=T(T-2I)=[U\oplus(U^*+2I)][(U-2I)\oplus U^*]$. Since U is Fredholm of index -1, and since U^*+2I and $U-2I$ are invertible it follows that T and $T-2I$ are Fredholm of indices -1 and $+1$, respectively. Therefore $p(T)$ is Weyl, so that $0\notin\omega(p(T))$, while $0=p(0)\in p(\omega(T))$. □

Lemma 2.26. *If $T\in B(X)$ is isoloid then for every polynomial p,*

$$p(\sigma(T)\setminus\pi_{00}(T))=\sigma(p(T))\setminus\pi_{00}(p(T)).$$

Proof. We first claim that with no restriction on T,

$$\sigma(p(T))\setminus\pi_{00}(p(T))\subset p(\sigma(T)\setminus\pi_{00}(T)). \tag{42}$$

Let $\lambda\in\sigma(p(T))\setminus\pi_{00}(p(T))=p(\sigma(T))\setminus\pi_{00}(p(T))$. There are two cases to consider.

Case 1. $\lambda\notin\operatorname{iso}p(\sigma(T))$. In this case, there exists a sequence (λ_n) in $p(\sigma(T))$ such that $\lambda_n\to\lambda$. So there exists a sequence (μ_n) in $\sigma(T)$ such that $p(\mu_n)=\lambda_n\to\lambda$. This implies that (μ_n) contains a convergent subsequence and we may assume that $\lim\mu_n=\mu_0$. Thus $\lambda=\lim p(\mu_n)=p(\mu_0)$. Since $\mu_0\in\sigma(T)\setminus\pi_{00}(T)$, it follows that $\lambda\in p(\sigma(T)\setminus\pi_{00}(T))$.

Case 2. $\lambda\in\operatorname{iso}p(\sigma(T))$. In this case either λ is not an eigenvalue of $p(T)$ or it is an eigenvalue of infinite multiplicity. Let $p(T)-\lambda I=a_0(T-\mu_1 I)\cdots(T-\mu_n I)$. If λ is not an eigenvalue of $p(T)$ then none of $\mu_1, \cdots, \mu_n$ can be an eigenvalue of T and at least one of $\mu_1, \cdots, \mu_n$ is in $\sigma(T)$. Therefore $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. If λ is an eigenvalue of $p(T)$ of infinite multiplicity then at least one of $\mu_1, \cdots, \mu_n$, say μ_1, is an eigenvalue of T of infinite multiplicity. Then $\mu_1 \in \sigma(T) \setminus \pi_{00}(T)$ and $p(\mu_1) = \lambda$, so that $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. This proves (42). For the reverse inclusion of (42), we assume $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. Since $p(\sigma(T)) = \sigma(p(T))$, we have $\lambda \in \sigma(p(T))$. If possible let $\lambda \in \pi_{00}(p(T))$. So $\lambda \in \operatorname{iso} \sigma(p(T))$. Let

$$p(T) - \lambda I = a_0(T - \mu_1 I) \cdots (T - \mu_n I). \tag{43}$$

The equality (43) shows that if any of $\mu_1, \cdots, \mu_n$ is in $\sigma(T)$ then it must be an isolated point of $\sigma(T)$ and hence an eigenvalue since T is isoloid. Since λ is an eigenvalue of finite multiplicity, any such μ must be an eigenvalue of finite multiplicity and hence belongs to $\pi_{00}(T)$. This contradicts the fact that $\lambda \in p(\sigma(T) \setminus \pi_{00}(T))$. Therefore $\lambda \notin \pi_{00}(T)$ and

$$p(\sigma(T) \setminus \pi_{00}(T)) \subset \sigma(p(T)) \setminus \pi_{00}(p(T)).$$

□

Theorem 2.27. *If $T \in B(X)$ is isoloid and Weyl's theorem holds for T then for every polynomial p, Weyl's theorem holds for $p(T)$ if and only if $p(\omega(T)) = \omega(p(T))$.*

Proof. By Lemma 2.26, $p(\sigma(T) \setminus \pi_{00}(T)) = \sigma(p(T)) \setminus \pi_{00}(p(T))$. If Weyl's theorem holds for T then $\omega(T) = \sigma(T) \setminus \pi_{00}(T)$, so that

$$p(\omega(T)) = p(\sigma(T) \setminus \pi_{00}(T)) = \sigma(p(T)) \setminus \pi_{00}(p(T)).$$

The result follows at once from this relationship. □

Example 2.28. Theorem 2.27 may fail if T is not isoloid. To see this define T_1 and T_2 on ℓ^2 by

$$T_1(x_1, x_2, \cdots) = (x_1, 0, x_2/2, x_3/2, \cdots)$$

and

$$T_2(x_1, x_2, \cdots) = (0, x_1/2, x_2/3, x_3/4, \cdots).$$

Let $T := T_1 \oplus (T_2 - I)$ on $X = \ell^2 \oplus \ell^2$. Then

$$\sigma(T) = \{1\} \cup \{z : |z| \leq 1/2\} \cup \{-1\}, \quad \pi_{00}(T) = \{1\},$$

and

$$\omega(T) = \{z : |z| \leq 1/2\} \cup \{-1\},$$

which shows that Weyl's theorem holds for T. Let $p(t) = t^2$. Then

$$\sigma(p(T)) = \{z : |z| \leq 1/4\} \cup \{1\}, \quad \pi_{00}(p(T)) = \{1\}$$

and

$$\omega(p(T)) = \{z : |z| \leq 1/4\} \cup \{1\}.$$

Thus $1 \in p(\sigma(T) \setminus \pi_{00}(T))$, but $1 \notin \sigma(p(T)) \setminus \pi_{00}(p(T))$. Also $\omega(p(T)) = p(\omega(T))$ but Weyl's theorem does not hold for $p(T)$.

Theorem 2.29. *If $p(\omega(T)) = \omega(p(T))$ for every polynomial p, then $f(\omega(T)) = \omega(f(T))$ for every $f \in \mathrm{Hol}(\sigma(T))$.*

Proof. Let $(p_n(T))$ be a sequence of polynomials converging uniformly in a neighborhood of $\sigma(T)$ to $f(t)$ so that $p_n(T) \to f(T)$. Since $f(T)$ commutes with each $p_n(T)$, it follows from Theorem 2.23 that

$$\omega(f(T)) = \lim \omega(p_n(T)) = \lim p_n(\omega(T)) = f(\omega(T)).$$

□

Theorem 2.30. *If $T \in B(X)$ then the following are equivalent:*

$$\mathrm{index}(T - \lambda I)\, \mathrm{index}(T - \mu I) \geq 0 \textit{ for each pair } \lambda, \mu \in \mathbb{C} \setminus \sigma_e(T); \tag{44}$$

$$f(\omega(T)) = \omega(f(T)) \textit{ for every } f \in \mathrm{Hol}(\sigma(T)). \tag{45}$$

Proof. The spectral mapping theorem for the Weyl spectrum may be rewritten as implication, for arbitrary $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^n$,

$$(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I) \text{ Weyl} \implies T - \lambda_j I \text{ Weyl for each } j = 1, 2, \cdots, n. \tag{46}$$

Now if $\mathrm{index}(T - zI) \geq 0$ on $\mathbb{C} \setminus \sigma_e(T)$ then we have

$$\sum_{j=1}^{n} \mathrm{index}(T - \lambda_j I) = \mathrm{index} \prod_{j=1}^{n} (T - \lambda_j I) = 0 \Longrightarrow \mathrm{index}(T - \lambda_j I) = 0 \ (j = 1, 2, \cdots, n),$$

and similarly if $\mathrm{index}\,(T - zI) \leq 0$ off $\sigma_e(T)$. If conversely there exist λ, μ for which

$$\mathrm{index}(T - \lambda I) = -m < 0 < k = \mathrm{index}(T - \mu I) \tag{47}$$

then

$$(T - \lambda I)^k (T - \mu I)^m \tag{48}$$

is a Weyl operator whose factors are not Weyl. This together with Theorem 2.29 proves the equivalence of the conditions (44) and (45). □

Corollary 2.31. *If H is a Hilbert space and $T \in B(H)$ is hyponormal then*

$$f(\omega(T)) = \omega(f(T)) \textit{ for every } f \in \mathrm{Hol}(\sigma(T)). \tag{49}$$

Proof. Immediate from Theorem 2.30 together with the fact that if T is hyponormal then $\mathrm{index}\,(T - \lambda I) \leq 0$ for every $\lambda \in \mathbb{C} \setminus \sigma_e(T)$. □

Corollary 2.32. *Let $T \in B(X)$. If*

(i) *Weyl's theorem holds for T;*
(ii) *T is isoloid;*
(iii) *T satisfies the spectral mapping theorem for the Weyl spectrum,*

then Weyl's theorem holds for $f(T)$ for every $f \in \mathrm{Hol}(\sigma(T))$.

Proof. A slight modification of the proof of Lemma 2.26 shows that if T is isoloid then

$$f\big(\sigma(T) \setminus \pi_{00}(T)\big) = \sigma(f(T)) \setminus \pi_{00}(f(T)) \quad \text{for every } f \in \text{Hol}(\sigma(T)).$$

It thus follows from Theorem 1.31 and Corollary 2.31 that

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f\big(\sigma(T) \setminus \pi_{00}(T)\big) = f(\omega(T)) = \omega(f(T)),$$

which implies that Weyl's theorem holds for $f(T)$. □

Corollary 2.33. *If $T \in B(X)$ has the SVEP then*

$$\omega(f(T)) = f(\omega(T)) \quad \textit{for every } f \in \text{Hol}(\sigma(T)).$$

Proof. If $\lambda \notin \sigma_e(T)$ then by Lemma 2.12, $T - \lambda I$ has a finite ascent. Since if $S \in B(X)$ is Fredholm of finite ascent then index $(S) \leq 0$: indeed, either if S has finite descent then S is Browder and hence index $(S) = 0$, or if S does not have finite descent then

$$n\,\text{index}\,(S) = \dim\,\ker(S^n) - \dim\,(\text{ran}\,S^n)^{\perp} \;\to\; -\infty \quad \text{as } n \to \infty,$$

which implies that index $(S) < 0$. Thus we have that index $(T - \lambda I) \leq 0$. Thus T satisfies the condition (44), which gives the result. □

Theorem 2.34. *If $T \in B(X)$ satisfies*

$$\mathcal{X}_T(\{\lambda\}) = \ker(T - \lambda I) \quad \textit{for every } \lambda \in \mathbb{C},$$

then Weyl's theorem holds for $f(T)$ for every $f \in \text{Hol}(\sigma(T))$.

Proof. By Corollary 2.18, Weyl's theorem holds for T, T is isoloid, and T has the SVEP. In particular by Corollary 2.33, T satisfies the spectral mapping theorem for the Weyl spectrum. Thus the result follows from Corollary 2.32. □

2.3 Perturbation theorems

In this section we consider how Weyl's theorem survives under "small" perturbations. Weyl's theorem is transmitted from $T \in B(X)$ to $T - K$ for commuting nilpotents $K \in B(X)$ To see this we need:

Lemma 2.35. *If $T \in B(X)$ and if N is a quasinilpotent operator commuting with T then $\omega(T + N) = \omega(T)$.*

Proof. It suffices to show that if $0 \notin \omega(T)$ then $0 \notin \omega(T + N)$. Let $0 \notin \omega(T)$ so that $0 \notin \sigma(\pi(T))$. For all $\lambda \in \mathbb{C}$ we have $\sigma(\pi(T + \lambda N)) = \sigma(\pi(T))$. Thus $0 \notin \sigma(\pi(T + \lambda N))$ for all $\lambda \in \mathbb{C}$, which implies $T + \lambda N$ is a Fredholm operator forall $\lambda \in \mathbb{C}$. But since T is Weyl, it follows that $T + N$ is also Weyl, that is, $0 \notin \omega(T + N)$. □

Theorem 2.36. *Let $T \in B(X)$ and let N be a nilpotent operator commuting with T. If Weyl's theorem holds for T then it holds for $T+N$.*

Proof. We first claim that

$$\pi_{00}(T+N) = \pi_{00}(T). \tag{50}$$

Let $0 \in \pi_{00}(T)$ so that $\ker(T)$ is finite dimensional. Let $(T+N)x = 0$ for some $x \neq 0$. Then $Tx = -Nx$. Since T commutes with N it follows that

$$T^m x = (-1)^m N^m x \quad \text{for every } m \in \mathbb{N}. \tag{51}$$

Let n be the nilpotency of N, i.e., n be the smallest positive integer such that $N^n = 0$. Then by (51) we have that for some r with $1 \leq r \leq n$, $T^r x = 0$ and then $T^{r-1}x \in \ker(T)$. Thus $\ker(T+N) \subset \ker(T^{n-1})$. Therefore $\ker(T+N)$ is finite dimensional. Also if for some x $(\neq 0)$ $Tx = 0$ then $(T+N)^n x = 0$, and hence 0 is an eigenvalue of $T+N$. Again since $\sigma(T+N) = \sigma(T)$ it follows that $0 \in \pi_{00}(T+N)$. By symmetry $0 \in \pi_{00}(T+N)$ implies $0 \in \pi_{00}(T)$, which proves (50). Thus we have

$$\begin{aligned} \omega(T+N) &= \omega(T) \quad \text{(by Lemma 2.35)} \\ &= \sigma(T) \setminus \pi_{00}(T) \quad \text{(since Weyl's theorem holds for } T) \\ &= \sigma(T+N) \setminus \pi_{00}(T+N), \end{aligned}$$

which shows that Weyl's theorem holds for $T+N$. □

Theorem 2.36 however does not extend to quasinilpotents: let

$$Q : (x_1, x_2, x_3, \cdots) \mapsto (\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \cdots) \text{ on } \ell^2$$

and set on $\ell^2 \oplus \ell^2$,

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}. \tag{52}$$

Evidently K is quasinilpotent commutes with T: but Weyl's Theorem holds for T because

$$\sigma(T) = \omega(T) = \{0, 1\} \text{ and } \pi_{00}(T) = \emptyset, \tag{53}$$

while Weyl's Theorem does not hold for $T+K$ because

$$\sigma(T+K) = \omega(T+K) = \{0, 1\} \text{ and } \pi_{00}(T+K) = \{0\}. \tag{54}$$

But if K is an *injective* quasinilpotent operator commuting with T then Weyl's theorem is transmitted from T to $T+K$.

Theorem 2.37. *If Weyl's theorem holds for $T \in B(X)$ then Weyl's theorem holds for $T+K$ if $K \in B(X)$ is an injective quasinilpotent operator commuting with T.*

Proof. First of all we prove that if there exists an injective quasinilpotent operator commuting with T, then

$$T \text{ is Weyl} \implies T \text{ is injective}. \tag{55}$$

To show this suppose K is an injective quasinilpotent operator commuting with T. Assume to the contrary that T is Weyl but not injective. Then there exists a nonzero vector $x \in X$

such that $Tx = 0$. Then by the commutativity assumption, $TK^n x = K^n T x = 0$ for every $n = 0, 1, 2, \cdots$, so that $K^n x \in N(T)$ for every $n = 0, 1, 2, \cdots$. We now claim that $\{K^n x\}_{n=0}^{\infty}$ is a sequence of linearly independent vectors in X. To see this suppose $c_0 x + c_1 K x + \cdots + c_n K^n x = 0$. We may then write $c_n(K - \lambda_1 I) \cdots (K - \lambda_n I)x = 0$. Since K is an injective quasinilpotent operator it follows that $(K - \lambda_1 I) \cdots (K - \lambda_n I)$ is injective. But since $x \neq 0$ we have that $c_n = 0$. By an induction we also have that $c_{n-1} = \cdots = c_1 = c_0 = 0$. This shows that $\{K^n x\}_{n=0}^{\infty}$ is a sequence of linearly independent vectors in X. From this we can see $N(T)$ is infinite-dimensional, which contradicts to the fact that T is Weyl. This proves (55). From (55) we can see that if Weyl's theorem holds for T then $\pi_{00}(T) = \emptyset$. We now claim that $\pi_{00}(T + K) = \emptyset$. Indeed if $\lambda \in \pi_{00}(T + K)$, then $0 < \dim \ker(T + K - \lambda I) < \infty$, so that there exists a nonzero vector $x \in X$ such that $(T + K - \lambda I)x = 0$. But since K commutes with $T + K - \lambda I$, the same argument as in the proof of (55) with $T + K - \lambda I$ in place of T shows that $\ker(T + K - \lambda I)$ is infinite-dimensional, a contradiction. Therefore $\pi_{00}(T + K) = \emptyset$ and hence Weyl's theorem holds for $T + K$ because $\varpi(T) = \varpi(T + K)$ with $\varpi = \sigma, \omega$. $\square$

In Theorem 2.37, "quasinilpotent" cannot be replaced by "compact". For example consider the following operators on $\ell^2 \oplus \ell^2$:

$$T = \begin{pmatrix} 0 & & & & \\ \frac{1}{2} & 0 & & 0 & \\ & \frac{1}{3} & 0 & & \\ 0 & & \frac{1}{4} & 0 & \\ & & & \ddots & \ddots \end{pmatrix} \oplus I \quad \text{and} \quad K = \begin{pmatrix} 1 & & & & \\ & -\frac{1}{2} & & 0 & \\ & & -\frac{1}{3} & & \\ & 0 & & -\frac{1}{4} & \\ & & & & \ddots \end{pmatrix} \oplus Q,$$

where Q is an injective compact quasinilpotent operator on ℓ^2. Observe that Weyl's theorem holds for T, K is an injective compact operator, and $TK = KT$. But

$$\sigma(T + K) = \{0, 1\} = \omega(T + K) \quad \text{and} \quad \pi_{00}(T + K) = \{1\},$$

which says that Weyl's theorem does not hold for $T + K$.

On the other hand, Weyl's theorem for T is not sufficient for Weyl's theorem for $T + F$ with finite rank F. To see this, let $X = \ell^2$ and let $T, F \in B(X)$ be defined by

$$T(x_1, x_2, x_3, \cdots) = (0, x_1/2, x_2/3, \cdots)$$

and

$$F(x_1, x_2, x_3, \cdots) = (0, -x_1/2, 0, 0, \cdots).$$

since the point spectrum of T is empty it follows Weyl's theorem holds for T. Also F is a nilpotent operator. Since $0 \in \pi_{00}(T + F) \cap \omega(T + F)$, it follows that Weyl's theorem fails for $T + F$.

Lemma 2.38. *Let $T \in B(X)$. If $F \in B(X)$ is a finite rank operator then*

$$\dim \ker(T) < \infty \iff \dim \ker(T + F) < \infty.$$

Further if $TF = FT$ then

$$\operatorname{acc} \sigma(T) = \operatorname{acc} \sigma(T + F).$$

Proof. This follows from a straightforward calculation. $\square$

Theorem 2.39. *Let $T \in B(X)$ be an isoloid operator and let $F \in B(X)$ be a finite rank operator commuting with T. If Weyl's theorem holds for T then it holds for $T+F$.*

Proof. We have to show that $\lambda \in \sigma(T+F)\setminus\omega(T+F)$ if and only if $\lambda \in \pi_{00}(T+F)$. Without loss of generality we may assume that $\lambda = 0$. We first suppose that $0 \in \sigma(T+F)\setminus\omega(T+F)$ and thus $T+F$ is Weyl but not invertible. It suffice to show that $0 \in \operatorname{iso}\sigma(T+F)$. Since T is Weyl and Weyl's theorem holds for T, it follows that $0 \in \rho(T)$ or $0 \in \operatorname{iso}\sigma(T)$. Thus by Lemma 2.38, $0 \notin \operatorname{acc}\sigma(T+F)$. But since $T+F$ is not invertible we have that $0 \in \operatorname{iso}\sigma(T+F)$.

Conversely, suppose that $0 \in \pi_{00}(T+F)$. We want to show that $T+F$ is Weyl. By our assumption, $0 \in \operatorname{iso}\sigma(T+F)$ and $0 < \dim\ker(T+F) < \infty$. By Lemma 2.38, we have

$$0 \notin \operatorname{acc}\sigma(T) \quad \text{and} \quad \dim\ker(T) < \infty. \tag{56}$$

If T is invertible then it is evident that $T+F$ is Weyl. If T is not invertible then by the first part of (56) we have $0 \in \operatorname{iso}\sigma(T)$. But since T is isoloid it follows that T is not one-one, which together with the second part of (56) gives $0 < \dim\ker(T) < \infty$. Since Weyl's theorem holds for T it follows that T is Weyl and so is $T+F$. □

Example 2.40. *There exists an operator $T \in B(X)$ and a finite rank operator $F \in B(X)$ commuting with T such that Weyl's theorem holds for T but it does not hold for $T+F$.*

Proof. Define on $\ell^2 \oplus \ell^2$, $T := I \oplus S$ and $F = K \oplus 0$, where $S : \ell^2 \to \ell^2$ is an injective quasinilpotent operator and $F : \ell^2 \to \ell^2$ is defined by

$$F(x_1, x_2, x_3, \cdots) = (-x_1, 0, 0, \cdots).$$

Then F is of finite rank and commutes with T. It is easy to see that

$$\sigma(T) = \omega(T) = \{0, 1\} \quad \text{and} \quad \pi_{00}(T) = \emptyset,$$

which implies that Weyl's theorem holds for T. We however have

$$\sigma(T+F) = \omega(T+F) = \{0, 1\} \quad \text{and} \quad \pi_{00}(T+F) = \{0\},$$

which implies that Weyl's theorem fails for $T+F$. □

Theorem 2.39 may fail if "finite rank" is replaced by "compact". In fact Weyl's theorem may fail even if K is both compact and quasinilpotent: for example, take $T = 0$ and K the operator on ℓ_2 defined by $K(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots)$. We will however show that if "isoloid" condition is strengthened slightly then Weyl's theorem is transmitted from T to $T+K$ if K is either a compact or a quasinilpotent operator commuting with T. To see this we observe:

Lemma 2.41. *If $K \in B(X)$ is a compact operator commuting with $T \in B(X)$ then*

$$\pi_{00}(T+K) \subseteq \operatorname{iso}\sigma(T) \cup \rho(T).$$

Proof. See [HanL]. □

An operator $T \in B(X)$ will be said to be *finite-isoloid* if $\operatorname{iso}\sigma(T) \subseteq \pi_{0f}(T)$. Evidently finite-isoloid $\Rightarrow$ isoloid. The converse is not true in general: for example, take $T = 0$. In particular if $\sigma(T)$ has no isolated points then T is finite-isoloid. We now have:

Theorem 2.42. *Suppose $T \in B(X)$ is finite-isoloid. If Weyl's theorem holds for T then Weyl's theorem holds for $T+K$ if $K \in B(X)$ commutes with T and is either compact or quasinilpotent.*

Proof. First we assume that K is a compact operator commuting with T. Suppose Weyl's theorem holds for T. We first claim that with no restriction on T,

$$\sigma(T+K) \setminus \omega(T+K) \subseteq \pi_{00}(T+K). \tag{57}$$

For (57), it suffices to show that if $\lambda \in \sigma(T+K) \setminus \omega(T+K)$ then $\lambda \in \operatorname{iso} \sigma(T+K)$. Assume to the contrary that $\lambda \in \operatorname{acc} \sigma(T+K)$. Then we have that $\lambda \in \sigma_b(T+K) = \sigma_b(T)$, so that $\lambda \in \sigma_e(T)$ or $\lambda \in \operatorname{acc} \sigma(T)$. Remember that the essential spectrum and the Weyl spectrum are invariant under compact perturbations. Thus if $\lambda \in \sigma_e(T)$ then $\lambda \in \sigma_e(T+K) \subseteq \omega(T+K)$, a contradiction. Therefore we should have that $\lambda \in \operatorname{acc} \sigma(T)$. But since Weyl's theorem holds for T and $\lambda \notin \omega(T+K) = \omega(T)$, it follows that $\lambda \in \pi_{00}(T)$, a contradiction. This proves (57). For the reverse inclusion suppose $\lambda \in \pi_{00}(T+K)$. Then by Lemma 2.41, either $\lambda \in \operatorname{iso} \sigma(T)$ or $\lambda \in \rho(T)$. If $\lambda \in \rho(T)$ then evidently $T+K-\lambda I$ is Weyl, i.e., $\lambda \notin \omega(T+K)$. If instead $\lambda \in \operatorname{iso} \sigma(T)$ then $\lambda \in \pi_{00}(T)$ whenever T is finite-isoloid. Since Weyl's theorem holds for T, it follows that $\lambda \notin \omega(T)$ and hence $\lambda \notin \omega(T+K)$. Therefore Weyl's theorem holds for $T+K$.

Next we assume that K is a quasinilpotent operator commuting with T. Then by Lemma 2.35, $\varpi(T) = \varpi(T+Q)$ with $\varpi = \sigma, \omega$. Suppose Weyl's theorem holds for T. Then

$$\sigma(T+K) \setminus \omega(T+K) = \sigma(T) \setminus \omega(T) = \pi_{00}(T) \subseteq \operatorname{iso} \sigma(T) = \operatorname{iso} \sigma(T+K),$$

which implies that $\sigma(T+K) \setminus \omega(T+K) \subseteq \pi_{00}(T+K)$. Conversely, suppose $\lambda \in \pi_{00}(T+K)$. If T is finite-isoloid then $\lambda \in \operatorname{iso} \sigma(T+K) = \operatorname{iso} \sigma(T) \subseteq \pi_{0f}(T)$. Thus $\lambda \in \pi_{00}(T) = \sigma(T) \setminus \omega(T) = \sigma(T+K) \setminus \omega(T+K)$. This completes the proof. □

Corollary 2.43. *Suppose H is a Hilbert space and $T \in B(H)$ is p-hyponormal. If T satisfies one of the following:*

(i) $\operatorname{iso} \sigma(T) = \emptyset$;
(ii) *T has finite-dimensional eigenspaces,*

then Weyl's theorem holds for $T+K$ if $K \in B(H)$ is either compact or quasinilpotent and commutes with T.

Proof. Observe that each of the conditions (i) and (ii) forces p-hyponormal operators to be finite-isoloid. Since by Corollary 2.5 Weyl's theorem holds for p-hyponormal operators, the result follows at once from Theorem 2.42. □

In the perturbation theory the "commutative" condition looks so rigid. Without the commutativity, the spectrum can however undergo a large change under even rank one perturbations. In spite of it, Weyl's theorem may hold for (non-commutative) compact perturbations of "good" operators. We now give such a perturbation theorem. To do this we need:

Lemma 2.44. *If $N \in B(X)$ is a quasinilpotent operator commuting with $T \in B(X)$ modulo compact operators (i.e., $TN - NT \in K(X)$) then $\sigma_e(T+N) = \sigma_e(T)$ and $\omega(T+N) = \omega(T)$.*

Proof. Immediate from Lemma 2.35. ☐

Theorem 2.45. *Suppose $T \in B(X)$ satisfies the following:*

(i) *T is finite-isoloid;*
(ii) *$\sigma(T)$ has no "holes" (bounded components of the complement), i.e., $\sigma(T) = \eta\,\sigma(T)$;*
(iii) *$\sigma(T)$ has at most finitely many isolated points;*
(iv) *Weyl's theorem holds for T.*

If $K \in B(X)$ is either compact or quasinilpotent and commutes with T modulo compact operators then Weyl's theorem holds for $T+K$.

Proof. By Lemma 2.44, we have that $\sigma_e(T+K) = \sigma_e(T)$ and $\omega(T+K) = \omega(T)$. Suppose Weyl's theorem holds for T and $\lambda \in \sigma(T+K)\setminus\omega(T+K)$. We now claim that $\lambda \in \operatorname{iso}\sigma(T+K)$. Assume to the contrary that $\lambda \in \operatorname{acc}\sigma(T+K)$. Since $\lambda \notin \omega(T+K) = \omega(T)$, it follows from the punctured neighborhood theorem that $\lambda \notin \partial\,\sigma(T+K)$. Also since the set of all Weyl operators forms an open subset of $B(X)$, we have that $\lambda \in \operatorname{int}\big(\sigma(T+K)\setminus\omega(T+K)\big)$. Then there exists $\epsilon > 0$ such that $\{\mu \in \mathbb{C} : |\mu-\lambda| < \epsilon\} \subseteq \operatorname{int}\big(\sigma(T+K)\setminus\omega(T+K)\big)$, and hence $\{\mu \in \mathbb{C} : |\mu-\lambda| < \epsilon\} \cap \omega(T) = \emptyset$. But since

$$\partial\,\sigma(T+K)\setminus \operatorname{iso}\sigma(T+K) \subseteq \sigma_e(T+K) = \sigma_e(T),$$

it follows from our assumption that

$$\begin{aligned}\{\mu \in \mathbb{C} : |\mu-\lambda| < \epsilon\} &\subseteq \operatorname{int}\big(\sigma(T+K)\setminus\omega(T+K)\big)\\ &\subseteq \eta\,\big(\partial\,\sigma(T+K)\setminus\operatorname{iso}\sigma(T+K)\big)\\ &\subseteq \eta\,\sigma_e(T) \subseteq \eta\,\sigma(T) = \sigma(T),\end{aligned}$$

which implies that $\{\mu \in \mathbb{C} : |\mu-\lambda| < \epsilon\} \subseteq \sigma(T)\setminus\omega(T)$. This contradicts to Weyl's theorem for T. Therefore $\lambda \in \operatorname{iso}\sigma(T+K)$ and hence $\sigma(T+K)\setminus\omega(T+K) \subseteq \pi_{00}(T+K)$. For the reverse inclusion suppose $\lambda \in \pi_{00}(T+K)$. Assume to the contrary that $\lambda \in \omega(T+K)$ and hence $\lambda \in \omega(T)$. Then we claim $\lambda \notin \partial\,\sigma(T)$. Indeed if $\lambda \in \operatorname{iso}\sigma(T)$ then by assumption $\lambda \in \pi_{00}(T)$, which contradicts to Weyl's theorem for T. If instead $\lambda \in \operatorname{acc}\sigma(T) \cap \partial\,\sigma(T)$ then since $\operatorname{iso}\sigma(T)$ is finite it follows that

$$\lambda \in \operatorname{acc}\big(\partial\,\sigma(T)\big) \subseteq \operatorname{acc}\sigma_e(T) = \operatorname{acc}\sigma_e(T+K),$$

which contradicts to the fact that $\lambda \in \operatorname{iso}\sigma(T+K)$. Therefore $\lambda \notin \partial\,\sigma(T)$. Also since $\lambda \in \operatorname{iso}\sigma(T+K)$, there exists $\epsilon > 0$ such that

$$\{\mu \in \mathbb{C} : 0 < |\mu-\lambda| < \epsilon\} \subseteq \sigma(T) \cap \rho(T+K),$$

so that $\{\mu \in \mathbb{C} : 0 < |\mu-\lambda| < \epsilon\} \cap \omega(T) = \emptyset$, which contradicts to Weyl's theorem for T. Thus $\lambda \in \sigma(T+K)\setminus\omega(T+K)$ and therefore Weyl's theorem holds for $T+K$. ☐

If, in Theorem 2.45, the condition "$\sigma(T)$ has no holes" is dropped then Theorem 2.45 may fail even though T is normal. For example, if on $\ell_2 \oplus \ell_2$

$$T = \begin{pmatrix} U & I-UU^* \\ 0 & U^* \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & I-UU^* \\ 0 & 0 \end{pmatrix},$$

where U is the unilateral shift on ℓ_2, then T is unitary (essentially the bilateral shift) with $\sigma(T) = \mathbb{T}$, K is a rank one nilpotent, and Weyl's theorem does not hold for $T - K$.

Also in Theorem 2.45, the condition "iso $\sigma(T)$ is finite" is essential in the cases where K is compact. For example, if on ℓ_2

$$T(x_1, x_2, \cdots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots) \quad \text{and} \quad Q(x_1, x_2, \cdots) = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots),$$

we define $K := -(T + Q)$. Then we have that (i) T is finite-isoloid; (ii) $\sigma(T)$ has no holes; (iii) Weyl's theorem holds for T; (iv) iso $\sigma(T)$ is infinite; (v) K is compact because T and Q are both compact; (vi) Weyl's theorem does not hold for $T + K$ $(= -Q)$.

Corollary 2.46. *If $\sigma(T)$ has no holes and at most finitely many isolated points and if K is a compact operator then Weyl's theorem is transmitted from T to $T + K$.*

Proof. Immediate from Theorem 2.45. □

Corollary 2.46 shows that if Weyl's theorem holds for T whose spectrum has no holes and at most finitely many isolated points then for every compact operator K, the passage from $\sigma(T)$ to $\sigma(T+K)$ is putting at most countably many isolated points outside $\sigma(T)$ which are Riesz points of $\sigma(T + K)$. Here we should note that this holds even if T is quasinilpotent because for every quasinilpotent operator T (more generally, "Riesz operators"), we have

$$\sigma(T + K) \subseteq \eta\, \sigma_e(T + K) \cup p_{00}(T + K) = \eta\, \sigma_e(T) \cup p_{00}(T + K) = \{0\} \cup p_{00}(T + K).$$

2.4 Hyponormal operators

Recall that if H is a Hilbert space then an operator $A \in B(H)$ is called *hyponormal* if

$$[A^*, A] \equiv A^*A - AA^* \geq 0.$$

Thus if $A \in B(H)$ then

$$A \text{ is hyponormal} \iff \|Ah\| \geq \|A^*h\| \text{ for all } h \in H.$$

If $A^*A \leq AA^*$, or equivalently, $\|A^*h\| \geq \|Ah\|$ for all h, then A is called a *cohyponormal* operator. Operators that are either hyponormal or cohyponormal are called *seminormal*.

Proposition 2.47. *Let $A \in B(H)$ be a hyponormal operator. Then we have:*

(a) *If A is invertible then A^{-1} is hyponormal.*
(b) *$A - \lambda I$ is hyponormal for every $\lambda \in \mathbb{C}$.*
(c) *If $\lambda \in \pi_0(A)$ and $Af = \lambda f$ then $A^*f = \overline{\lambda} f$, i.e., $\ker(A - \lambda I) \subseteq \ker(A - \lambda I)^*$.*
(d) *If f and g are eigenvectors corresponding to distinct eigenvalues of A then $f \perp g$.*
(e) *If $\mathcal{M} \in \text{Lat}\, A$ then $A|_{\mathcal{M}}$ is hyponormal.*

Proof. (a) Recall that if T is positive and invertible then

$$T \geq 1 \Longrightarrow T^{-1} \leq 1 :$$

because if $T \in C^*(T) \equiv C(X)$ then $T = f \geq 1 \Rightarrow T^{-1} = \frac{1}{f} \leq 1$. Since $A^*A \geq AA^*$ and A is invertible,

$$\begin{aligned}
&A^{-1}(A^*A)(A^*)^{-1} \geq A^{-1}(AA^*)(A^*)^{-1} = 1 \\
&\Longrightarrow A^*A^{-1}(A^*)^{-1}A \leq 1 \\
&\Longrightarrow A^{-1}(A^*)^{-1} = (A^*)^{-1}(A^*A^{-1}A^{*-1}A)A^{-1} \leq (A^*)^{-1}A^{-1} \\
&\Longrightarrow A^{-1} \text{ is hyponormal.}
\end{aligned}$$

(b) $(A-\lambda I)(A^*-\overline{\lambda}I) = AA^*-\lambda A^*-\overline{\lambda}A+|\lambda|^2 I \leq A^*A-\lambda A^*-\overline{\lambda}A+|\lambda|^2 I = (A^*-\overline{\lambda}I)(A-\lambda I)$.

(c) Immediate from the fact that $\|(A^*-\overline{\lambda}I)f\| \leq \|(A-\lambda I)f\|$.

(d) $Af = \lambda f,\ Ag = \mu g \Rightarrow \lambda\langle f, g\rangle = \langle Af, g\rangle = \langle f, A^*g\rangle = \langle f, \overline{\mu}g\rangle = \mu\langle f, g\rangle$.

(e) If $\mathcal{M} \in \operatorname{Lat} A$ then

$$\begin{aligned}
&A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^\perp \end{matrix} \text{ is hyponormal} \\
&\Longrightarrow 0 \leq [A^*, A] = \begin{pmatrix} [B^*, B] - CC^* & * \\ * & * \end{pmatrix} \\
&\Longrightarrow [B^*, B] \geq CC^* \geq 0 \\
&\Longrightarrow B \text{ is hyponormal.}
\end{aligned}$$

□

Corollary 2.48. *If $A \in B(H)$ is hyponormal and $\lambda \in \pi_0(A)$ then $\ker(A-\lambda I)$ reduces A. Hence if A is a pure hyponormal then $\pi_0(A) = \emptyset$.*

Proof. From Proposition 2.47(c), if $f \in \ker(A-\lambda I)$ then $Af = \lambda f \in \ker(A-\lambda I)$ and $A^*f = \overline{\lambda}f \in \ker(A-\lambda I)$. □

Proposition 2.49. [Sta] *If $A \in B(H)$ is hyponormal then $\|A^n\| = \|A\|^n$, so*

$$\|A\| = \gamma(A), \text{ where } r(\cdot) \text{ denoted the spectral radius,}$$

in other words, A is normaloid.

Proof. Observe

$$\|A^n f\|^2 =< A^n f, A^n f >=< A^*A^n f, A^{n-1}f >\leq \|A^*A^n f\| \cdot \|A^{n-1}f\| \leq \|A^{n+1}f\| \cdot \|A^{n-1}f\|.$$

Hence $\|A^n\|^2 \leq \|A^{n+1}\| \cdot \|A^{n-1}\|$. We use an induction. Clearly, it is true for $n = 1$. Suppose $\|A^k\| = \|A\|^k$ for $1 \leq k \leq n$. Then $\|A\|^{2n} = \|A^n\|^2 \leq \|A^{n+1}\| \cdot \|A^{n-1}\| = \|A^{n+1}\| \cdot \|A\|^{n-1}$, so $\|A\|^{n+1} \leq \|A^{n+1}\|$. Also $r(A) = \lim \|A^n\|^{\frac{1}{n}} = \|A\|$. □

Corollary 2.50. *If $A \in B(H)$ is hyponormal and $\lambda \notin \sigma(A)$ then*

$$\frac{1}{\|(A-\lambda I)^{-1}\|} = \operatorname{dist}\big(\lambda, \sigma(A)\big).$$

Proof. Observe

$$||\frac{1}{(A-\lambda I)^{-1}}|| = \frac{1}{\max_{\mu\in\sigma(A-\lambda I)^{-1}}|\mu|} = \min_{\mu\in\sigma(A-\lambda I)}|\mu| = \operatorname{dist}\big(\lambda, \sigma(A)\big).$$

□

Proposition 2.51. [Sta] *If $A \in B(H)$ is hyponormal then A is isoloid, i.e.,* $\operatorname{iso}\sigma(A) \subseteq \pi_0(A)$. *The pure hyponormal operators have no isolated points in their spectrum.*

Proof. Replacing A by $A-\lambda I$ we may assume that $\lambda = 0$. Observe that the only quasinilpotent hyponormal operator is zero. Consider the spectral decomposition of A:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \text{ where } \sigma(A_1) = \{0\},\ \sigma(A_2) = \sigma(A)\backslash\{0\}.$$

Then $A_1 = 0$, so $0 \in \pi_0(A)$.

The second assertion comes from the fact that $\ker(A-\lambda I)$ is a reducing subspaces of a hyponormal operator A. □

Corollary 2.52. *The only compact hyponormal operator is normal.*

Proof. Recall that if K is compact then every nonzero point of $\sigma(K)$ is isolated. So if K is hyponormal then every eigenspaces reduces K and the restriction of K to each eigenspace is normal. Consider the restriction of K to the orthogonal complement of the span of all the eigenvectors. The resulting operator is hyponormal and quasinilpotent, and hence 0. Therefore K is normal. □

Proposition 2.53. *Let $A \in B(H)$ be a hyponormal operator. Then we have:*

(a) *A is invertible $\Longleftrightarrow$ A is right invertible.*
(b) *A is Fredholm $\Longleftrightarrow$ A is right Fredholm.*
(c) *$\sigma(A) = \sigma_r(A)$ and $\sigma_e(A) = \sigma_{re}(A)$.*
(d) *A is pure, $\lambda \in \sigma(A)\backslash\sigma_e(A) \Longrightarrow$* $\operatorname{index}(A-\lambda I) \le -1$.

Proof. (a) Observe that

$$\begin{aligned} A \text{ is right invertible} &\Longrightarrow \exists\, B \text{ such that } AB = 1 \\ &\Longrightarrow A \text{ is onto and hence } \ker A^* = (\operatorname{ran} A)^\perp = \{0\} \\ &\Longrightarrow \ker A = \{0\} \\ &\Longrightarrow A \text{ is invertible.} \end{aligned}$$

(b) Similar to (a).

(c) From (a) and (b).

(d) Observe that

$$
\begin{aligned}
A \text{ is pure hyponormal} &\Longrightarrow A-\lambda I \text{ is pure hyponormal}\\
&\Longrightarrow \ker(A-\lambda I)=\{0\} \text{ (by Proposition 2.51)}\\
&\Longrightarrow A-\lambda I \text{ is not onto since } \lambda\in\sigma(A)\\
&\Longrightarrow \operatorname{index}(A-\lambda I)=\dim(\ker(A-\lambda I))-\dim\big(\operatorname{ran}(A-\lambda I)^{\perp}\big)\\
&\qquad\qquad = -\dim\big(\operatorname{ran}(A-\lambda I)^{\perp}\big)\le -1.
\end{aligned}
$$

□

Write $\mathcal{F}$ denotes the set of Fredholm operators. We here give a direct proof showing that Weyl's theorem holds for hyponormal operators.

Proposition 2.54. *If $A\in B(H)$ is hyponormal then*

$$\sigma(A)\backslash\omega(A)=\pi_{00}(A),$$

where $\pi_{00}(A)=$ the set of isolated eigenvalues of finite multiplicity.

Proof. ($\Leftarrow$) If $\lambda\in\pi_{00}(A)$ then $\ker(A-\lambda I)$ reduces A. So

$$A=\lambda I\bigoplus B,$$

where I is the identity on a finite dimensional space, B is hyponormal and $\lambda\notin\sigma(B)$. So $\lambda\notin\omega(A)$.

($\Rightarrow$) Suppose $\lambda\in\sigma(A)\backslash\omega(A)$, and so $A-\lambda I$ not invertible, Fredholm with index $(A-\lambda I)=0$. We may assume $\lambda=0$. Since $A\in\mathcal{F}$ and $\operatorname{index}A=0$, it follows that 0 is an eigenvalue of finite multiplicity.

It remains to show that $0\in\operatorname{iso}\sigma(A)$. Observe that

$$\ker(A)\subseteq\ker(A^*)=(\operatorname{ran}A)^{\perp} \text{ and } 0=\operatorname{index}(A)=\dim\ker(A)-\dim\big(\operatorname{ran}A\big)^{\perp},$$

so that $\ker(A)=(\operatorname{ran}A)^{\perp}$. So

$$A=0\bigoplus B,$$

where B is invertible. Since $\sigma(A)=\{0\}\cup\sigma(B)$, 0 must be an isolated point of $\sigma(A)$. □

Corollary 2.55. *If $A\in B(H)$ is a pure hyponormal then*

$$\|A\|\le\|A+K\| \text{ for every compact operator } K.$$

Proof. Since A is pure, $\pi_0(A)=\emptyset$. So $\sigma(A)=\omega(A)=\bigcap_{K\in K(H)}\sigma(A+K)$. Thus for every compact operator K, $\sigma(A)\subseteq\sigma(A+K)$. Therefore, $\|A\|=r(A)\le r(A+K)\le\|A+K\|$. □

2.5 p-hyponormal operators

Recall that if H is a Hilbert space, the *numerical range* of $T \in B(H)$ is defined by

$$W(T) := \Big\{\langle Tx, x\rangle : \ ||x|| = 1\Big\}$$

and the *numerical radius* of T is defined by

$$w(T) := \sup\Big\{|\lambda| : \ \lambda \in W(T)\Big\}.$$

It was well-known (cf. [Ha3]) that

(a) $W(T)$ is convex (Toeplitz-Haussdorff theorem);
(b) $\text{conv}\,\sigma(T) \subset \text{cl}\,W(T)$;
(c) $r(T) \le w(T) \le ||T||$;
(d) $\frac{1}{\text{dist}\,(\lambda, \sigma(T))} \le ||(T-\lambda)^{-1}|| \le \frac{1}{\text{dist}\,(\lambda, \text{cl}\,W(T))}$.

Definition 2.56. (a) T is called *normaloid* if $||T|| = r(T)$;
(b) T is called *spectraloid* if $w(T) = r(T)$;
(c) T is called *convexoid* if $\text{conv}\,\sigma(T) = \text{cl}\,W(T)$;
(d) T is called *transaloid* if $T - \lambda I$ is normaloid for any λ;
(e) T is siad to satisfy (G_1)-condition if

$$||(T-\lambda I)^{-1}|| \le \frac{1}{\text{dist}\,(\lambda,\ \sigma(T))}, \quad \text{in fact, } ||(T-\lambda I)^{-1}|| = \frac{1}{\text{dist}\,(\lambda,\ \sigma(T))}.$$

(f) T is called *paranormal* if $||T^2x|| \ge ||Tx||^2$ for any x with $||x|| = 1$.

It was well-known that it T is paranormal then
(i) T^n is paranormal for any n;
(ii) T is normaloid;
(iii) T^{-1} is paranormal if it exists;
and that

$$\text{hyponormal} \subset \text{paranormal} \subset \text{normaloid} \subset \text{spectraloid}.$$

Theorem 2.57. *If $T \in B(H)$ then*

(a) *T is convexoid $\Longleftrightarrow$ $T - \lambda I$ is spectraloid for any λ, i.e., $w(T-\lambda I) = r(T-\lambda I)$;*

(b) *T is convexoid $\Longleftrightarrow$ $||(T-\lambda I)^{-1}|| \le \frac{1}{\text{dist}\,(\lambda,\ \text{conv}\,\sigma(T))}$ for any $\lambda \notin \text{conv}\,\sigma(T)$.*

Proof. (a) Note that

$$\begin{aligned}\text{conv}\,M &= \text{the intersection of all disks containing } M\\ &= \bigcap_{\mu}\Big\{\lambda : \ |\lambda-\mu| \le \sup_{x\in M}|x-\mu|\Big\}.\end{aligned}$$

Since $\operatorname{cl} W(T)$ is convex,

$$\operatorname{cl} W(T) = \bigcap_{\mu} \Big\{ \lambda : \; |\lambda - \mu| \le w(T - \mu I) \Big\};$$
$$\operatorname{conv} \sigma(T) = \bigcap_{\mu} \Big\{ \lambda : \; |\lambda - \mu| \le r(T - \mu I) \Big\}.$$

so the result immediately follows.

(b) ($\Rightarrow$) Clear from the preceding remark.

($\Leftarrow$) Suppose

$$||(T - \lambda I)^{-1}|| \le \frac{1}{\operatorname{dist}(\lambda, \; \operatorname{conv} \sigma(T))} \text{ for any } \lambda \notin \operatorname{conv} \sigma(T),$$

or equivalently,

$$||(T - \lambda I)\, x|| \ge \frac{1}{\operatorname{dist}(\lambda, \; \operatorname{conv} \sigma(T))} \text{ for any } \lambda \notin \operatorname{conv} \sigma(T) \text{ and } ||x|| = 1.$$

Thus

$$||Tx||^2 - 2 \operatorname{Re} \langle Tx, x \rangle \overline{\lambda} + |\lambda|^2 \ge \inf_{s \in \operatorname{conv} \sigma(T)} \Big(|s|^2 - 2 \operatorname{Re} s \overline{\lambda} + |\lambda|^2 \Big).$$

Taking $\lambda = |\lambda| e^{-i(\theta + \pi)}$, dividing by $|\lambda|$ and letting $\lambda \to \infty$, we have

$$\operatorname{Re} \langle Tx, x \rangle e^{i\theta} \;\ge \inf_{s \in \operatorname{conv} \sigma(T)} \operatorname{Re} \left(s e^{i\theta} \right) \quad \text{for } ||x|| = 1,$$

which implies $\operatorname{cl} W(T) \subset \operatorname{conv} \sigma(T)$. Therefore $\operatorname{cl} W(T) = \operatorname{conv} \sigma(T)$. □

Corollary 2.58. *We have:*

(a) transaloid $\Rightarrow$ convexoid;

(b) $(G_1) \Rightarrow$ convexoid.

Proof. (a) Clear.

(b) $||(T - \lambda I)^{-1}|| = \frac{1}{\operatorname{dist}(\lambda, \; \sigma(T))} \le \frac{1}{\operatorname{dist}(\lambda, \; \operatorname{conv} \sigma(T))}$ □

Definition 2.59. An operator $T \in B(H)$ for a Hilbert space H is said to satisfy the *projection property* if $\operatorname{Re} \sigma(T) = \sigma(\operatorname{Re} T)$, where $\operatorname{Re} T := \frac{1}{2}(T + T^*)$.

Theorem 2.60. *Let H be a Hilbert space. An operator $T \in B(H)$ is convexoid if and only if*

$$\operatorname{Re} \operatorname{conv} \sigma \left(e^{i\theta} T \right) = \operatorname{conv} \sigma \left(\operatorname{Re} (e^{i\theta} T) \right) \quad \textit{for any } \theta \in [0, 2\pi).$$

Proof. Observe that

$$\begin{aligned} \operatorname{Re} \left(e^{i\theta} \operatorname{conv} \sigma(T) \right) &= \operatorname{conv} \sigma \left(\operatorname{Re} (e^{i\theta} T) \right) \\ &= \operatorname{cl} W \left(\operatorname{Re} (e^{i\theta} T) \right) \\ &= \operatorname{Re} \operatorname{cl} W (e^{i\theta} T) \\ &= \operatorname{Re} \left(e^{i\theta} \operatorname{cl} W(T) \right). \end{aligned}$$

which implies that $\operatorname{conv} \sigma(T) = \operatorname{cl} W(T)$ and this argument is reversible. □

Example 2.61. There exist convexoid operators which are not normaloid and vice versa. (see [Ha2, Problem 219]).

Example 2.62. (An example of a non-convexoid and papranormal operator) Let U be the unilateral shift on ℓ^2, $P = \text{diag}(1, 0, 0, \ldots)$ and put

$$T = \begin{pmatrix} U + I & P \\ 0 & 0 \end{pmatrix}.$$

Then $\sigma(T) = \sigma(U + I) \cup \{0\} = \{\lambda : |\lambda - 1| \le 1\}$. But if $x = (-\frac{1}{2}, 0, 0, \ldots)$ and $y = (\frac{\sqrt{3}}{2}, 0, 0, \ldots)$ then $\left|\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|\right| = 1$ and

$$W(T) \ni \langle T(x \oplus y),\ x \oplus y \rangle = \frac{1}{4} - \frac{\sqrt{3}}{4} < 0.$$

Therefore T is not convexoid, but T is papranormal (see [Fur]).

Definition 2.63. [ChH] Let H be a Hilbert space. An operator $T \in B(H)$ is called *p-hyponormal* if

$$(T^*T)^p \ge (TT^*)^p.$$

If $p = 1$, T is hyponormal ad if $p = \frac{1}{2}$, T is called *seminormal*. It was known that q-hyponormal $\Rightarrow$ p-hyponormal for $p \le q$ by Löner-Heinz inequality.

Theorem 2.64. *p-hyponormal $\Longrightarrow$ paranormal.*

Proof. See [An]. □

It was also well-known that if $T \in B(H)$ is p-hyponormal then
(i) T is normaloid;
(ii) T is reduced by its eigenspaces;
(iii) T^{-1} is paranormal if it exists.

However p-hyponormal operators need not be transaloid. In fact, p-hyponormality is not translation-invariant. To see this we first recall:

Lemma 2.65. *If T is p-hyponormal then T^n is $\frac{p}{n}$-hyponormal for $0 < p \le 1$.*

Proof. See [AW]. □

Theorem 2.66. [ChL] *There exists an operator T satisfying*
(i) *T is semi-hyponormal;*
(ii) *$T - \lambda I$ is not p-hyponormal for any $p > 0$ and some $\lambda \in \mathbb{C}$.*

Proof. Let

$$S \equiv 4U^2 + U^{*2} + 2UU^* + 2I \quad (U = \text{the unilateral shift on } \ell^2).$$

Then we claim that

(a) S is semi-hyponormal;
(b) $S - 4I$ is not p-hyponormal for any $p > 0$, in fact $S - 4I$ is not paranormal.

Indeed, if we put $\varphi(z) = 2z + z^{-1}$ the T_φ is hyponormal but T_φ^2 is not because Since $T_\varphi^2 = S$, so S is semi-hyponormal. On the other hand, observe that

$$||(S-4I)e_0||^2 = 20 \quad \text{and} \quad ||(S-4I)^2 e_0|| = \sqrt{384},$$

so

$$||(S-4I)e_0||^2 > ||(S-4I)^2 e_0||,$$

which is not paranormal. □

2.6 Browder's theorem

One fall day of 1995, Robin Harte and I have traveled to Zion Canyon and Bryce Canyon in the Utah, USA. Two canyons had a similar feature but a subtle difference in our feeling. At that time, we have discussed about Weyl's theorem all the way in the car and recognized that Weyl's theorem is close to, but not quite the same as, equality between the Weyl spectrum and the "Browder spectrum", which in turn ought to, but does not, guarantee the spectral mapping theorem for the Weyl spectrum of polynomials in T. After a return from the trip, we have defined a new notion of the "Browder's theorem," which has a little different feature with the Weyl's theorem like a subtle difference between the Zion and the Bryce. This was first appeared in [HaL2] in 1997. Nowadays, this theorem has extensively studied by many authors.

We first recall that if $T \in B(X)$ then

$$\sigma_{\text{ess}}(T) \subseteq \omega_{\text{ess}}(T) \subseteq \sigma_b(T) = \sigma_{\text{ess}}(T) \cup \text{acc } \sigma(T) \tag{58}$$

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) \tag{59}$$

for the *Riesz points* of T and

$$\text{iso } \sigma(T) \setminus \sigma_{\text{ess}}(T) = \text{iso } \sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T). \tag{60}$$

We begin with:

Definition 2.67. We say that *Browder's theorem holds for T* if

$$\sigma(T) \setminus \omega(T) = p_{00}(T). \tag{61}$$

Evidently "Weyl's theorem" implies "Browder's theorem":

Theorem 2.68. *Each of the following conditions is equivalent to Browder's theorem for $T \in B(X)$:*

$$\sigma(T) = \omega(T) \cup \pi_{00}(T); \tag{62}$$

$$\omega(T) = \sigma_b(T). \tag{63}$$

Necessary and sufficient for Weyl's theorem is Browder's theorem together with either of the following:

$$\omega(T) \cap \pi_{00}(T) = \emptyset; \tag{64}$$

$$\pi_{00}(T) \subseteq p_{00}(T). \tag{65}$$

Proof. Implication (61) $\Longrightarrow$ (62) is the last part of (60). Conversely if (61) holds then $\sigma(T) \setminus \omega(T) = \pi_{00}(T) \setminus \omega(T) \subseteq p_{00}(T)$, giving (61). Equivalence (61) $\Longleftrightarrow$ (63) is (59). Implication (63) $\Longrightarrow$ (61) is the middle part of (60). Towards the second part of the theorem notice that (65) always implies (64): we claim that Browder's theorem together with (64) implies Weyl's theorem, and that Weyl's theorem implies (65). Indeed using the last part of (60) Browder's theorem says that the complement in $\sigma(T)$ of the Weyl spectrum is a subset of $\pi_{00}(T)$, while (64) ensures that $\pi_{00}(T)$ is a subset of this complement. On the other hand the second part of (60) together the inclusion $\pi_{00}(T) \subseteq \text{iso}\ \sigma(T)$ and Weyl's theorem gives (65). □

The disjointness condition (64) can fail whether or not Browder's theorem holds [Ob1]:

Example 2.69. If $X = \ell_p$ or $X = c_0$ and

$$T = vw : (x_1, x_2, x_3, \cdots) \mapsto (\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \cdots) \text{ on } X \tag{66}$$

is the product of the backward shift v and the standard weight w then

$$\sigma(T) = \sigma_{\text{ess}}(T) = \omega(T) = \sigma_b(T) = \{0\} \tag{67}$$

and

$$\pi_{00}(T) = \{0\}. \tag{68}$$

Proof. T is quasinilpotent, and compact so not Fredholm, giving (67), while

$$T^{-1}(0) = \mathbb{C}\delta_1 = \{(\lambda, 0, 0, \cdots) : \lambda \in \mathbb{C}\}$$

is of dimension 1. □

In general the spectral mapping theorem is liable to fail for the Weyl spectrum ([Be2, Example 3.3]): there is only inclusion, since the product of Weyl operators is Weyl,

$$\omega p(T) \subseteq p\omega(T). \tag{69}$$

Similarly the Weyl spectrum of a direct sum need not be the union of the Weyl spectra of the components: we only have in general, since the direct sum of Weyl operators is Weyl and the index additive on direct sums,

$$\omega(T) \setminus \omega(S) \subseteq \omega(S \oplus T) \subseteq \omega(S) \cup \omega(T). \tag{70}$$

By contrast ([Har4, Theorem 9.8.2]) the spectral mapping theorem holds for the Browder spectrum, and the Browder spectrum of a direct sum is the union of the Browder spectrum of the components. This might suggest that Browder's theorem for S and T is sufficient for equality in (69) and the second part of (70):

Theorem 2.70. *If Browder's theorem holds for $T \in B(X)$ and $S \in B(Y)$ and if p is a polynomial then*

$$\text{Browder's theorem holds for } p(T) \iff p(\omega(T)) \subseteq \omega(p(T)), \tag{71}$$

and

$$\text{Browder's theorem holds for } S \oplus T \iff \omega(S) \cup \omega(T) \subseteq \omega(S \oplus T). \tag{72}$$

Proof. If $\sigma_b(p(T)) \subseteq \omega(p(T))$ then, with no other restriction on T,

$$p(\omega(T)) \subseteq p(\sigma_b(T)) = \sigma_b(p(T)) \subseteq \omega(p(T)),$$

which is the right hand side of (71); conversely if Browder's theorem holds for T as well as this inclusion then $\sigma_b(p(T)) = p(\sigma_b(T)) \subseteq p(\omega(T)) \subseteq \omega(p(T))$. Similarly if Browder's theorem holds for $S \oplus T$ then, with no other restriction on either S or T,

$$\omega(S) \cup \omega(T) \subseteq \sigma_b(S) \cup \sigma_b(T) = \sigma_b(S \oplus T) \subseteq \omega(S \oplus T),$$

which is the right hand side of (72); conversely if Browder's theorem holds for S and for T as well as this inclusion then $\sigma_b(S \oplus T) = \sigma_b(S) \cup \sigma_b(T) \subseteq \omega(S) \cup \omega(T) \subseteq \omega(S \oplus T)$. □

We have a familiar example of an operator for which the spectral mapping theorem holds for the Weyl spectrum, which does not coincide with the Browder spectrum:

Example 2.71. *If*

$$T = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : \begin{pmatrix} Y \\ Y \end{pmatrix} \to \begin{pmatrix} Y \\ Y \end{pmatrix} \tag{73}$$

with $Y = \ell_p$ or $Y = c_0$ and the forward and backward shifts u and v, and if $|\lambda| < 1$, then

$$T - \lambda I \text{ is Weyl and not Browder}, \tag{74}$$

but there is inclusion

$$p(\omega(T)) \subseteq \omega(p(T)) \text{ for each polynomial } p. \tag{75}$$

At the same time Browder's theorem holds for each of u and v, but not for $T = u \oplus v$.

Proof. It is clear that $T - \lambda I$ is Fredholm and has index zero, therefore is Weyl; alternatively (cf. [Har4, (7.7.6.6)-(7.7.6.9)]) if

$$S = \begin{pmatrix} u - \lambda & 1 - uv \\ 0 & v - \lambda \end{pmatrix} \text{ and } S' = \begin{pmatrix} v(1 - \lambda v)^{-1} & 0 \\ (1 - \lambda u)^{-1}(1 - uv)(1 - \lambda v)^{-1} & (1 - \lambda u)^{-1}u \end{pmatrix} \tag{76}$$

then

$$S'S = I = SS' \text{ and } T - \lambda I - S \text{ is finite rank}; \tag{77}$$

note to make the calculations

$$(1-\lambda v)^{-1}(u-\lambda) = u\ ;\ (v-\lambda)(1-\lambda u)^{-1} = v\ ;\ (1-\lambda v)^{-1}(1-uv) = 1-uv = (1-uv)(1-\lambda u)^{-1}.$$

To see that $T - \lambda I$ is not Browder recall the eigenvector

$$\delta_1 = (1, 0, 0, \cdots) \in v^{-1}(0) :$$

we claim

$$y = u^n(1-\lambda u)^{-(n+1)}\delta_1 \Longrightarrow (v-\lambda)^{n+1}y = 0 \neq (v-\lambda)^n y,$$

noting that

$$(v-\lambda)^n(1-\lambda u)^{-n} = v^n,$$

and hence

$$x = \begin{pmatrix} 0 \\ y \end{pmatrix} \Longrightarrow (T-\lambda I)^{n+1}x = 0 \neq (T-\lambda I)^n x.$$

Since there is only one hole in the essential spectrum it follows that T satisfies the condition (5.34) and hence (75). □

We have a very similar example in the opposite direction:

Example 2.72. *If $Y = \ell_p$ or $Y = c_0$ and*

$$T = \begin{pmatrix} u+1 & 0 \\ 0 & v-1 \end{pmatrix} : \begin{pmatrix} Y \\ Y \end{pmatrix} \to \begin{pmatrix} Y \\ Y \end{pmatrix} \tag{78}$$

with the forward and backward shifts u and v on Y, then Browder's theorem holds for T while the spectral mapping theorem for the Weyl spectrum fails.

Proof. We claim ([Har4, (7.6.4.9)]), using the first part of (70),

$$\sigma(T) = \omega(T) = \{|1-z| \leq 1\} \cup \{|1+z| \leq 1\},$$

since both the spectrum and the Weyl spectrum of each of the shifts is the closed unit disc. Thus Browder's theorem certainly holds for T; to see the failure of the spectral mapping theorem with the polynomial $p = z^2$ observe

$$1 \in p\omega(T) \supseteq \sigma_{\text{ess}}p(T) \text{ and } 1 \notin \omega p(T) = \omega(T^2) :$$

for this last part observe that $\text{index}(T-I) = -1 = -\text{index}(T+I)$, or alternatively make a direct calculation ([Har4, (7.6.4.13)]) as for Example 2.71. □

Weyl's theorem may or may not hold for quasinilpotent operators, and is not transmitted to or from dual operators: for example it fails for the quasinilpotent $T = vw$ of Example 2.69, but holds for its adjoint $T^* = wu$: if $T = vw$ on ℓ_2 then

$$\sigma(T^*) = \omega(T^*) = \{0\} \text{ and } \pi_{00}(T^*) = \emptyset. \tag{79}$$

Once again Browder's theorem performs better:

Theorem 2.73. *If $T \in B(X)$ then*

$$\textit{Browder's theorem holds for } T \Longleftrightarrow \textit{Browder's theorem holds for } T^*. \tag{80}$$

Proof. Observe that

$$\omega(T^*) = \omega(T) \text{ and iso } \sigma(T^*) = \text{iso } \sigma(T), \tag{81}$$

which together with (60) and (63) gives (80). □

Combining (68) and (79) shows that the Riesz points need not coincide with the intersection of the isolated eigenvalues of finite multiplicity for the operator and its dual:

$$T = vw \oplus wu \Longrightarrow \pi_{00}(T) \cap \pi_{00}(T^*) = \{0\} \neq \pi_{00}(T) = \emptyset. \tag{82}$$

Theorem 2.74. *Necessary and sufficient for Browder's theorem to hold for $T \in B(X)$ is that*

$$acc\ \sigma(T) \subseteq \omega(T). \tag{83}$$

Hence in particular Browder's theorem holds for quasinilpotent operators, compact operators and algebraic operators.

Proof. If (83) holds then

$$\sigma(T) \setminus \omega(T) \subseteq \text{iso}\ \sigma(T),$$

giving Browder's theorem by (60); the converse is (58). If $\sigma(T)$ consists of isolated points then T satisfies (83); thus Browder's theorem holds for quasinilpotents, algebraic operators and compact operators with finite spectrum. For general compact operators (more generally, "Riesz operators"), we have (in infinite dimensions)

$$\text{acc}\ \sigma(T) \subset \{0\} \subseteq \sigma_{\text{ess}}(T), \tag{84}$$

giving again (83). □

An example of Berberian shows that on a Hilbert space H it is not sufficient, for Weyl's theorem for $T \in B(H)$, that T is reduced by its finite dimensional eigenspaces ([Be2, Example 1]): take $T = T_1 \oplus T_2$, where T_1 is the one-dimensional zero operator and T_2 is an injective quasinilpotent compact operator. This condition is however sufficient for Browder's theorem:

Theorem 2.75. *If H is a Hilbert space and $T \in B(H)$ is reduced by its finite dimensional eigenspaces then Browder's theorem holds for T.*

Proof. If T is reduced by its finite dimensional eigenspaces then $T = T_1 \oplus T_2$ with

$$T_1 \text{ is normal and } \omega(T_2) = \sigma(T_2) : \tag{85}$$

for take ([Be1, Example 5])

$$X_2^\perp = X_1 = \sum_{\lambda \in \Lambda} (T - \lambda I)^{-1}(0)$$

to be the sum of the (not necessarily isolated) eigenvalues of finite multiplicity. Evidently both the condition

$$\omega(T) = \sigma_e(T)$$

and Browder's theorem hold for each of T_1 and T_2. □

Theorem 2.75 shows that Browder's theorem holds for hyponormal operators, since hyponormal operators are reduced by their eigenspaces.

As we have seen in Theorem 2.30, Weyl's theorem is transmitted from $T \in B(X)$ to $T-K$ for commuting nilpotents $K \in B(X)$. This however does not extend to quasinilpotents (cf. the example above Theorem 2.37) Browder's theorem behaves better, at least for commuting perturbations:

Theorem 2.76. *If Browder's theorem holds for $T \in B(X)$ then Browder's theorem holds for $T + K$ if K commutes with T and is either quasinilpotent or compact.*

Proof. For the first part recall the argument of Oberai ([Ob2, Lemma 2]): if K is quasinilpotent and commutes with a Weyl operator T then $0 \notin \sigma_{\text{ess}}(T + \lambda K)$ for arbitrary $\lambda \in \mathbb{C}$, which by index continuity forces $T + \lambda K$ to have index zero for all $\lambda \in \mathbb{C}$, in particular with $\lambda = 1$. Thus if K is quasinilpotent and commutes with T then

$$\omega(T + K) = \omega(T). \tag{86}$$

It is also clear that, for the same K,

$$\sigma(T + K) = \sigma(T) \text{ and } \sigma_e(T + K) = \sigma_e(T) : \tag{87}$$

and hence also the accumulation points of the spectrum coincide. By (58) it follows that also

$$\sigma_b(T + K) = \sigma_b(T) \tag{88}$$

whenever K is quasinilpotent and commutes with T. If instead K is a commuting compact remember that the Weyl spectrum is invariant under compact purturbations, giving again (86), while the Browder spectrum is invariant under commuting compact perturbations giving (88). □

This may fail if K is not assumed to commute with T, even if K both compact and nilpotent:

Example 2.77. *If*

$$T = \begin{pmatrix} u & 1 - uv \\ 0 & v \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 1 - uv \\ 0 & 0 \end{pmatrix} \tag{89}$$

then K is a rank one nilpotent, T is unitary, and Browder's theorem does not hold for $T - K$.

Proof. It is clear that K is both rank one and square zero. The operator T is unitary (essentially the bilateral shift), so that Weyl's theorem holds for T; we saw in Example 2.71 that Browder's theorem fails for $T - K$. □

2.7 Concluding remarks and open problems

(a) Transaloid and SVEP. For an operator $T \in B(H)$ for a Hilbert space H, denote

$$W(T) = \{(Tx, x) : \ ||x|| = 1\}$$

for the numerical range of T and

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$$

for the numerical radius of T. An operator T is called *convexoid* if $\operatorname{conv}\sigma(T) = \operatorname{cl} W(T)$ and is called *spectraloid* if $w(T) = r(T) =$ the spectral radius. We call an operator $T \in B(H)$ *transaloid* if $T - \lambda I$ is normaloid for all $\lambda \in \mathbb{C}$. It was well known that

$$\text{transaloid} \implies \text{convexoid} \implies \text{spectraloid},$$
$$(G_1) \implies \text{convexoid} \quad \text{and} \quad (G_1) \implies \text{reguloid}.$$

We would like to expect that Corollary 2.16 remains still true if "reguloid" is replaced by "transaloid"

Problem 2.78. *If $T \in B(H)$ is transaloid and has the SVEP, does Weyl's theorem hold for T ?*

The following question is a strategy to answer Problem 2.78.

Problem 2.79. *Does it follow that*

$$\textit{transaloid} \implies \textit{reguloid}?$$

If the answer to Problem 2.79 is affirmative then the answer to Problem 2.78 is affirmative by Corollary 2.16.

(b) $*$-paranormal operators. An operator $T \in B(H)$ for a Hilbert space H is said to be $*$-*paranormal* if

$$||T^*x||^2 \le ||T^2x||\,||x|| \quad \text{for every } x \in H.$$

It was [AT] known that if $T \in B(H)$ is $*$-paranormal then the following hold:

$$T \text{ is normaloid;} \tag{90}$$

$$\ker(T - \lambda I) \subset \ker(T - \lambda I)^*. \tag{91}$$

So if $T \in B(H)$ is $*$-paranormal then by (91), $T - \lambda I$ has finite ascent for every $\lambda \in \mathbb{C}$. Thus $*$-paranormal operators have the SVEP ([La]). On the other hand, we can easily show that if $T \in B(H)$ is $*$-paranormal then

$$\sigma(T) \setminus \omega(T) \subset \pi_{00}(T). \tag{92}$$

However we were unable to decide:

Problem 2.80. *Does Weyl's theorem hold for $*$-paranormal operators?*

The following question is a strategy to answer Problem 2.80.

Problem 2.81. *Is every $*$-paranormal operator isoloid?*

If the answer to Problem 2.81 is affirmative then the answer to Problem 2.80 is affirmative. To see this suppose $T \in B(H)$ is $*$-paranormal. In view of (92), it suffices to show that $\pi_{00}(T) \subseteq \sigma(T) \setminus \omega(T)$. Assume $\lambda \in \pi_{00}(T)$. By (185), $T - \lambda I$ is reduced by its eigenspaces. Thus we can write

$$T - \lambda I = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} : \begin{pmatrix} \ker(T-\lambda I) \\ (\ker(T-\lambda I))^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} \ker(T-\lambda I) \\ (\ker(T-\lambda I))^\perp \end{pmatrix}.$$

Thus $T = \left(\begin{smallmatrix} \lambda I & 0 \\ 0 & S+\lambda I \end{smallmatrix}\right)$. We now claim that S is invertible. Assume to the contrary that S is not invertible. Then $0 \in \operatorname{iso}\sigma(S)$ since $\lambda \in \operatorname{iso}\sigma(T)$. Thus $\lambda \in \operatorname{iso}\sigma(S+\lambda I)$. But since $S+\lambda I$ is also $*$-paranormal, it follows from our assumption that λ is an eigenvalue of $S+\lambda I$. Thus $0 \in \pi_0(S)$, which contradicts to the fact that S is one-one. Therefore S should be invertible. Note that $\ker(T-\lambda I)$ is finite-dimensional. Thus evidently $T - \lambda I$ is Weyl, so that $\lambda \in \sigma(T) \setminus \omega(T)$. This gives a proof.

(c) Subclasses of paranormal operators. An operator $T \in B(H)$ for a Hilbert space H is said to be *quasihyponormal* if $T^*(T^*T - TT^*)T \geq 0$ and is said to be *class A*-operator if $|T^2| \geq |T|^2$ (cf. [FIY]). Let $T = U\,|T|$ be the polar decomposition of T and $\widetilde{T} := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ be the Aluthge transformation of T (cf. [Al]). An operator $T \in B(H)$ for a Hilbert space H is called *w-hyponormal* if $|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|$ (cf. [AW]). It was well known that

$$\text{hyponormal} \Longrightarrow \text{quasihyponormal} \Longrightarrow \text{class } A \Longrightarrow \text{paranormal} \tag{93}$$

$$\text{hyponormal} \Longrightarrow p\text{-hyponormal} \Longrightarrow w\text{-hyponormal} \Longrightarrow \text{paranormal.} \tag{94}$$

Since by Theorem 2.20, Weyl's theorem holds for paranormal operators on an arbitrary Banach space, all classes of operators in (93) and (94) enjoy Weyl's theorem.

The following problem on p-hyponormal operators remains still open:

Problem 2.82. (a) *Is every p-hyponormal operator convexoid?*
(b) *Does every p-hyponormal operator satisfy the (G_1)-condition?*
(c) *Does every p-hyponormal operator satisfy the projection property?*

In fact,

$$\text{Yes to (b)} \Longrightarrow \text{Yes to (a)} \Longrightarrow \text{Yes to (c).}$$

It was known that the projection property holds for every hyponormal operator. For a proof, see [Put].

For a partial answer, see [CHKL].

It is easily check that every p-hyponormal weighted shift is hyponormal. However we were unable to answer the following:

Problem 2.83. *Is every p-hyponormal Toeplitz operator hyponormal ?*

We conclude with a problem of hyponormal operators with finite rank self-commutators. In general it is quite difficult to determine the subnormality of an operator by definition. An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in H$ ([Bra], [Con1, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1\ ). \tag{95}$$

Condition (95) provides a measure of the gap between hyponormality and subnormality. An operator $T \in B(H)$ is called *k-hyponormal* if the $(k+1) \times (k+1)$ operator matrix in (95) is positive; the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([CMX]). It now seems to be interesting to consider the following problem:

$$\text{Which 2-hyponormal operators are subnormal?} \tag{96}$$

The first inquiry involves the self-commutator. The self-commutator of an operator plays an important role in the study of subnormality. B. Morrel [Mor] showed that a pure subnormal operator with rank-one self-commutator (pure means having no normal summand) is unitarily equivalent to a linear function of the unilateral shift. Morrel's theorem can be essentially stated (also see [Con2, p.162]) that if

$$\begin{cases} \text{(i) } T \text{ is hyponormal;} \\ \text{(ii) } [T^*, T] \text{ is of rank-one; and} \\ \text{(iii) } \ker [T^*, T] \text{ is invariant for } T, \end{cases} \tag{97}$$

then $T - \beta$ is quasinormal for some $\beta \in \mathbb{C}$. Now remember that every pure quasinormal operator is unitarily equivalent to $U \otimes P$, where U is the unilateral shift and P is a positive operator with trivial kernel. Thus if $[T^*, T]$ is of rank-one (and hence so is $[(T-\beta)^*, (T-\beta)]$), we must have $P \cong \alpha\ (\neq 0) \in \mathbb{C}$, so that $T - \beta \cong \alpha U$, or $T \cong \alpha U + \beta$. It would be interesting (in the sense of giving a simple sufficiency for the subnormality) to note that Morrel's theorem gives that if T satisfies the condition (97) then T is subnormal. On the other hand, it was shown ([CuL, Lemma 2.2]) that if T is 2-hyponormal then $T(\ker [T^*, T]) \subseteq \ker [T^*, T]$. Therefore by Morrel's theorem, we can see that

$$\text{every 2-hyponormal operator with rank-one self-commutator is subnormal.} \tag{98}$$

On the other hand, M. Putinar [Pu2] gave a matricial model for the hyponormal operator $T \in B(H)$ with finite rank self-commutator, in the cases where

$$H_0 := \bigvee_{k=0}^{\infty} T^{*k}\big(\text{ran}\, [T^*, T]\big) \text{ has finite dimension } d \quad \text{and} \quad H = \bigvee_{n=0}^{\infty} T^n H_0.$$

In this case, if we write

$$H_n := G_n \ominus G_{n-1} \quad (n \geq 1) \quad \text{and} \quad G_n := \bigvee_{k=0}^{n} T^k H_0 \quad (n \geq 0),$$

then T has the following two-diagonal structure relative to the decomposition $H = H_0 \oplus H_1 \oplus \cdots$:

$$T = \begin{pmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{99}$$

where

$$\begin{cases} \dim(H_n) = \dim(H_{n+1}) = d \quad (n \geq 0); \\ [T^*, T] = ([B_0^*, B_0] + A_0^* A_0) \oplus 0_\infty; \\ [B_{n+1}^*, B_{n+1}] + A_{n+1}^* A_{n+1} = A_n A_n^* \quad (n \geq 0); \\ A_n^* B_{n+1} = B_n A_n^* \quad (n \geq 0). \end{cases} \tag{100}$$

We will refer the operator (99) to the *Putinar's matricial model of rank d.* This model was also introduced in [GuP], [Pu1], [Xi], [Ya1], and etc.

We then have

Theorem 2.84. *Let $T \in B(H)$. If*

(i) *T is a pure hyponormal operator;*
(ii) *$[T^*, T]$ is of rank-two; and*
(iii) *$\ker [T^*, T]$ is invariant for T,*

then the following hold:

1. *If $T|_{\ker [T^*,T]}$ has the rank-one self-commutator then T is subnormal;*
2. *If $T|_{\ker [T^*,T]}$ has the rank-two self-commutator then T is either a subnormal operator or the Putinar's matricial model (99) of rank two.*

Proof. See [LeL2]. □

Since the operator (99) can be constructed from the pair of matrices $\{A_0, B_0\}$, we know that the pair $\{A_0, B_0\}$ is a complete set of unitary invariants for the operator (99). Many authors used the following Xia's unitary invariants $\{\Lambda, C\}$ to describe pure subnormal operators with finite rank self-commutators:

$$\Lambda := \left(T^*|_{\operatorname{ran}[T^*,T]}\right)^* \quad \text{and} \quad C := [T^*, T]|_{\operatorname{ran}[T^*,T]}.$$

Consequently,

$$\Lambda = B_0 \quad \text{and} \quad C = [B_0^*, B_0] + A_0^2.$$

We know that given Λ and C (or equivalently, A_0 and B_0) corresponding to a pure subnormal operator we can reconstruct T. Now the following question naturally arises: "what are the restrictions on matrices A_0 and B_0 such that they represent a subnormal operator?" In the cases where A_0 and B_0 operate on a finite dimensional Hilbert space, D. Yakubovich

[Ya1] showed that such a description can be given in terms of a topological property of a certain algebraic curve, associated with A_0 and B_0. However there is a subtle difference between Yakubovich's criterion and the Putinar's model operator (99). In fact, in some sense, Yakubovich gave conditions on A_0 and B_0 such that the operator (99) can be constructed so that the condition (100) is satisfied. By comparison, the Putinar's model operator (99) was already constructed so that it satisfies the condition (100). Thus we would guess that if the operator (99) can be constructed so that the condition (100) is satisfied then two matrices $\{A_0, B_0\}$ in (99) must satisfy the Yakubovich's criterion. In this viewpoint, we have the following:

Conjecture 2.85. *The Putinar's matricial model* (99) *of rank two is subnormal.*

An affirmative answer to Conjecture 2.85 would show that if T is a hyponormal operator with rank-two self-commutator and satisfying that ker $[T^*, T]$ is invariant for T then T is subnormal.

3 Upper triangles

We may ask whether a property is transmitted from the diagonals to the upper triangles. For examples, Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, when does it hold for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$? We explore the passage from the diagonals to the upper triangles for various spectra and Weyl's theorem.

3.1 Spectra of upper triangles

In this section we consider spectra of upper triangular operator matrices.

Recall [Har4] that $T \in B(X, Y)$ is called regular if there is an operator $T' \in B(Y, X)$ for which

$$T = TT'T; \tag{101}$$

then T' is called a generalized inverse for T. If $T \in B(X, Y)$ is left or right invertible, then, evidently, T is regular: in this case a left or right inverse is just a generalized inverse. Also if $T \in B(X, Y)$ is regular with a generalized inverse T', then X and Y can be decomposed as follows (cf. [Har4]):

$$T^{-1}(0) \oplus T'T(X) = X \quad \text{and} \quad T(X) \oplus (TT')^{-1}(0) = Y \tag{102}$$

It is familiar with that if X and Y are Hilbert spaces and $T \in B(X, Y)$ then T is regular if and only if T has closed range. An operator$T \in B(X, Y)$ is called *relatively Weyl* if there is an invertible operator $T' \in B(Y, X)$ for which $T = TT'T$. It is known ([Har4], Theorem 3.8.6) that $T \in B(X, Y)$ is relatively Weyl if and only if T is regular and $T^{-1}(0) \cong Y/TX$.

When $A \in B(X)$ and $B \in B(Y)$ are given we denote by M_C an operator acting on $X \oplus Y$ of the form

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{103}$$

where $C \in B(Y, X)$.

Recall that a sequence of module-homomorphisms $f_j : \mathcal{A}_{j-1} \to \mathcal{A}_j$ and

$$\mathcal{A}_0 \to \mathcal{A}_1 \to \mathcal{A}_2 \to \cdots \to \mathcal{A}_{n-1} \to \mathcal{A}_n$$

is said to be *exact* if $\operatorname{ran} f_i = \ker f_{i+1}$ $(i = 1, \cdots, n-1)$.

We begin with:

Lemma 3.1. *Suppose $T_j : \mathcal{A}_{j-1} \to \mathcal{A}_j$ and the sequence*

$$0 \to \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_n \to 0 \tag{104}$$

is an exact sequence of Banach spaces. If each T_i $(1 \le i \le n)$ is regular then

$$\begin{cases} \bigoplus_{i=0}^{\frac{n}{2}} \mathcal{A}_{2i} \cong \bigoplus_{i=0}^{\frac{n}{2}-1} \mathcal{A}_{2i+1} & (n \text{ even}) \\ \bigoplus_{i=0}^{\frac{n-1}{2}} \mathcal{A}_{2i} \cong \bigoplus_{i=0}^{\frac{n-1}{2}} \mathcal{A}_{2i+1} & (n \text{ odd}). \end{cases} \tag{105}$$

Hence in particular if the sequence (104) *is an exact sequence of Hilbert spaces then (105) holds.*

Proof. If $T_j = T_j T_j' T_j$ with $T_j' \in B(\mathcal{A}_j, \mathcal{A}_{j-1})$ $(1 \le j \le n)$, then each space $\mathcal{A}_j$ can be decomposed as follows:

$$\mathcal{A}_{j-1} = T_j^{-1}(0) \oplus T_j' T_j(\mathcal{A}_{j-1}) \quad (1 \le j \le n+1), \tag{106}$$

where $T_{n+1} : \mathcal{A}_n \to 0$ is the zero operator. Since the given sequence is exact, we have that $T_j(\mathcal{A}_{j-1}) = T_{j+1}^{-1}(0)$ $(1 \le j \le n)$. Since the restriction $T_j^{\#}$ of T_j to $T_j' T_j(\mathcal{A}_{j-1})$ is one-one and $T_j^{\#}(T_j' T_j(\mathcal{A}_{j-1})) = T_j(\mathcal{A}_{j-1})$, it follows that $T_j^{\#} : T_j' T_j(\mathcal{A}_{j-1}) \to T_{j+1}^{-1}(0)$ is an isomorphism, i.e.,

$$T_j' T_j(\mathcal{A}_{j-1}) \cong T_{j+1}^{-1}(0) \quad (1 \le j \le n). \tag{107}$$

Now (105) follows from (106) and (107). The second assertion follows from the first together with the observation that the exactness of the sequence gives that T_j has closed range, so that T_j is regular for $1 \le j \le n$. □

From Lemma 1 we can see that if $0 \to \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots \to \mathcal{A}_n \to 0$ is an exact sequence of finite dimensional spaces then $\sum_{i=0}^{n} (-1)^i \dim(A_i) = 0$ (cf. [Ya2], Theorem A.6).

We now have:

Theorem 3.2. *Suppose X, Y, Z are Banach spaces. If $T : X \to Y$, $S : Y \to Z$ and $ST : X \to Z$ are regular then there is isomorphism*

$$T^{-1}(0) \oplus S^{-1}(0) \oplus Z/ST(X) \;\cong\; (ST)^{-1}(0) \oplus Y/T(X) \oplus Z/S(Y). \tag{108}$$

Proof. From the "one-diagram" proof of the index theorem due to Yang [Ya1], we can see that the sequence

$$0 \to T^{-1}(0) \to (ST)^{-1}(0) \to S^{-1}(0) \to Y/T(X) \to Z/ST(X) \to Z/S(Y) \to 0 \tag{109}$$

is exact. For a concrete representation of T_j $(1 \le j \le 5)$, we give

$T_1 :=$ the natural injection from $T^{-1}(0)$ to $(ST)^{-1}(0)$;

$T_2 :=$ the restriction of T to $(ST)^{-1}(0)$;

$T_3 : y \mapsto y + T(X)$ for each $y \in S^{-1}(0)$;

$T_4 : y + T(X) \mapsto Sy + ST(X)$ for each $y \in Y$;

$T_5 : z + ST(X) \mapsto z + S(Y)$ for each $z \in Z$:

then we have

$$\begin{aligned}\ker T_1 &= \{0\};\\ \operatorname{ran} T_1 &= T^{-1}(0) = \ker T_2;\\ \operatorname{ran} T_2 &= S^{-1}(0) \cap T(X) = \ker T_3;\\ \operatorname{ran} T_3 &= S^{-1}(0) + T(X) = \ker T_4;\\ \operatorname{ran} T_4 &= S(Y)/ST(X) = \ker T_5;\\ \operatorname{ran} T_5 &= Z/S(Y).\end{aligned}$$

Thus evidently, T_1 and T_5 are regular. To show that each T_j $(2 \le j \le 4)$ is regular it suffices to show that $S^{-1}(0) \cap T(X)$ and $S^{-1}(0) + T(X)$ are complemented in Y. Indeed if S, T, and ST are regular then we can arrange (cf. [Har4])

$$\begin{aligned}&T = TT'T, \quad S = SS'S, \quad ST = STUST;\\ &T' = T'TT', \quad S' = S'SS', \quad U = USTU, \quad \text{and} \quad U = T'VS'.\end{aligned}$$

Then we can see that $(I - TUS)TT'$ and $S'S(I - TUS)$ are projections with

$$S^{-1}(0) \cap T(X) = (I - TUS)TT'(Y) \quad \text{and} \quad [S'S(I - TUS)]^{-1}(0) = S^{-1}(0) + T(X),$$

which shows that $S^{-1}(0) \cap T(X)$ and $S^{-1}(0) + T(X)$ are complemented in Y (cf. [Har4]). Therefore each T_j $(1 \le j \le 5)$ is regular, so the result follows from Lemma 1.33. □

Lemma 3.3. *If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ are both invertible, then M_C is invertible for every $C \in \mathcal{B}(Y, X)$.*

Proof. The inverse of M_C is $\begin{pmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{pmatrix}$. □

Theorem 3.4. *A 2×2 operator matrix M_C is invertible for some $C \in B(Y, X)$ if and only if $A \in B(X)$ and $B \in B(Y)$ satisfy the following conditions:*

(i) *A is left invertible;*

(ii) *B is right invertible;*

(iii) *$X/A(X) \cong B^{-1}(0)$.*

Proof. Suppose $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is invertible for some $C \in B(Y, X)$ and write

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{110}$$

Then A is left invertible and B is right invertible. On the other hand since $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ are both regular, Theorem 1.34 gives

$$\begin{aligned}&\ker \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \times \ker(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}) \times \begin{pmatrix} X \\ Y \end{pmatrix} /\operatorname{ran} M_C\\ \cong \quad &\ker M_C \times \begin{pmatrix} X \\ Y \end{pmatrix} /\operatorname{ran} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \times \begin{pmatrix} X \\ Y \end{pmatrix} /\operatorname{ran}(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}),\end{aligned}$$

which implies
$$A^{-1}(0) \times B^{-1}(0) \times \{0\} \cong \{0\} \times X/A(X) \times Y/B(Y),$$
which gives that $B^{-1}(0) \cong X/A(X)$ because A is left invertible and B is right invertible. For the converse observe that if A' is a left inverse of A and if B' is a right inverse of B, then, as in (102), X and Y can be decomposed as

$$A(X) \oplus (AA')^{-1}(0) = X \quad \text{and} \quad B^{-1}(0) \oplus B'B(Y) = Y.$$

Then, by (iii), we have that $(AA')^{-1}(0) \cong B^{-1}(0)$. Thus there is an isomorphism $J : B^{-1}(0) \to (AA')^{-1}(0)$. Define an operator $C : Y \to X$ by

$$C := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} B^{-1}(0) \\ B'B(Y) \end{pmatrix} \to \begin{pmatrix} (AA')^{-1}(0) \\ A(X) \end{pmatrix}.$$

Then we have that $C \in B(Y, X), C(Y) = (AA')^{-1}(0)$ and $C^{-1}(0) = B'B(Y)$. We now claim that M_C is one-one and onto, and hence invertible. Indeed we have

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A(X) + C(Y) \\ B(Y) \end{pmatrix} = \begin{pmatrix} A(X) + (AA')^{-1}(0) \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

and

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} Ax + Cy = 0 \\ By = 0 \end{cases} \Rightarrow \begin{cases} Ax = 0 \\ By = 0 = Cy \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the second and the third implications follow from the facts that $A(X) \cap C(Y) = \{0\}$ and $B^{-1}(0) \cap C^{-1}(0) = \{0\}$, respectively. This completes the proof. ☐

Corollary 3.5. *For a given pair (A, B) of operators we have*

$$\bigcap_{C \in B(Y,X)} \sigma(M_C) = \sigma^{left}(A) \cup \sigma^{right}(B) \cup \{\lambda \in \mathbb{C} : (B - \lambda)^{-1}(0) \ncong Y/(A - \lambda)(X)\}.$$

Proof. This follows at once from Theorem 3.4. ☐

The following two corollaries are also immediate results from Theorem 3.4.

Corollary 3.6. *For a given pair (A, B) of operators we have*

$$(\sigma(A) \cup \sigma(B)) \setminus (\sigma(A) \cap \sigma(B)) \subseteqq \sigma(M_C) \subseteqq \sigma(A) \cup \sigma(B) \quad \textit{for every } C \in B(Y, X).$$

Proof. The second inclusion comes from Lemma 1.33. The first inclusion follows from the observation

$$M_C - \lambda I \quad \text{is invertible} \Rightarrow (A - \lambda I \quad \text{is invertible} \Leftrightarrow B - \lambda I \quad \text{is invertible})$$

for each $\lambda \in \mathbb{C}$. ☐

Corollary 3.7. *If M_C is Fredholm and if either A or B is Fredholm, then A and B are both Fredholm with*

$$\operatorname{index} M_C = \operatorname{index} A + \operatorname{index} B. \tag{111}$$

Proof. The first assertion follows by applying Theorem 3.4 with the pair $(\pi(A), \pi(B))$, where π is the Calkin homomorphism. The second assertion follows from applying the index product theorem to (110). □

The equality (111) is called the "snake lemma". From this we can also see that if M_C is Weyl, in the sense of Fredholm of index zero, and if either A or B is Fredholm, then A is Weyl if and only if B is Weyl.

From Corollary 3.6 we see that, in perturbing a nilpotent matrix $\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $\sigma(M_C)$ shrinks from $\sigma(A)\cup\sigma(B)$. How much of $\sigma(A)\cup\sigma(B)$ survives? The following theorem provides a clue:

Theorem 3.8. *For a given pair (A, B) of operators we have*

$$\eta(\sigma(M_C)) = \eta(\sigma(A) \cup \sigma(B)) \quad \textit{for every} \quad C \in B(Y, X), \tag{112}$$

where $\eta(\cdot)$ denotes the "polynomially convex hull".

Proof. By Corollary 3.6 we have

$$\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) \quad \text{for every} \quad C \in B(Y, X). \tag{113}$$

We now claim that

$$\partial(\sigma(A) \cup \sigma(B)) \subseteq \partial\sigma(M_C). \tag{114}$$

Since

$$\operatorname{int}\sigma(M_C) \subseteq \operatorname{int}(\sigma(A) \cup \sigma(B)),$$

it suffices to show that $\partial(\sigma(A) \cup \sigma(B)) \subseteq \sigma(M_C)$. Indeed we have

$$\partial(\sigma(A) \cup \sigma(B)) \subseteq \partial\sigma(A) \cup \partial\sigma(B) \subseteq \sigma_{ap}(A) \cup \sigma_\delta(B) \subseteq \sigma^{left}(A) \cup \sigma^{right}(B) \subseteq \sigma(M_C),$$

where the second inclusion follows from the fact that if $T \in B(Z)$ for a Banach space Z, then $\partial\sigma(T) \subseteq \sigma_{ap}(T) \cap \sigma_\delta(T)$ and the last inclusion follows from Corollary 3.5. This proves (114). Now the Maximum Modulus Theorem with (113) and (114) gives (112). □

The following corollary says that the passage from $\sigma(A)\cup\sigma(B)$ to $\sigma(M_C)$ is the punching of some open sets in $\sigma(A) \cap \sigma(B)$:

Corollary 3.9. [HLL] *For a given pair (A, B) of operators we have $\sigma(A)\cup\sigma(B) = \sigma(M_C)\cup W$; where W is the union of certain of the holes in $\sigma(M_C)$ which happen to be subsets of $\sigma(A) \cap \sigma(B)$.*

Proof. Theorem 3.8 says that the passage from $\sigma(M_C)$ to $\sigma(A)\cup\sigma(B)$ is the filling in certain of the holes in $\sigma(M_C)$. But since, by Corollary 3.6, $(\sigma(A) \cup \sigma(B)) \setminus \sigma(M_C)$ is contained in $\sigma(A) \cap \sigma(B)$, the filling some holes in $\sigma(M_C)$ should occur in $\sigma(A) \cap \sigma(B)$. This gives the result. □

The following is a generalization of [Ha3, Problem 72].

Corollary 3.10. *If $\sigma(A) \cap \sigma(B)$ has no interior points, then*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad \textit{for every} \quad C \in B(Y, X). \tag{115}$$

In particular if either $A \in B(X)$ or $B \in B(Y)$ is a compact operator, then (115) *holds.*

Proof. The equality (115) immediately follows from Corollary 3.9. The second assertion follows from the fact that the spectrum of a compact operator is at most countable. ☐

One might guess that the closure of each member of W in Corollary 3.9 is a connected component of $\sigma(A) \cap \sigma(B)$. But this is not the case. See the following:

Example 3.11. *Let $U : \ell_2 \to \ell_2$ be the unilateral shift and let $D : \ell_2 \to \ell_2$ be a diagonal operator whose diagonals form a countable dense subset of the annulus $\{z \in \mathbb{C} : 1 \le |z| \le 2\}$. Define the operators A, B and C acting on $\ell_2 \oplus \ell_2$ by*

$$A = \begin{pmatrix} U & 0 \\ 0 & D \end{pmatrix}, \quad B = \begin{pmatrix} U^* & 0 \\ 0 & D \end{pmatrix} \quad \textit{and} \quad C = \begin{pmatrix} 1 - UU^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we have $\sigma(A) = \sigma(B) = \{z \in \mathbb{C} : |z| \le 2\}$ and $\sigma(M_C) = \{z \in \mathbb{C} : 1 \le |z| \le 2\}$, which shows that the closure of the hole,$\{z \in \mathbb{C} : |z| < 1\}$, of $\sigma(M_C)$ is not a component of $\sigma(A) \cap \sigma(B)$.

We now consider another case in which the equality in (113) holds. To do this write, for $T \in B(X)$,

$$\rho^\ell_\sigma(T) = \sigma(T) \setminus \sigma^l(T) \quad \text{and} \quad \rho^r_\sigma(T) = \sigma(T) \setminus \sigma^r(T).$$

Thus by Corollary 3.10 and Theorem 3.4 we can see that the holes in $\sigma(M_C)$ should lie in $\rho^\ell_\sigma(T) \cap \rho^r_\sigma(T)$. Thus we have:

Corollary 3.12. *If $\rho^\ell_\sigma(T) \cap \rho^r_\sigma(T) = \varnothing$, then*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad \textit{for every} \quad C \in B(Y, X).$$

We conclude with an application of Corollary 3.12.

Corollary 3.13. *Suppose H and K are Hilbert spaces. If either $A \in B(H)$ is cohyponormal or $B \in B(K)$ is hyponormal, then*

$$\sigma(M_C) = \sigma(A) \cup \sigma(B) \quad \textit{for every} \quad C \in B(K, H). \tag{116}$$

Proof. If B is hyponormal and so is $B - \lambda I$ for every $\lambda \in \mathbb{C}$, then $\ker(B - \lambda) \subseteq \ker(B - \lambda I)^*$. Thus if $B - \lambda I$ is right invertible, then it must be invertible, which implies $\rho^r_\sigma(B) = \varnothing$. If instead A is cohyponormal, then a similar argument gives $\rho^l_\sigma(A) = \varnothing$. Thus (116) follows from Corollary 3.12. ☐

3.2 Weyl spectra of upper triangles

In this section we consider Weyl spectra of upper triangular operator matrices. When $A \in B(X)$ and $B \in B(Y)$ write on $X \oplus Y$,

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{117}$$

where $C \in B(Y, X)$.

Lemma 3.14. *For a given pair (A, B) of operators, if $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is Weyl then M_C is Weyl for every $C \in B(Y, X)$. Hence, in particular, we have*

$$\omega(M_C) \subseteq \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \subseteq \omega(A) \cup \omega(B). \tag{118}$$

Proof. If $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is Weyl then A and B are both Fredholm, and $\text{index}\, A + \text{index}\, B = 0$. Write

$$M_C = \left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right). \tag{119}$$

Since $\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ is invertible for every $C \in B(Y, X)$, and since $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ are both Fredholm, it follows that M_C is Fredholm. Furthermore we have that $\text{index}\, M_C = \text{index}\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right) + \text{index}\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right) = 0$ and therefore M_C is Weyl for every $C \in B(Y, X)$. The inclusions in (118) are evident from the first assertion. □

The following lemma gives a necessary condition for M_C to be Weyl:

Lemma 3.15. *If M_C is Weyl for some $C \in B(Y, X)$ then $A \in B(X)$ and $B \in B(Y)$ satisfy the following conditions:*

(i) *A is left Fredholm;*

(ii) *B is right Fredholm;*

(iii) $A^{-1}(0) \oplus B^{-1}(0) \cong X/AX \oplus Y/BY$.

which in turn implies that $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is relatively Weyl.

Proof. From (110) we can see that if M_C is Fredholm then $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ is left-Fredholm and $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)$ is right-Fredholm, so that A is left-Fredholm and B is right-Fredholm. On the other hand since $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ are both regular, Theorem 1.34 gives

$$\begin{aligned} &\ker \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \times \ker(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}) \times \begin{pmatrix} X \\ Y \end{pmatrix}/\text{ran} M_C \\ \cong\ & \ker M_C \times \begin{pmatrix} X \\ Y \end{pmatrix}/\text{ran}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \times \begin{pmatrix} X \\ Y \end{pmatrix}/\text{ran}(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}). \end{aligned}$$

Thus if M_C is Weyl then we get (iii). For the second assertion, noting that if the pair (A, B) satisfies the conditions (i) and (ii) then (A, B) has a pair of generalized inverses (A', B'), we have that $\left(\begin{smallmatrix} A' & 0 \\ 0 & B' \end{smallmatrix}\right)$ is a generalized inverse of $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ and the condition (iii) is just the equivalence $\ker \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)/\text{ran}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, which implies that $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is relatively Weyl. □

By the argument of Lemma 3.15, we can see that if any two of A, B, and $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ are Fredholm then so is the other and, in that case, $\operatorname{index}\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \operatorname{index} A + \operatorname{index} B$.

The first inclusion in (118) may be proper. But we have a large class of operators for which the first inclusion in (118) is reversible.

Corollary 3.16. *If either $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ has no pseudoholes then, for every $C \in B(Y, X)$,*

$$\omega(M_C) = \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right). \tag{120}$$

Hence, in particular, if either $A \in B(X)$ or $B \in B(Y)$ is essentially normal (i.e., the self-commutator is a compact operator) then (120) *holds.*

Proof. From Lemma 3.14, we have that $\omega(M_C) \subseteq \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. For the reverse, observe that if $\mathcal{SP}(A)$ has no pseudoholes then, for every $\lambda \in \mathbb{C}$,

$$A - \lambda I \text{ is left-Fredholm} \Longrightarrow A - \lambda I \text{ is Fredholm.} \tag{121}$$

Thus if $\lambda \notin \omega(M_C)$ then by the remark after Lemma 3.15 and (121), $A - \lambda I$ and $B - \lambda I$ are both Fredholm. Further since $\left(\begin{smallmatrix} A-\lambda I & 0 \\ 0 & B-\lambda I \end{smallmatrix}\right)$ is relatively Weyl we must have that $\lambda \notin \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. If instead $\mathcal{SP}(B)$ has no pseudoholes then the same argument gives the result. $\square$

The condition "either $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ has no pseudoholes" is essential in Corollary 3.16. For example consider the following operators on $\ell_2 \otimes \ell_2$:

$$A = U \otimes 1, \quad B = U^* \otimes 1 \quad \text{and} \quad C = (1 - UU^*) \otimes 1, \tag{122}$$

where U is the unilateral shift on ℓ_2. Then $\omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \mathbb{D}$ and $\omega\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \mathbb{T}$.

The following theorem says that the passage from $\omega(A) \cup \omega(B)$ to $\omega(M_C)$ is accomplished by removing certain open subsets of $\omega(A) \cap \omega(B)$ from the former:

Theorem 3.17. *For a given pair (A, B) of operators there is equality, for every $C \in B(Y, X)$,*

$$\omega(A) \cup \omega(B) = \omega(M_C) \cup \mathfrak{S},$$

where $\mathfrak{S}$ is the union of certain of the holes in $\omega(M_C)$ which happen to be subsets of $\omega(A) \cap \omega(B)$.

Proof. We first claim that, for every $C \in B(Y, X)$,

$$\big(\omega(A) \cup \omega(B)\big) \setminus \big(\omega(A) \cap \omega(B)\big) \subseteq \omega(M_C) \subseteq \omega(A) \cup \omega(B). \tag{123}$$

Indeed the second inclusion in (123) follows from Lemma 3.14. For the first inclusion suppose that $\lambda \notin \omega(M_C)$. Then by Lemma 3.15, $\left(\begin{smallmatrix} A-\lambda I & 0 \\ 0 & B-\lambda I \end{smallmatrix}\right)$ is relatively Weyl, so that by the remark after Lemma 3.15, $A - \lambda I$ is Weyl if and only if $B - \lambda I$ is Weyl. Therefore if $\lambda \in \big(\omega(A) \cup \omega(B)\big) \setminus \omega(M_C)$ then $\lambda \in \omega(A) \cap \omega(B)$, which proves (123). We next claim that, for every $C \in B(Y, X)$,

$$\eta\left(\omega(M_C)\right) = \eta\left(\omega(A) \cup \omega(B)\right), \tag{124}$$

where $\eta\mathfrak{C}$ denotes the "polynomially convex hull" of the compact set $\mathfrak{C} \subseteq \mathbb{C}$. Since by (123), $\omega(M_C) \subseteq \omega(A) \cup \omega(B)$ for every $C \in B(Y, X)$, we need to show that $\partial\big(\omega(A) \cup \omega(B)\big) \subseteq \partial\,\omega(M_C)$, where $\partial\,\mathfrak{C}$ denotes the topological boundary of the compact set $\mathfrak{C} \subseteq \mathbb{C}$. But since

$\operatorname{int}\omega(M_C) \subseteq \operatorname{int}\big(\omega(A)\cup\omega(B)\big)$, it suffices to show that $\partial\left(\omega(A)\cup\omega(B)\right) \subseteq \omega(M_C)$. Indeed there are inclusions

$$\partial\big(\omega(A)\cup\omega(B)\big) \subseteq \partial\omega(A)\cup\partial\omega(B) \subseteq \sigma_e^+(A)\cup\sigma_e^-(B) \subseteq \omega(M_C),$$

where the last inclusion follows from Lemma 3.15 and the second inclusion follows from the punctured neighborhood theorem ([Har4, Theorem 9.8.9]): for every operator T,

$$\partial\omega(T) \subseteq \partial\sigma_e(T) \subseteq \sigma_e^+(T)\cap\sigma_e^-(T).$$

This proves (124). Consequently, (124) says that the passage from $\omega(M_C)$ to $\omega(A)\cup\omega(B)$ is the filling in certain of the holes in $\omega(M_C)$. But since, by (123), $\big(\omega(A)\cup\omega(B)\big)\setminus\omega(M_C)$ is contained in $\omega(A)\cap\omega(B)$, it follows that the filling in certain of the holes in $\omega(M_C)$ should occur in $\omega(A)\cap\omega(B)$. This completes the proof. ☐

Corollary 3.18. [Le2] *If $\omega(A)\cap\omega(B)$ has no interior points then, for every $C \in B(Y,X)$,*

$$\omega(M_C) = \omega(A)\cup\omega(B). \tag{125}$$

In particular if either $A \in B(X)$ or $B \in B(Y)$ is a compact operator (more generally, a "Riesz operator"), then (125) *holds.*

Proof. The first assertion follows at once from Theorem 3.17. The second assertion follows from the fact that the Weyl spectrum of a Riesz operator is contained in $\{0\}$. ☐

Let $r(\cdot)$ and $r_\omega(\cdot)$ denote the spectral radius and the "Weyl spectral radius", respectively. Corollary 3.9 shows that for a given pair (A,B) of operators, $r(M_C)$ is a constant. We also have an analogue for r_ω:

Corollary 3.19. *For a given pair (A,B) of operators, $r_\omega(M_C)$ is a constant. Furthermore if $\pi_{00}\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) = \emptyset$ then, for every $C \in B(Y,X)$,*

$$r\left(\begin{smallmatrix} A & C\\ 0 & B\end{smallmatrix}\right) = r\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) = r_\omega\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) = r_\omega\left(\begin{smallmatrix} A & C\\ 0 & B\end{smallmatrix}\right). \tag{126}$$

Proof. The first assertion follows at once from Theorem 3.17. For the second assertion we claim that

$$\eta\,\sigma\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) \setminus \eta\,\omega\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) \subseteq \pi_{00}\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right). \tag{127}$$

Indeed if $\lambda \in \eta\,\sigma\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) \setminus \eta\,\omega\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right)$ then there exists $\epsilon > 0$ such that $\{\mu : |\lambda-\mu| < \epsilon\} \cap \eta\,\omega\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) = \emptyset$, which forces that $\lambda \in \operatorname{iso}\sigma\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right)$ because if it were not so then λ would be in $\eta\,\omega\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right)$, a contradiction. This proves (127) and hence the second equality in (126). ☐

If $\left(\begin{smallmatrix} A & C\\ 0 & B\end{smallmatrix}\right)$ is normaloid (i.e., norm equals spectral radius) and if $\pi_{00}\left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right) = \emptyset$ then

$$\left|\left|\left(\begin{smallmatrix} A & C\\ 0 & B\end{smallmatrix}\right)\right|\right| \le r\left(\begin{smallmatrix} A & C\\ D & B\end{smallmatrix}\right) \quad \text{for every compact operator } D \in B(Y,X): \tag{128}$$

for we can also argue, by (126),

$$\left|\left|\left(\begin{smallmatrix} A & C\\ 0 & B\end{smallmatrix}\right)\right|\right| = r\left(\begin{smallmatrix} A & C\\ 0 & B\end{smallmatrix}\right) = r_\omega\left(\begin{smallmatrix} A & C\\ 0 & B\end{smallmatrix}\right) = r_\omega\left(\begin{smallmatrix} A & C\\ D & B\end{smallmatrix}\right) \le r\left(\begin{smallmatrix} A & C\\ D & B\end{smallmatrix}\right).$$

Note that (128) may, in general, fail for even finite dimensional matrices: for example,

$$\left|\left|\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)\right|\right| = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r\begin{pmatrix} 1 & 1\\ \frac{1}{4} & 1\end{pmatrix} = \frac{3}{2}.$$

Lemma 3.20. *Suppose Weyl's theorem holds for $A \in B(X)$ and $B \in B(Y)$.*

(a) *If Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ then*

$$\omega \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \omega(A) \cup \omega(B). \tag{129}$$

(b) *If A and B are isoloid then the converse of (a) is true.*

Proof. By (118), $\omega \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \subset \omega(A) \cup \omega(B)$. If Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ then

$$\omega(A) \cup \omega(B) \subset \sigma_b(A) \cup \sigma_b(B) \subset \sigma_b \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \subset \omega \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right),$$

which gives (129). For the statement (b) observe that if A and B are isoloid then

$$\pi_{00} \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \Bigl(\pi_{00}(A) \cap \rho(B)\Bigr) \cup \Bigl(\rho(A) \cap \pi_{00}(B)\Bigr) \cup \Bigl(\pi_{00}(A) \cap \pi_{00}(B)\Bigr), \tag{130}$$

where $\rho(\cdot)$ denotes the resolvent set. If Weyl's theorem holds for A and B then the right-hand side of (130) must be just the set $\bigl(\sigma(A) \cup \sigma(B)\bigr) \setminus \bigl(\omega(A) \cup \omega(B)\bigr)$. Thus if (129) holds then $\pi_{00} \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \sigma \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \setminus \omega \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, which says that Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. $\square$

The assumption "A and B are isoloid" is essential in the statement (b) of Lemma 3.20. For example if $A, B : \ell_2 \to \ell_2$ are defined by

$$A(x_1, x_2, \cdots) = (0, x_2, x_3, x_4, \cdots) \quad \text{and} \quad B(x_1, x_2, \cdots) = (0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots),$$

then we have that (i) Weyl's theorem holds for A and B; (ii) $\omega(A) = \{1\}$ and $\omega(B) = \{0\}$; (iii) $\sigma \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \omega \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \{0, 1\}$; (iv) $\pi_{00} \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \{0\}$; (v) B is not isoloid.

Corollary 3.21. *Suppose $A \in B(X)$ and $B \in B(Y)$ are isoloid. If Weyl's theorem holds for A and B, and if $\omega(A) \cap \omega(B)$ has no interior points then Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.*

Proof. This follows from Lemma 3.20 together with applying Corollary 3.18 with $C = 0$. $\square$

It is familiar (cf. [GGK]) that for given operators $A \in B(X), B \in B(Y)$ and $C \in B(Y, X)$, if the operator equation

$$AZ - ZB = C \qquad \text{(where } Z \in B(Y, X) \text{ is the unknown)} \tag{131}$$

is solvable then $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$: in fact, $\left(\begin{smallmatrix} 1 & Z \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -Z \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. Also it is known (cf. [GGK, Theorem I.4.1]) that if $\sigma(A) \cap \sigma(B) = \emptyset$ then the operator equation (131) is solvable. Thus if $\sigma(A) \cap \sigma(B) = \emptyset$ then for most of the familiar kinds of spectrum ϖ there is equality

$$\varpi \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \varpi \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right). \tag{132}$$

Note that (132) for $\varpi = \omega$ is a special case of Corollary 3.18. But evidently the condition "$\sigma(A) \cap \sigma(B)$ has no interior points" does not imply the solvability of the operator equation (131): for example, take $X = Y, A = B = 0$ and $C = I$.

3.3 Weyl's theorem for upper triangles

As we have noticed in Lemma 3.20, Weyl's theorem is liable to fail for 2×2 (even diagonal) operator matrices even though Weyl's theorem holds for the entries in the operator matrices. In this section we consider Weyl's theorem for 2×2 operator matrices (cf. [Le1]).

For operator matrices we observe that for most of the familiar kinds of spectrum ϖ there is equality, for $T \in B(X)$ and $S \in B(Y)$,

$$\varpi \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} = \varpi(T) \cup \varpi(S) \tag{133}$$

also, for $V \in B(X, Y)$ and $U \in B(Y, X)$,

$$\varpi \begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix} = \sqrt{\varpi(UV) \cup \varpi(VU)}, \tag{134}$$

the set of those $\lambda \in \mathbb{C}$ for which λ^2 is in the spectrum of one of the products. When the entries in an operator matrix commute then ([Ha3, Solution 70]; [Har4, Theorem 11.7.7]) spectra can be calculated by determinants: if $\{S, T, U, V\} \subseteq B(X)$ is commutative then

$$\varpi \begin{pmatrix} T & U \\ V & S \end{pmatrix} = \{\lambda \in \mathbb{C} : 0 \in \varpi\big((S - \lambda I)(T - \lambda I) - UV\big)\}. \tag{135}$$

Indeed (133) is easily checked for the ordinary spectrum, the left and the right spectrum, the essential spectrum, the eigenvalues and the approximate eigenvalues while, for the same spectra ϖ, (134) follows from (133) together with the spectral mapping theorem for the polynomial z^2: simply observe that

$$\begin{pmatrix} 0 & U \\ V & 0 \end{pmatrix}^2 = \begin{pmatrix} UV & 0 \\ 0 & VU \end{pmatrix}. \tag{136}$$

For commutative matrices (135) just write down the "classical adjoint":

$$\begin{pmatrix} S & -U \\ -V & T \end{pmatrix} \begin{pmatrix} T & U \\ V & S \end{pmatrix} = \begin{pmatrix} ST - UV & 0 \\ 0 & ST - UV \end{pmatrix} = \begin{pmatrix} T & U \\ V & S \end{pmatrix} \begin{pmatrix} S & -U \\ -V & T \end{pmatrix}. \tag{137}$$

We first consider Weyl's theorem for the 2×2 skew-diagonal operator matrix of the form $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$. Weyl's theorem for the skew-diagonal matrices is more delicate in comparison with the diagonal matrices.

Lemma 3.22. *If $A \in B(X, Y)$ and $B \in B(Y, X)$, then the non-zero elements of $\varpi(AB)$ and $\varpi(BA)$ are the same for each $\varpi \in \{\sigma, \sigma_{ess}, \omega\}$.*

Proof. Remember (cf. [GGK, p.38]) that if $\lambda \neq 0$ then

$$\begin{pmatrix} AB - \lambda I & 0 \\ 0 & I \end{pmatrix} = F(\lambda) \begin{pmatrix} BA - \lambda I & 0 \\ 0 & I \end{pmatrix} E(\lambda), \tag{138}$$

where $E(\lambda)$ and $F(\lambda)$ are both invertible for each $\lambda \neq 0$. It thus follows from (138) that if $\lambda \neq 0$,

$$(AB - \lambda I)^{-1}(0) \cong (BA - \lambda I)^{-1}(0) \quad \text{and} \quad Y/(AB - \lambda I)Y \cong X/(BA - \lambda I)X,$$

which gives the result. $\square$

Although $\sigma(AB) = \sigma(BA)$, we need not expect that $\omega(AB) = \omega(BA)$. To see this, let $\dim X < \infty$ and $Y = \ell_2$, let $S, T : Y \to Y$ be defined by

$$S(x_1, x_2, \cdots) = (0, x_1, 0, x_2, 0, x_3, \cdots) \quad \text{and} \quad T(x_1, x_2, \cdots) = (x_2, x_4, x_6, \cdots), \tag{139}$$

and put $A = 0_X \oplus S$ and $B = 0_X \oplus T$, then $\sigma(AB) = \sigma(BA) = \{0, 1\}$, while $\omega(AB) = \{0, 1\} \neq \omega(BA) = \{1\}$. However one might be tempted to guess that if $\sigma(AB) = \sigma(BA)$ and if in addition $\sigma(AB)$ is connected, then $\omega(AB) = \omega(BA)$. But this guess is also wrong. For an example, if on $\ell_2 \otimes \ell_2$

$$A = \begin{pmatrix} 0 & I & & & & \\ & 0 & U^* & & & \\ & & 0 & U^* & & \\ & & & 0 & U^* & \\ & & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & & & & & \\ U & 0 & & & & \\ & I & 0 & & & \\ & & U & 0 & & \\ & & & U & 0 & \\ & & & & \ddots & \ddots \end{pmatrix},$$

where U is the unilateral shift on ℓ_2, then a straightforward calculation shows that $\sigma(AB) = \sigma(BA) = \mathbb{D}$, while $\omega(AB) = \mathbb{T} \neq \omega(BA) = \mathbb{T} \cup \{0\}$, where $\mathbb{D}$ and $\mathbb{T}$ denote the closed unit disk and the unit circle, respectively.

Lemma 3.23. *If $A \in B(Y, X)$ and $B \in B(X, Y)$, then there is equality*

$$\omega \left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix} \right) = \omega(AB) \cup \omega(BA). \tag{140}$$

Hence, in particular, if AB and BA are isoloid and if Weyl's theorem holds for AB and BA then Weyl's theorem holds for $\left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix} \right)$.

Proof. The inclusion "$\subseteq$" in (140) follows from the fact that the index of a direct sum is the sum of the indices. For the inclusion "$\supseteq$", suppose that $\left(\begin{smallmatrix} AB-\lambda I & 0 \\ 0 & BA-\lambda I \end{smallmatrix} \right)$ is Weyl. Then $AB - \lambda I$ and $BA - \lambda I$ are both Fredholm, and $\text{index}\,(AB - \lambda I) + \text{index}\,(BA - \lambda I) = 0$. But if $\lambda \neq 0$ then by Lemma 3.22, $\text{index}\,(AB - \lambda I) = \text{index}\,(BA - \lambda I)$, so that we must have that $AB - \lambda I$ and $BA - \lambda I$ are both Weyl. If instead $\lambda = 0$ then since AB and BA are both Fredholm it follows from the continuity of the index that for sufficiently small $|\mu|$ ($\mu \neq 0$),

$$\text{index}\,(AB) = \text{index}\,(AB - \mu I) = \text{index}\,(BA - \mu I) = \text{index}\,(BA),$$

which also forces that AB and BA are both Weyl. This proves (140). The second assertion follows at once from Lemma 3.20. $\square$

Example 3.24. (a) If $X = Y$ in Lemma 3.23, one might expect to replace the condition "Weyl's theorem holds for AB and BA" by the condition "Weyl's theorem holds for A and B". But this is not the case: for example, Weyl's theorem may fail for T^2 when it holds for the operator T (see Example 2.28).

(b) Since

$$\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix} \right)^2 = \left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix} \right),$$

one might also expect that "Weyl's theorem for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix} \right)$" is inherited from "Weyl's theorem for $\left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix} \right)$". But in general Weyl's theorem need not be transmitted from the square of the operator T to T. For example, if U is the unilateral shift on ℓ_2 and $K : \ell_2 \to \ell_2$ is defined by

$$K(x_1, x_2, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \cdots), \tag{141}$$

put on $\ell_2 \oplus \ell_2$

$$T = \left(\begin{smallmatrix} U+1 & 0 \\ 0 & K-1 \end{smallmatrix}\right) :$$

then $\sigma(T) = \omega(T) = \{z : |z-1| \le 1\} \cup \{-1\}$ and $\pi_{00}(T) = \{-1\}$, while

$$\sigma(T^2) = \omega(T^2) = \{re^{i\theta} : r \le 2(1+\cos\theta)\} \quad \text{and} \quad \pi_{00}(T^2) = \emptyset,$$

which says that Weyl's theorem holds for T^2, but fails for T.

(c) In general "Weyl's theorem holds for AB" does not imply "Weyl's theorem holds for BA". For example if the operators K, S and T on ℓ_2 are defined as in (145) and (143), put on $\ell_2 \oplus \ell_2$

$$A = K \oplus S \quad \text{and} \quad B = 1 \oplus T :$$

then $\sigma(AB) = \omega(AB) = \sigma(BA) = \omega(BA) = \{0, 1\}$, while $\pi_{00}(AB) = \emptyset \neq \pi_{00}(BA) = \{0\}$.

In spite of Example 3.24 (b), we have:

Theorem 3.25. *Let $A \in B(Y,X)$ and $B \in B(X,Y)$ be such that AB and BA are isoloid. If Weyl's theorem holds for AB and BA then it holds for $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$.*

Proof. We first claim that with no restriction on either A or B,

$$\varpi\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) = \sqrt{\varpi(AB) \cup \varpi(BA)} \quad \text{for each} \quad \varpi \in \{\sigma, \sigma_e, \omega\}, \tag{142}$$

where $\sqrt{\mathbf{K}}$ denotes the set of square roots of complex numbers in $\mathbf{K} \subseteq \mathbb{C}$. For (142) notice that if $0 \neq \lambda \in \mathbb{C}$

$$\begin{pmatrix} -\lambda I & A \\ B & -\lambda I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda I \\ B \end{pmatrix} (AB - \lambda^2 I)^{-1}(0) = \begin{pmatrix} A \\ \lambda I \end{pmatrix} (BA - \lambda^2 I)^{-1}(0) \tag{143}$$

and

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} B^{-1}(0) \\ A^{-1}(0) \end{pmatrix}. \tag{144}$$

Taking adjoints and combining shows that, whether or not $\lambda = 0 \in \mathbb{C}$,

$$\operatorname{index} \begin{pmatrix} -\lambda I & A \\ B & -\lambda I \end{pmatrix} = \operatorname{index}(AB - \lambda^2 I) = \operatorname{index}(BA - \lambda^2 I) \tag{145}$$

if U and V are Fredholm. This proves (142). Thus if Weyl's theorem holds for AB and BA then by Lemma 3.23, there is equality

$$\begin{aligned} \sigma\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \setminus \omega\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) &= \sqrt{\sigma(AB) \cup \sigma(BA)} \setminus \sqrt{\omega(AB) \cup \omega(BA)} \\ &= \sqrt{\big(\sigma(AB) \cup \sigma(BA)\big) \setminus \big(\omega(AB) \cup \omega(BA)\big)} \\ &= \sqrt{\sigma\left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix}\right) \setminus \omega\left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix}\right)} \\ &= \sqrt{\pi_{00}\left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix}\right)}. \end{aligned}$$

Now it will be shown that

$$\sqrt{\pi_{00}\left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix}\right)} = \pi_{00}\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right). \tag{146}$$

But since, in view of (142), $\sigma\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ is symmetric with respect to the origin, it follows from the spectral mapping theorem that

$$\sqrt{\operatorname{iso}\sigma\left(\begin{smallmatrix} AB & 0 \\ 0 & BA \end{smallmatrix}\right)} = \sqrt{\operatorname{iso}\left(\sigma\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)\right)^2} = \operatorname{iso}\sigma\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right).$$

Thus for (146) it suffices to show that, for any $\mu \in \mathbb{C}$,

$$0 < \dim \begin{pmatrix} AB-\mu I & 0 \\ 0 & BA-\mu I \end{pmatrix}^{-1} \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) < \infty \Longleftrightarrow 0 < \dim \begin{pmatrix} -\sqrt{\mu} I & A \\ B & -\sqrt{\mu} I \end{pmatrix}^{-1} \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) < \infty. \tag{147}$$

If $\mu = 0$, then (147) follows from the observation

$$0 < \dim \Big(A^{-1}(0) \oplus B^{-1}(0)\Big) < \infty \iff 0 < \dim \Big((AB)^{-1}(0) \oplus (BA)^{-1}(0)\Big) < \infty. \tag{148}$$

If instead $\mu \neq 0$, then (147) follows from the observation

$$\begin{aligned} &\bigvee\{(x, \frac{1}{\sqrt{\mu}}Bx) : x \in (AB-\mu I)^{-1}(0)\} \bigcup \bigvee\{(\frac{1}{\sqrt{\mu}}Ay, y) : y \in (BA-\mu I)^{-1}(0)\} \\ &\qquad \subseteq \begin{pmatrix} -\sqrt{\mu} I & A \\ B & -\sqrt{\mu} I \end{pmatrix}^{-1} \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \subseteq (AB-\mu I)^{-1}(0) \oplus (BA-\mu I)^{-1}(0), \end{aligned} \tag{149}$$

where $\bigvee G$ denotes the closed linear span of G. This proves (146) and completes the proof. □

Note that A^2 may not be hyponormal when A is hyponormal ([Ha3, Problem 209]). In spite of it, if A is hyponormal then Weyl's theorem holds for $f(A)$, where $f \in \operatorname{Hol}(\sigma(A))$ (see Corollary 2.32). Thus we have:

Corollary 3.26. *If A is hyponormal then Weyl's theorem holds for $\begin{pmatrix} 0 & f(A) \\ g(A) & 0 \end{pmatrix}$ for every $f, g \in \operatorname{Hol}(\sigma(A))$.*

Proof. It is easy to show that if A is isoloid then $h(A)$ is also isoloid for every $h \in \operatorname{Hol}(\sigma(A))$. Thus the result follows at once from Theorem 3.25. □

The assumption of Theorem 3.25 can easily be satisfied by Toeplitz operators. Evidently, every Toeplitz operator is isoloid. If either φ is analytic (i.e., $\varphi \in H^\infty(\mathbb{T}) := L^\infty \cap H^2(\mathbb{T})$) or ψ is coanalytic (i.e., $\overline{\psi} \in H^\infty(\mathbb{T})$) then $T_\psi T_\varphi = T_{\psi\varphi}$ (cf. [Do1, Proposition 7.5]). Also if $\varphi \in C(\mathbb{T})$ then (cf. [Do1, Proposition 7.22])

$$T_\varphi T_\psi - T_{\varphi\psi} \text{ and } T_\psi T_\varphi - T_{\psi\varphi} \text{ are compact operators for every } \psi \in L^\infty(\mathbb{T}). \tag{150}$$

We then have:

Example 3.27. *If φ is either analytic or coanalytic and if $\psi \in C(\mathbb{T})$ then Weyl's theorem holds for $\begin{pmatrix} 0 & T_\varphi \\ T_\psi & 0 \end{pmatrix}$.*

Proof. Suppose that $\varphi \in H^\infty(\mathbb{T})$. Then $T_\psi T_\varphi = T_{\psi\varphi}$, and hence $T_\psi T_\varphi$ is isoloid and Weyl's theorem holds for $T_\psi T_\varphi$. Therefore, in view of Theorem 3.25, it suffices to prove that $T_\varphi T_\psi$ is isoloid and Weyl's theorem holds for $T_\varphi T_\psi$. Suppose that $\psi \in C(\mathbb{T})$. If $\sigma(T_\varphi T_\psi) = \sigma(T_\psi T_\varphi)$ then by (150),

$$\omega(T_\varphi T_\psi) = \omega(T_{\varphi\psi}) = \sigma(T_{\psi\varphi}) = \sigma(T_\psi T_\varphi) = \sigma(T_\varphi T_\psi), \tag{151}$$

because the Weyl spectrum is invariant under the compact perturbations. Since $\sigma(T_\varphi T_\psi) = \sigma(T_\psi T_\varphi)$ and hence $\sigma(T_\varphi T_\psi)$ is connected, (151) implies that $T_\varphi T_\psi$ is isoloid and Weyl's theorem holds for $T_\varphi T_\psi$. If instead $\sigma(T_\varphi T_\psi) \neq \sigma(T_\psi T_\varphi)$ then since $\sigma(T_\psi T_\varphi)$ is connected, it follows from Lemma 3.22 that

$$0 \notin \sigma(T_\psi T_\varphi) \quad \text{and} \quad 0 \in \operatorname{iso} \sigma(T_\varphi T_\psi).$$

But then $T_\psi T_\varphi$ is invertible and hence, by (150), $T_\varphi T_\psi$ is Weyl but not invertible, which implies that $0 \in \pi_{00}(T_\varphi T_\psi)$ and hence $T_\varphi T_\psi$ is isoloid. Therefore there is equality

$$\begin{aligned}\sigma(T_\varphi T_\psi) \setminus \omega(T_\varphi T_\psi) &= \big(\sigma(T_\psi T_\varphi) \setminus \omega(T_\psi T_\varphi)\big) \cup \{0\} \\ &= \pi_{00}(T_\psi T_\varphi) \cup \{0\} \\ &= \pi_{00}(T_\varphi T_\psi),\end{aligned}$$

which says that Weyl's theorem holds for $T_\varphi T_\psi$. The argument for the case of the coanalytic symbol φ is the same. □

Example 3.28. (a) If φ and ψ are in $C(\mathbb{T})$, then Weyl's theorem need not hold for $\begin{pmatrix} 0 & T_\varphi \\ T_\psi & 0 \end{pmatrix}$. For example if φ is defined by

$$\varphi(e^{i\theta}) = \begin{cases} -e^{2i\theta} + 1 & (0 \le \theta \le \pi) \\ e^{-2i\theta} - 1 & (\pi \le \theta \le 2\pi) \end{cases}$$

then a straightforward calculation shows (cf. Example 2.11)

$$\sigma \begin{pmatrix} 0 & T_\varphi \\ T_\varphi & 0 \end{pmatrix} = \sqrt{\sigma(T_\varphi^2)} = \sqrt{\sigma(T_\varphi)^2} = \sqrt{\{re^{i\theta} : r \le 2(1+\cos\theta)\}}$$

and

$$\omega \begin{pmatrix} 0 & T_\varphi \\ T_\varphi & 0 \end{pmatrix} = \sqrt{\omega(T_\varphi^2)} = \sqrt{\omega(T_{\varphi^2})} = \sqrt{\{re^{i\theta} : r = 2(1+\cos\theta)\}},$$

which implies that Weyl's theorem does not hold for $\begin{pmatrix} 0 & T_\varphi \\ T_\varphi & 0 \end{pmatrix}$.

(b) As we noticed, if U is the unilateral shift on ℓ_2 then Weyl's theorem fails for $\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}$. But Example 3.27 guarantees that Weyl's theorem holds for $\begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}$.

We have seen in Theorem 2.37 that Weyl's theorem is transmitted from $T \in B(X)$ to $T + K$ for commuting nilpotents $K \in B(X)$. But this may fail if K is not assumed to commute with T even if K is both compact and nilpotent: for example, consider $T = \begin{pmatrix} U & 1-UU^* \\ 0 & U^* \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 1-UU^* \\ 0 & 0 \end{pmatrix}$. We now consider the following question: if Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, when does it hold for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$?

Although the passage from $\sigma(M_C)$ to $\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is the filling in certain of the holes in $\sigma(M_C)$, we cannot expect that $\text{iso}\,\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \text{iso}\,\sigma(M_C)$ even when both $\mathcal{SP}(A)$ and $\mathcal{SP}(B)$ have no pseudoholes. For example if on $\ell_2 \oplus \ell_2$

$$A = U \oplus 0, \quad B = U^* \oplus 0, \quad \text{and} \quad C = (1 - UU^*) \oplus 0,$$

where U is the unilateral shift on ℓ_2, then $\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \mathbb{D}$ and $\sigma\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \mathbb{T} \cup \{0\}$, while $\omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \mathbb{T} \cup \{0\} = \omega\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$.

We might expect that if either $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ has no pseudoholes then for every $C \in B(Y, X)$,

$$\text{Weyl's theorem holds for } \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \implies \text{Weyl's theorem holds for } \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right). \tag{152}$$

However (152) may fail for "Weyl's theorem" even with the additional assumption that Weyl's theorem holds for A and B. To see this let the operators A, B and C on ℓ_2 be defined by

$$A(x_1, x_2, \cdots) = (0, x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, 0, \frac{1}{4}x_4, \cdots); \tag{153}$$

$$B(x_1, x_2, \cdots) = (0, x_2, 0, x_4, 0, x_6, 0, x_8, \cdots); \tag{154}$$

$$C(x_1, x_2, \cdots) = (0, 0, x_2, 0, x_3, 0, x_4, 0, \cdots) : \tag{155}$$

then

$$\sigma(A) = \omega(A) = \{0\}, \quad \sigma(B) = \omega(B) = \{0, 1\}, \quad \text{and} \quad \pi_{00}(A) = \pi_{00}(B) = \emptyset, \tag{156}$$

which says that Weyl's theorem holds for A and B. Also a straightforward calculation shows that

$$\begin{aligned} \sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) &= \sigma\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \{0, 1\}, \\ \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) &= \omega\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \{0, 1\}, \\ \pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) &= p_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \emptyset, \end{aligned}$$

while

$$\pi_{00}\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \{0\} \neq p_{00}\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = \emptyset,$$

which implies that Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, but fails for $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$.

We now have:

Theorem 3.29. *If either $\mathcal{SP}(A)$ or $\mathcal{SP}(B)$ has no pseudoholes and if A is an isoloid operator for which Weyl's theorem holds then for every $C \in B(Y, X)$,*

$$\textit{Weyl's theorem holds for } \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \implies \textit{Weyl's theorem holds for } \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right).$$

Proof. By assumption we have that $\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \setminus \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. But since $\sigma(M_C)$ shrinks from $\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, Corollary 3.9, Corollary 3.16 and Theorem 3.17 give

$$\sigma(M_C) \setminus \omega(M_C) \subseteq \pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right). \tag{157}$$

Thus noting that $\text{iso}\,\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \subseteq \text{iso}\,\sigma(M_C)$ passing to Corollary 3.9, it follows that

$$\sigma(M_C) \setminus \omega(M_C) \subseteq \pi_{00}(M_C). \tag{158}$$

For the reverse inclusion of (158) suppose that $\lambda \in \pi_{00}(M_C)$. But if $\lambda \in \text{iso}\sigma(M_C) \setminus \text{iso}\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ then, in view of Corollary 3.16, λ should lie in $\big(\sigma(A) \cap \sigma(B)\big) \setminus \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, and hence $\lambda \notin \omega(M_C)$. Therefore it suffices to show that, for each $\lambda \in \pi_{00}(M_C) \cap \text{iso}\,\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$,

$$0 < \dim\,(M_C - \lambda I)^{-1}(0) < \infty \implies 0 < \dim\big((A - \lambda I)^{-1}(0) \oplus (B - \lambda I)^{-1}(0)\big) < \infty : \quad (159)$$

because (159) implies that if λ is in $\pi_{00}(M_C) \cap \text{iso}\,\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ then λ is in $\pi_{00}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, so that $\lambda \notin \omega\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \omega(M_C)$ since Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. For (159) suppose that $0 < \dim\,(M_C - \lambda I)^{-1}(0) < \infty$. First of all observe that there is inclusion, in general,

$$(M_C - \lambda I)^{-1}(0) \subseteq (A - \lambda I)^{-1}\big(C\left((B - \lambda I)^{-1}(0)\right)\big) \oplus (B - \lambda I)^{-1}(0), \qquad (160)$$

which forces that $(A - \lambda I)^{-1}(0) \oplus (B - \lambda I)^{-1}(0)$ is non-trivial because if it were not so then $(M_C - \lambda I)^{-1}(0)$ would be trivial, a contradiction. Now we must show that $(A - \lambda I)^{-1}(0) \oplus (B - \lambda I)^{-1}(0)$ is finite dimensional. To the contrary we assume that $(A - \lambda I)^{-1}(0) \oplus (B - \lambda I)^{-1}(0)$ is infinite dimensional. But since

$$(A - \lambda I)^{-1}(0) \oplus \{0\} \subseteq (M_C - \lambda I)^{-1}(0), \qquad (161)$$

it follows that $\dim\,(A - \lambda I)^{-1}(0) < \infty$, so that $(B - \lambda I)^{-1}(0)$ must be infinite dimensional. Now there are two cases to consider.

Case 1. Suppose that $C\big((B-\lambda I)^{-1}(0)\big)$ is finite dimensional. Then $C^{-1}(0)$ must contain a sequence $\{z_j\}$ of linear independent vectors in $(B - \lambda I)^{-1}(0)$. But then

$$\begin{pmatrix} A-\lambda I & C \\ 0 & B-\lambda I \end{pmatrix} \begin{pmatrix} 0 \\ z_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for each } j = 1, 2, \cdots,$$

which implies that $(M_C - \lambda I)^{-1}(0)$ is infinite dimensional, a contradiction.

Case 2. Suppose that $C\big((B - \lambda I)^{-1}(0)\big)$ is infinite dimensional. Since

(i) $\lambda \in \text{iso}\,\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ and hence $\lambda \in \rho(A) \cup \text{iso}\,\sigma(A)$;

(ii) $\dim\,(A - \lambda I)^{-1}(0) < \infty$;

(iii) A is isoloid,

it follows that $\lambda \in \rho(A) \cup \pi_{00}(A)$. But since Weyl's theorem holds for A, we have that $A - \lambda I$ is Weyl, and hence $(A - \lambda I)(X)$ is finite co-dimensional. Therefore $C\big((B - \lambda I)^{-1}(0)\big) \cap (A - \lambda I)(X)$ is infinite dimensional. Thus we can find a sequence $\{y_j\}$ of linearly independent vectors in $(B - \lambda I)^{-1}(0)$ for which there exists a sequence $\{x_j\}$ in X such that

$$(A - \lambda I)x_j = Cy_j \quad \text{for each } j = 1, 2, \cdots.$$

But then

$$\begin{pmatrix} A-\lambda I & C \\ 0 & B-\lambda I \end{pmatrix} \begin{pmatrix} x_j \\ -y_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for each } j = 1, 2, \cdots,$$

which implies that $(M_C - \lambda I)^{-1}(0)$ is infinite dimensional, a contradiction. This completes the proof. ∎

The "isoloid" condition is essential in Theorem 3.29. For an example, consider the matrix $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, where A, B and C are given by (153), (154) and (155), respectively: in fact, the operator A in (153) is not isoloid. Also the condition "Weyl's theorem holds for A" cannot be dropped in Theorem 3.29. For example if on ℓ_2

$$\begin{aligned} A(x_1, x_2, \cdots) &= (0, 0, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, 0, \frac{1}{4}x_4, \cdots); \\ B(x_1, x_2, \cdots) &= (0, x_2, 0, x_4, 0, x_6, 0, x_8, \cdots); \\ C(x_1, x_2, \cdots) &= (x_1, 0, x_2, 0, x_3, 0, x_4, 0, \cdots) : \end{aligned}$$

then the all the spectra are the same as (156) except

$$\pi_{00}(A) = \{0\}.$$

Therefore Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, but fails for $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$. Here note that Weyl's theorem does not hold for A, while A is isoloid.

Corollary 3.30. *If H is a Hilbert space and $A \in B(H)$ is essentially normal isoloid operator for which Weyl's theorem holds then for every $C \in B(K, H)$ for a Hilbert space K,*

$$\textit{Weyl's theorem holds for } \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \Longrightarrow \textit{Weyl's theorem holds for } \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right).$$

Hence, in particular, if $A \in B(H)$ is normal then Weyl's theorem is transmitted from $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ for every $C \in B(K, H)$.

Proof. The first assertion follows from Theorem 3.29 together with the fact ([Pe, Proposition 2.16]) that the spectral picture of every essentially normal operator has no pseudoholes. The second assertion follows at once from the first. □

When the entries in an operator matrix commute then the most of the familiar kinds of spectrum ϖ can be calculated by determinants (cf. [GGK, Theorem XI.7.2]): if $\left(\begin{smallmatrix} A & C \\ D & B \end{smallmatrix}\right)$ is a commutative operator matrix on $X \oplus X$ then

$$\varpi \left(\begin{smallmatrix} A & C \\ D & B \end{smallmatrix}\right) = \{\lambda \in \mathbb{C} : 0 \in \varpi\big((A - \lambda I)(B - \lambda I) - CD\big)\}. \tag{162}$$

Indeed (162) holds for the ordinary spectrum, the essential spectrum and the eigenvalues. We now have:

Theorem 3.31. *If $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is a commutative operator matrix acting on $X \oplus X$ then Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ if and only if it holds for $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$.*

Proof. Observe, by (162), that there is equality

$$\varpi \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \varpi \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \qquad \text{for each } \ \varpi \in \{\sigma, \pi_0, \sigma_e, \omega\}, \tag{163}$$

where $\pi_0(\cdot)$ denotes the set of eigenvalues: the equality for ω follows from the fact that index $\left(\begin{smallmatrix} A-\lambda I & 0 \\ 0 & B-\lambda I \end{smallmatrix}\right) =$ index $\left(\begin{smallmatrix} A-\lambda I & C \\ 0 & B-\lambda I \end{smallmatrix}\right)$ for every $\lambda \in \mathbb{C} \setminus \sigma_e \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. We now claim that (163) also holds with $\varpi = \pi_{00}$. In view of (163) it suffices to show

$$\dim\big((A - \lambda I)^{-1}(0) \oplus (B - \lambda I)^{-1}(0)\big) < \infty \Longleftrightarrow \dim (M_C - \lambda I)^{-1}(0) < \infty. \tag{164}$$

The forward implication follows from (160). For the backward implication suppose that $\dim (M_C - \lambda I)^{-1}(0) < \infty$. Then in view of (162), it suffices to show that $\dim (B - \lambda I)^{-1}(0) <$

∞. To the contrary we assume that $(B-\lambda I)^{-1}(0)$ contains a sequence $\{y_j\}$ of linearly independent vectors. Then since $CA = AC$, we have

$$\begin{pmatrix} A-\lambda I & C \\ 0 & B-\lambda I \end{pmatrix} \begin{pmatrix} Cy_j \\ (\lambda I - A)y_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for every } j = 1, 2, \cdots .$$

Thus we must have that $\dim \{Cy_j : j = 1, 2, \cdots\} < \infty$, and hence we can find a sequence $\{z_j\}$ of linearly independent vectors in $C^{-1}(0) \cap (B-\lambda I)^{-1}(0)$. But then

$$\begin{pmatrix} A-\lambda I & C \\ 0 & B-\lambda I \end{pmatrix} \begin{pmatrix} 0 \\ z_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for every } j = 1, 2, \cdots ,$$

which implies that $(M_C - \lambda I)^{-1}(0)$ is infinite dimensional, a contradiction. This proves (164) and completes the proof. $\square$

3.4 Boundedness below for upper triangles

In this section we consider the approximate point spectra of upper triangular operator matrices in the setting of Hilbert spaces. In this section, H and K denote Hilbert spaces.

Recall that an operator $A \in B(H, K)$ is said to be *bounded below* if there exists $k > 0$ for which $||x|| \le k\, ||Ax||$ for each $x \in H$. If $A \in B(H)$ then the approximate point spectrum, $\sigma_{ap}(A)$, and the defect spectrum, $\sigma_\delta(A)$, of A are defined by

$$\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not bounded below}\};$$
$$\sigma_d(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not onto}\}.$$

If $T \in B(H, K)$ then the *reduced minimum modulus* of T is defined by (cf. [Ap])

$$\gamma(T) = \begin{cases} \inf\{||Tx|| : \ \operatorname{dist}(x, N(T)) = 1\} & \text{if } T \neq 0 \\ 0 & \text{if } T = 0. \end{cases}$$

Thus $\gamma(T) > 0$ if and only if T has closed non-zero range (cf. [Ap],[Go]). If $T \in B(H)$ is a non-zero operator then we can see ([Ap]) that $\gamma(T) = \inf\big(\sigma(|T|) \setminus \{0\}\big)$, where $|T|$ denotes $(T^*T)^{\frac{1}{2}}$. Thus we have that $\gamma(T) = \gamma(T^*)$. From the definition we can also see that if T is bounded below then $||x|| \le \frac{1}{\gamma(T)}\, ||Tx||$ for each $x \in H$.

Recall ([Har4, Theorem 3.3.2]) that if $S \in B(K, H)$ and $T \in B(H, K)$ then

$$S, T \text{ bounded below} \Longrightarrow ST \text{ bounded below} \Longrightarrow T \text{ bounded below}. \tag{165}$$

Write $M_C := \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ is invertible for every $C \in B(K, H)$, noting the equation $M_C = \left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ gives

$$A, B \text{ bounded below} \Longrightarrow M_C \text{ bounded below} \Longrightarrow A \text{ bounded below}. \tag{166}$$

The following lemma is a result of independent interest.

Lemma 3.32. *Let $T \in B(H)$ and $T \neq 0$. Then T satisfies one of the following two conditions:*

(i) *There exists a unit vector x in $N(T)^\perp$ such that $||Tx|| = \gamma(T)$;*

(ii) *There exists an orthonormal sequence $\{x_n\}$ in $(\ker T)^\perp$ such that $||Tx_n|| \to \gamma(T)$.*

In particular, if $\text{ran}(T)$ is not closed then T must satisfy the condition (ii) with $\gamma(T) = 0$.

Proof. Suppose $T \neq 0$ and write $\alpha := \gamma(T) = \inf\big(\sigma(|T|) \setminus \{0\}\big)$. Let E be the spectral measure on the Borel subsets of $\sigma(|T|)$ such that $|T| = \int z\, dE(z)$. There are two cases to consider.

Case 1: $\alpha \in \text{acc}\big(\sigma(|T|) \setminus \{0\}\big)$. In this case, there exists a strictly decreasing sequence $\{\alpha_n\}$ of elements in $\sigma(|T|) \setminus \{0\}$ such that $\alpha_n \to \alpha$. Since the α_n's are distinct, there exists a sequence $\{U_n\}$ of mutually disjoint open intervals such that $\alpha_n \in U_n$ for all $n \in \mathbb{Z}^+$. Define $F_n := U_n \cap \sigma(|T|)$ $(n \in \mathbb{Z}^+)$. Then the F_n's are nonempty relatively open subsets of $\sigma(|T|)$. Thus $E(F_n)H \neq \{0\}$ for each $n \in \mathbb{Z}^+$. For each $n \in \mathbb{Z}^+$, choose a unit vector x_n in $E(F_n)H$. Since the F_n's are mutually disjoint, it follows that $\{x_n\}$ is an orthonormal sequence. We will show that $x_n \in (\ker T)^\perp$ $(n \in \mathbb{Z}^+)$. If $|T|$ is invertible then $(\ker T)^\perp = (\ker |T|)^\perp = H$, so evidently, $x_n \in (\ker T)^\perp$. Now suppose $|T|$ is not invertible. Since $|T|$ is a normal operator, $|T|$ is unitarily equivalent to a multiplication operator M_φ. But since our argument below depends only on the inner product, we may assume without loss of generality that $|T|$ is a multiplication operator. Let $|T| := M_\varphi$. If $F_0 := \{0\}$ then $E(F_0)$ is the multiplication by $\chi_{\varphi^{-1}(0)}$. Thus if $f \in \ker |T|$ then $\varphi f = 0$ and hence

$$\big(\chi_{\varphi^{-1}(0)} f\big)(x) = \begin{cases} 0 & \text{if } f(x) = 0, \\ f(x) & \text{if } f(x) \neq 0, \end{cases}$$

which shows that $E(F_0)f = f$. Therefore if $f \in \ker |T|$ then for each $n \in \mathbb{Z}^+$,

$$(f,\, x_n) = (E(F_0)f,\, E(F_n)x_n) = (f,\, E(F_0 \cap F_n)x_n) = (f, 0) = 0,$$

which shows that $x_n \in (\ker |T|)^\perp$ for all $n \in \mathbb{Z}^+$. It thus follows that $x_n \in (\ker T)^\perp$. On the other hand, for each $n \geq 2$,

$$\begin{aligned} ||Tx_n||^2 &= (T^*Tx_n,\, x_n) \leq ||(T^*T)|_{E(F_n)H}|| = r\big((T^*T)|_{E(F_n)H}\big) \\ &\leq \big(\sup\, F_n\big)^2 \leq \big(\sup\, U_n\big)^2 \leq \alpha_{n-1}^2, \end{aligned}$$

where $r(\cdot)$ denotes the spectral radius. Therefore we have that $\alpha \leq ||Tx_n|| \leq \alpha_{n-1}$ $(n \geq 2)$, which implies that $||Tx_n|| \to \alpha = \gamma(T)$.

Case 2: $\alpha \in iso\,\big(\sigma(|T|) \setminus \{0\}\big)$. Let $\mathfrak{L} := E(\{\alpha\})$ and $\mathfrak{M} := E\big(\sigma(|T|) \setminus \{\alpha\}\big)$. Then H can be decomposed as $H = \mathfrak{L} \oplus \mathfrak{M}$, where $\mathfrak{L}$ and $\mathfrak{M}$ are $|T|$-invariant subspaces, $\sigma(|T|\,|_{\mathfrak{L}}) = \{\alpha\}$ and $\sigma(|T|\,|_{\mathfrak{M}}) = \sigma(|T|) \setminus \{\alpha\}$: more precisely, we can write

$$|T| = \begin{pmatrix} \alpha & 0 \\ 0 & |T|\,|_{\mathfrak{M}} \end{pmatrix} :\ \mathfrak{L} \oplus \mathfrak{M} \longrightarrow \mathfrak{L} \oplus \mathfrak{M}.$$

But since $||Tx|| = ||\,|T|x||$ for all $x \in H$, it follows that for every unit vector x_0 in $\mathfrak{L}$, $||Tx_0|| = ||\,|T|x_0|| = ||\alpha x_0|| = \alpha$.

For the second assertion suppose $\gamma(T) = 0$ and $T \neq 0$. If T satisfies the condition (i) then there exists a unit vector $x \in (\ker T)^\perp$ such that $Tx = 0$, giving a contradiction. This shows that T must satisfy the condition (ii). □

The following theorem is a characterization of the boundedness below of M_C.

Theorem 3.33. [HwL2] *A* 2×2 *operator matrix* $M_C := \left(\begin{smallmatrix} A & C \\ 0 & B\end{smallmatrix}\right)$ *is bounded below for some* $C \in B(K,H)$ *if and only if* A *is bounded below and*

$$\begin{cases} \alpha(B) \le \beta(A) & \text{if } \operatorname{ran}(B) \text{ is closed,} \\ \beta(A) = \infty & \text{if } \operatorname{ran}(B) \text{ is not closed.}\end{cases}$$

Proof. We first claim that if A is bounded below and $\operatorname{ran}(B)$ is closed, then

$$\alpha(B) \le \beta(A) \iff M_C \text{ is bounded below for some } C \in B(K,H). \tag{167}$$

To show this suppose $\alpha(B) \le \beta(A)$. Since $\dim \ker B \le \dim R(A)^\perp$, there exists a isometry $J : \ker B \to (\operatorname{ran} A)^\perp$. Define an operator $C : K \to H$ by

$$C := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \ker B \\ (\ker B)^\perp \end{pmatrix} \to \begin{pmatrix} (\operatorname{ran} A)^\perp \\ \operatorname{ran} A \end{pmatrix}.$$

Then M_C is one-one. Assume to the contrary that M_C is not bounded below. Then there exists a sequence $\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)$ of unit vectors in $H \oplus K$ for which

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} Ax_n + Cy_n \\ By_n \end{pmatrix} \longrightarrow 0.$$

Write $y_n := \alpha_n + \beta_n$ for $n \in \mathbb{Z}^+$, where $\alpha_n \in \ker B$ and $\beta_n \in (\ker B)^\perp$. Since $\gamma(B) > 0$ and $By_n \to 0$, it follows that $\beta_n \to 0$. Also by the definition of C, $Cy_n = C(\alpha_n + \beta_n) = C\alpha_n \to 0$ and hence $\alpha_n \to 0$. Therefore $y_n \to 0$ and $||x_n|| \to 1$. But since $Ax_n \to 0$, it follows that A is not bounded below, giving a contradiction. This proves that M_C is bounded below.

Conversely, suppose M_C is bounded below for some $C \in B(K,H)$. Write M_C as in (117). Since $\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ have closed ranges, it follows from Theorem 3.8 that

$$\begin{aligned} &\ker \left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right) \bigoplus \ker \left(\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)\right) \bigoplus \operatorname{ran}(M_C)^\perp \\ &\qquad \cong \ \ker(M_C) \bigoplus \operatorname{ran}\left(\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)\right)^\perp \bigoplus \operatorname{ran}\left(\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)\right)^\perp, \end{aligned}$$

which implies that $\alpha(B) + \beta(M_C) = \beta(A) + \beta\left(\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)\right)$. Since

$$\beta(M_C) \ge \beta\left(\left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)\right),$$

it follows that $\alpha(B) \le \beta(A)$. This proves (167). We next claim that if A is bounded below and $\operatorname{ran}(B)$ is not closed, then

$$\beta(A) = \infty \iff M_C \text{ is bounded below for some } C \in B(K,H). \tag{168}$$

To show this suppose $\beta(A) = \infty$. Then with no restriction on $\operatorname{ran} B$, M_C is bounded below for some $C \in B(K,H)$. To see this, observe $\dim(\operatorname{ran} A)^\perp = \infty$, so there exists an isomorphism $C_0 : K \to (\operatorname{ran} A)^\perp$. Define an operator $C : K \to H$ by

$$C := \begin{pmatrix} C_0 & 0 \end{pmatrix} : K \to \begin{pmatrix} (\operatorname{ran} A)^\perp \\ \operatorname{ran} A) \end{pmatrix}.$$

Then a straightforward calculation shows that M_C is one-one and

$$\begin{aligned}\gamma(M_C) &= \inf_{||x||^2+||y||^2=1} || \begin{pmatrix} Ax+Cy \\ By \end{pmatrix} || \\ &\geq \inf_{||x||^2+||y||^2=1} \left(||Ax||^2+||Cy||^2\right)^{\frac{1}{2}} \\ &\geq \inf_{||x||^2+||y||^2=1} \left(\gamma(A)^2||x||^2+||y||^2\right)^{\frac{1}{2}} \\ &\geq \min\{1,\ \gamma(A)\} > 0,\end{aligned}$$

which implies that M_C is bounded below. For the converse, assume $\beta(A) = N < \infty$. Since $\operatorname{ran} B$ is not closed it follows from Lemma 3.32 that there exists an orthonormal sequence $\{y_n\}$ in $(\ker B)^{\perp}$ such that $By_n \to 0$. But since M_C is bounded below we have

$$\inf_{||x||^2+||y||^2=1} || \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) || = \inf_{||x||^2+||y||^2=1} || \left(\begin{smallmatrix} Ax+Cy \\ By \end{smallmatrix}\right) || > 0.$$

We now argue that there exist $\epsilon > 0$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ for which

$$\operatorname{dist}\left(\operatorname{ran} A,\ Cy_{n_k}\right) > \epsilon \quad \text{for all } k \in \mathbb{Z}^+. \tag{169}$$

Indeed, assume to the contrary that $\operatorname{dist}\left(\operatorname{ran} A,\ Cy_n\right) \to 0$ as $n \to \infty$. Thus there exists a sequence $\{x_n\}$ in H such that $\operatorname{dist}\left(Ax_n,\ Cy_n\right) \to 0$. Let $z_n := || \left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right) ||^{-1} x_n$ and $w_n := || \left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right) ||^{-1}(-y_n)$. Then $|| \left(\begin{smallmatrix} z_n \\ w_n \end{smallmatrix}\right) || = 1$ and $|| \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \left(\begin{smallmatrix} z_n \\ w_n \end{smallmatrix}\right) || = || \left(\begin{smallmatrix} Az_n+Cw_n \\ Bw_n \end{smallmatrix}\right) || \longrightarrow 0$, giving a contradiction. This proves (169). There is no loss in simplifying the notation and assuming that

$$\operatorname{dist}\left(\operatorname{ran} A,\ Cy_n\right) > \epsilon \quad \text{for all } n \in \mathbb{Z}^+. \tag{170}$$

Since $\beta(A) = N$, there exists an orthonormal basis $\{e_1, \cdots, e_N\}$ for $(\operatorname{ran} A)^{\perp}$. Let P_m be the projection from H to $\vee\{e_m\}$ for $m = 1, \cdots, N$, where $\vee(\cdot)$ denotes the closed linear span. If we let $Cy_n := \alpha_n + \beta_n$ $(n \in \mathbb{Z}^+)$, where $\alpha_n \in \operatorname{ran} A$ and $\beta_n \in (\operatorname{ran}, A)^{\perp}$, then by (170), $||\beta_n|| > \epsilon$ for all $n \in \mathbb{Z}^+$. Observe that $\sum_{n=1}^{\infty} ||\frac{1}{n}\beta_n|| = \infty$ and hence $||\sum_{n=1}^{\infty} P_{m_0}(\frac{1}{n}\beta_n e^{i\theta_n})|| = \infty$ for some $m_0 \in \{1, \cdots, N\}$ and for some $\theta_n \in [0, 2\pi)$ $(n \in \mathbb{Z}^+)$. Now if we write $y := \sum_{n=1}^{\infty} \frac{1}{n} y_n e^{i\theta_n}$, then $||y||^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ and hence $y \in K$. But

$$||Cy|| \geq ||P_{m_0}(Cy)|| = ||\sum_{n=1}^{\infty} P_{m_0} C(\frac{1}{n} y_n e^{i\theta_n})|| = ||\sum_{n=1}^{\infty} P_{m_0}(\frac{1}{n}\beta_n e^{i\theta_n})|| = \infty,$$

giving a contradiction. Therefore we must have that $\beta(A) = \infty$. This proves (168). Now the result follows from (166), (167) and (168). □

The following corollary is immediate from Theorem 3.33.

Corollary 3.34. *For a given pair (A, B) of operators we have*

$$\begin{aligned}&\bigcap_{C \in B(K,H)} \sigma_{ap}(M_C) \\ &= \sigma_{ap}(A) \bigcup \{\lambda \in \mathbb{C} : \operatorname{ran}(B - \lambda I) \textit{ is closed and } \beta(A - \lambda I) < \alpha(B - \lambda I)\} \\ &\quad \bigcup \{\lambda \in \mathbb{C} : \operatorname{ran}(B - \lambda I) \textit{ is not closed and } \beta(A - \lambda I) < \infty\}.\end{aligned}$$

The following is the dual statement of Corollary 3.34.

Corollary 3.35. *For a given pair (A, B) of operators we have*

$$\bigcap_{C\in B(K,H)} \sigma_d(M_C) = \sigma_d(B) \bigcup \{\lambda \in \mathbb{C} : \operatorname{ran}(A-\lambda I) \text{ is closed and } \alpha(B-\lambda I) < \beta(A-\lambda I)\}$$
$$\bigcup \{\lambda \in \mathbb{C} : \operatorname{ran}(A-\lambda I) \text{ is not closed and } \alpha(B-\lambda I) < \infty\}.$$

Combining Corollaries 3.34 and 3.35 gives:

Corollary 3.36. ([DP, Theorem 2]) *For a given pair (A, B) of operators we have*

$$\bigcap_{C\in B(K,H)} \sigma(M_C) = \sigma_{ap}(A) \bigcup \sigma_d(B) \bigcup \{\lambda \in \mathbb{C} : \ \alpha(B-\lambda) \neq \beta(A-\lambda)\}.$$

It was shown in Corollary 3.9 that the passage from $\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ to $\sigma(M_C)$ is accomplished by removing certain open subsets of $\sigma(A) \cap \sigma(B)$ from the former, that is, there is equality

$$\sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \sigma(M_C) \cup W, \tag{171}$$

where W is the union of certain of the holes in $\sigma(M_C)$ which happen to be subsets of $\sigma(A)\cap\sigma(B)$. However we need not expect the case for the approximate point spectrum (see Examples 3.39 and 3.40 below). The passage from $\sigma_{ap}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ to $\sigma_{ap}(M_C)$ is more delicate.

Theorem 3.37. *For a given pair (A, B) of operators we have that for every $C \in B(K, H)$,*

$$\eta\big(\sigma_{ap}(A) \cup \sigma_{ap}(B)\big) = \eta\big(\sigma_{ap}(M_C)\big). \tag{172}$$

More concretely,

$$\sigma_{ap}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \sigma_{ap}(M_C) \cup W, \tag{173}$$

where W lies in certain holes in $\sigma_{ap}(A)$, which happen to be subsets of $\sigma_d(A) \cap \sigma_{ap}(B)$. Hence, in particular, $r_{ap}(M_C)$ is a constant, and furthermore for every $C \in B(K, H)$,

$$r\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) = r\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = r_{ap}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = r_{ap}\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right), \tag{174}$$

where $r(\cdot)$ and $r_{ap}(\cdot)$ denote the spectral radius and the "approximate point spectral radius".

Proof. First, observe that for a given pair (A, B) of operators we have that for every $C \in B(K, H)$,

$$\sigma_{ap}(A) \subseteq \sigma_{ap}(M_C) \subseteq \sigma_{ap}(A) \cup \sigma_{ap}(B) = \sigma_{ap}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right): \tag{175}$$

the first and the second inclusions follow from (166) and the last equality is obvious. We now claim that for every $T \in B(H)$,

$$\eta\big(\sigma(T)\big) = \eta\big(\sigma_{ap}(T)\big). \tag{176}$$

Indeed since $\operatorname{int} \sigma_{ap}(T) \subseteq \operatorname{int} \sigma(T)$ and $\partial\,\sigma(T) \subseteq \sigma_{ap}(T)$, we have that $\partial\,\sigma(T) \subseteq \partial\,\sigma_{ap}(T)$, which implies that the passage from $\sigma_{ap}(T)$ to $\sigma(T)$ is filling in certain holes in $\sigma_{ap}(T)$, proving (176). Now suppose $\lambda \in \big(\sigma_{ap}(A) \cup \sigma_{ap}(B)\big) \setminus \sigma_{ap}(M_C)$. Thus by (175), $\lambda \in \sigma_{ap}(B) \setminus \sigma_{ap}(A)$. Since $M_C - \lambda I$ is bounded below it follows from Theorem 3.33 that if $\operatorname{ran}(B - \lambda I)$ is not closed then $\beta(A - \lambda I) = \infty$, and if instead $\operatorname{ran}(B - \lambda I)$ is closed then $\beta(A - \lambda I) \geq$

$\alpha(B-\lambda I) > 0$, where the last inequality comes from the fact that $B-\lambda I$ is not one-one since $B-\lambda I$ is not bounded below. Therefore $\lambda \in \sigma_d(A)$. On the other hand, λ should be in one of the holes in $\sigma_{ap}(A)$: for if this were not so then by (176), $A-\lambda I$ would be invertible, a contradiction. This proves (172) and (173). The equality (174) follows at once from (172) and (176). ☐

Corollary 3.38. *If A is a quasitriangular operator (e.g., A is either compact or cohyponormal) then for every $B \in K$ and $C \in B(K,H)$,*

$$\sigma_{ap}(M_C) = \sigma_{ap}(A) \cup \sigma_{ap}(B).$$

Proof. The inclusion $\subseteq$ is the second inclusion in (175). For the reverse inclusion suppose $\lambda \in \sigma_{ap}(A) \cup \sigma_{ap}(B)$. If $\lambda \in \big(\sigma_{ap}(A) \cup \sigma_{ap}(B)\big) \setminus \sigma_{ap}(M_C)$ then by Theorem 3.37, $\lambda \in \sigma_\delta(A) \cap \sigma_{ap}(B)$ and $A-\lambda I$ is bounded below. But since A is quasitriangular, we have that $\beta(A-\lambda I) \le \alpha(A-\lambda I) = 0$. Therefore $A-\lambda I$ is invertible, a contradiction. ☐

Example 3.39. One might expect that in Theorem 3.37, W is the union of certain of the holes in $\sigma_{ap}(M_C)$ together with the closure of some isolated points of $\sigma_{ap}(B)$. But this is not the case. To see this, let $\varphi \in H^\infty$ be an inner function (i.e., $|\varphi| = 1$ a.e.) with $\dim(\varphi H^2)^\perp = \infty$ (e.g., $\varphi(z) = \exp\left(\frac{z+\lambda}{z-\lambda}\right)$ with $|\lambda| = 1$), let ψ be any function in $C(\mathbb{T})$ with $||\psi||_\infty < 1$, and let J be an isometry from H^2 to $(\varphi H^2)^\perp$. Define

$$M_J := \begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix}.$$

Note that T_φ is a non–normal isometry and hence $\sigma_{ap}(T_\varphi) = \mathbb{T}$. Since $\operatorname{ran}(T_\varphi) \perp \operatorname{ran}(J)$, it follows that $||M_J \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)|| \ge ||\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)||$ for all $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in H^2 \oplus H^2$, which says that M_J is bounded below. Observe

$$\gamma(M_J) = \inf_{||\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)||=1} ||M_J \left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)|| \ge 1.$$

Thus by [Go, Theorem V.1.6], we have that for all $|\lambda| < 1 (\le \gamma(M_J))$,

(i) $M_J - \lambda I$ is semi-Fredholm;

(ii) $\alpha(M_J - \lambda I) \le \alpha(M_J) = 0$,

which implies that $M_J - \lambda I$ is bounded below for all $|\lambda| < 1$. But since $\sigma(T_\psi)$ is contained in the polynomially–convex hull of the range of ψ, it follows from our assumption that $\sigma_{ap}(T_\psi) \subseteq \mathbb{D}$. Thus by Theorem 3.37 we have that $\sigma_{ap}(M_J) = \mathbb{T}$. Note that $\sigma_{ap}(T_\psi)$ has disappeared in the passage from $\sigma_{ap}\left(\begin{smallmatrix} T_\varphi & 0 \\ 0 & T_\psi \end{smallmatrix}\right)$ to $\sigma_{ap}(M_J)$.

Example 3.40. We need not expect a general information for removing in the passage from $\sigma_{ap}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ to $\sigma_{ap}(M_C)$. To see this, let T_φ, T_ψ, and J be given as in Example 3.39. Also let ζ be a function in $C(\mathbb{T})$ such that $\sigma_{ap}(T_\zeta)$ is a compact subset σ of $\sigma_{ap}(T_\psi)$. We define, on $H^2 \oplus H^2$, $A := T_\varphi \oplus T_\varphi$, $B := T_\psi \oplus T_\zeta$, $C := J \oplus 0$ and in turn

$$M_C := \begin{pmatrix} T_\varphi & 0 & J & 0 \\ 0 & T_\varphi & 0 & 0 \\ 0 & 0 & T_\psi & 0 \\ 0 & 0 & 0 & T_\zeta \end{pmatrix}.$$

A straightforward calculation shows

$$\sigma_{ap}\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \sigma_{ap}(A) \cup \sigma_{ap}(B) = \mathbb{T} \cup \sigma_{ap}(T_\psi).$$

On the other hand, M_C is unitarily equivalent to the operator

$$\begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix} \bigoplus \begin{pmatrix} T_\varphi & 0 \\ 0 & T_\zeta \end{pmatrix}.$$

By Example 3.39 above, $\sigma_{ap}\begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix} = \mathbb{T}$. It therefore follows that

$$\sigma_{ap}(M_C) = \sigma_{ap}\begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix} \bigcup \sigma_{ap}\begin{pmatrix} T_\varphi & 0 \\ 0 & T_\zeta \end{pmatrix} = \mathbb{T} \cup \sigma.$$

One might conjecture that if M_C is bounded below then $\operatorname{ran} B$ is closed. But this is not the case. For example, in Example 3.40, take a function $\psi \in C(\mathbb{T})$ whose range includes 0, and consider M_C.

3.5 Concluding remarks and open problems

In connection with upper triangles, we have a chance to consider weak subnormality. Note that the operator T is subnormal if and only if there exist operators A and B such that

$$\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$$

is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases} \tag{177}$$

The operator $\widehat{T}$ is called a *normal extension* of T. We also say that $\widehat{T}$ in $B(K)$ is a *minimal normal extension* (briefly, m.n.e.) of T if K has no proper subspace containing H to which the restriction of $\widehat{T}$ is also a normal extension of T. It is known that

$$\widehat{T} = \text{ m.n.e.}(T) \iff K = \bigvee \{\widehat{T}^{*n} h : \ h \in H, \ n \geq 0\},$$

and the m.n.e.(T) is unique.

Recall ([CuL]) that an operator $T \in B(H)$ is said to be *weakly subnormal* if there exist operators $A \in B(H', H)$ and $B \in B(H')$ such that the first two conditions in (177) hold:

$$[T^*, T] = AA^* \quad \text{and} \quad A^*T = BA^*, \tag{178}$$

or equivalently, there is an extension $\widehat{T}$ of T such that $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \quad$ for all $f \in H$. The operator $\widehat{T}$ is called a *partially normal extension* (briefly, p.n.e.) of T. We also say that $\widehat{T}$ in $B(K)$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of T if K has no proper subspace containing H to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known ([CuL, Lemma 2.5 and Corollary 2.7]) that

$$\widehat{T} = \text{ m.p.n.e.}(T) \iff K = \bigvee \{\widehat{T}^{*n} h : \ h \in H, \ n = 0, 1\},$$

and the m.p.n.e.(T) is unique. For convenience, if $\widehat{T} =$ m.p.n.e. (T) is also weakly subnormal then we write $\widehat{T}^{(2)} := \widehat{\widehat{T}}$ and more generally, $\widehat{T}^{(n)} := \widehat{\widehat{T}^{(n-1)}}$, which will be called the *n-th minimal partially normal extension of* T. It was ([CuL], [CJP]) shown that

$$\text{2-hyponormal} \implies \text{weakly subnormal} \implies \text{hyponormal} \tag{179}$$

and the converses of both implications in (179) are not true in general. It was ([CuL]) known that

$$T\text{is weakly subnormal} \Longrightarrow T(\ker[T^*,T]) \subseteq \ker[T^*,T] \tag{180}$$

and it was ([CJP]) known that if $\widehat{T} :=$ m.p.n.e.(T) then for any $k \geq 1$,

$$T \text{ is } (k+1)\text{-hyponormal} \Longleftrightarrow T \text{ is weakly subnormal and } \widehat{T} \text{ is } k\text{-hyponormal.} \tag{181}$$

So, in particular, one can see that if T is subnormal then $\widehat{T}$ is subnormal. It is worth to noticing that in view of (179) and (180), Morrel's theorem gives that every weakly subnormal operator with rank-one self-commutator is subnormal.

On the other hand, in 2000, M. Dritschel and S. McCullough [DrM] have developed a model theory for hyponormal contractions in the context of the Agler's abstract model theory [Ag]. The purpose is to find a small, representative subcollection of a given family of operators, a so–called *model*, with the property that any member of the family extends to a member of the subcollection. Following Agler [Ag], a *family* $\mathcal{F}$ is a bounded collection of Hilbert space operators which is closed with respect to arbitrary direct sums, restrictions to invariant subspaces, and unital $*$-representations. There are many examples of such families: subnormal contractions, contractions, isometries, etc. The *extremals* $\operatorname{ext}\mathcal{F}$ of $\mathcal{F}$ are those operators T in $\mathcal{F}$ whose only extensions in $\mathcal{F}$ are obtained by adding a direct summand to T. The extremals have a role in finding the smallest possible model for $\mathcal{F}$, the *boundary* $\partial\mathcal{F}$ of $\mathcal{F}$. In [Ag, Propositions 5.9 and 5.10], it was shown that the extremals belong to every model, and that every element of $\mathcal{F}$ lifts to an element of $\operatorname{ext}\mathcal{F}$. In [DrM] it was proved that if T is a contractive n-hyponormal operator and if

$$\operatorname{ran}(T^{*k}A) \cap \operatorname{ran} A = \{0\} \tag{182}$$

and

$$\ker T^{*k} \cap \operatorname{ran} A = \{0\} \tag{183}$$

for some $1 \leq k \leq n$, where $[T^*,T] = AA^*$, then T is extremal. The following corollary shows that if T is weakly subnormal then conditions (182) and (183) force T to be normal.

Theorem 3.41. *Let $T \in B(H)$ be a weakly subnormal operator satisfying* (182) *and* (183) *for some $1 \leq k \leq n$. Then T must be normal, and therefore T is extremal for the collection $\mathcal{F}_{ws}$ of contractive weakly subnormal operators.*

Proof. Suppose T is weakly subnormal. Then there exists a partially normal extension $\widehat{T}$ of T such that

$$\widehat{T} = \begin{pmatrix} T & A \\ 0 & B \end{pmatrix} \quad \text{with } [T^*,T] = AA^* \text{ and } T^*A = AB^*.$$

Thus by induction, $T^{*k}A = AB^{*k}$, so $\operatorname{ran} T^{*k}A \subseteq \operatorname{ran} A$ for $1 \leq k \leq n$. Thus $\mathfrak{M}_k := \operatorname{ran}(T^{*k}A) \cap \operatorname{ran} A = \operatorname{ran}(T^{*k}A)$. By (182) we have that $\mathfrak{M}_k = \{0\}$, i.e., $T^{*k}A = 0$ for some

$1 \leq k \leq n$. Let $f \in \mathcal{H}$, and let $g := Af$. We have

$$\begin{aligned} T^{*k}g = T^{*k}Af = 0 &\Longrightarrow g \in \ker T^{*k} \cap \operatorname{ran} A \\ &\Longrightarrow g = 0 \quad \text{(by (183))} \\ &\Longrightarrow Af = 0. \end{aligned}$$

It follows that $A = 0$, which implies that T is normal. The extremality of normal operators for $\mathcal{F}_{ws}$ follows by looking at self-commutators. □

Corollary 3.42. *Let T be a contractive 2-hyponormal operator with closed range self-commutator. Assume that T satisfies* (182) *and* (183). *Then T must be normal, and therefore T is extremal for $\mathfrak{h}_2$, the family of 2–hyponormal contractions.*

A natural question arises: *Is every 2-hyponormal operator satisfying* (182) *and* (183) *normal?*

Finally, we examine five additional problems.

Problem 3.43. *Does every 2-hyponormal operator have a partially normal extension which is also 2-hyponormal?*

Let us suppose that the answer is affirmative. Let $\mathcal{N}$, $\mathcal{S}$, and $\mathfrak{h}_2$ denote the collections of normal, subnormal, and 2-hyponormal contractions, respectively. We now claim that if every element of $\mathfrak{h}_2$ has a partially normal extension in $\mathfrak{h}_2$, then $\operatorname{ext}\mathfrak{h}_2 = \mathcal{N}$. The inclusion $\mathcal{N} \subseteq \operatorname{ext}\mathfrak{h}_2$ is evident, and was mentioned in Corollary 3.42. For the converse, suppose $T \in \operatorname{ext}\mathfrak{h}_2$. By our assumption T has a partially normal extension $\widehat{T}$ which is 2-hyponormal:

$$\widehat{T} = \begin{pmatrix} T & A \\ 0 & S \end{pmatrix} \in \mathfrak{h}_2.$$

By extremality, we have $A = 0$, so weak subnormality forces T to be normal. Therefore $\operatorname{ext}\mathfrak{h}_2 = \mathcal{N}$. By [Ag, Proposition 5.10], every element in $\mathfrak{h}_2$ would then have a normal extension, and hence $\mathfrak{h}_2 = \mathcal{S}$, which leads to a contradiction because we know that there are non–subnormal 2-hyponormal operators. We have thus obtained the following result, which answers Problem 3.43 in the negative.

Proposition 3.44. *There exists a 2–hyponormal operator T which either does not have a partially normal extension, or such that* m.p.n.e. (T) *is not 2–hyponormal.*

Problem 3.45. *Does the collection $\mathcal{F}_{ws}$ of weakly subnormal contractions form a family?*

Note that $\mathcal{F}_{ws}$ is closed with respect to (i) restrictions to invariant subspaces (c.f. basic facts below Definition 1.1); (ii) unital $*$-representations (evident from the definition); and (iii) *finite* direct sums, by the following observation: if T_1 and T_2 have partially normal extensions $\left(\begin{smallmatrix} T_1 & A \\ 0 & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} T_2 & C \\ 0 & D \end{smallmatrix}\right)$, then

$$\begin{pmatrix} T_1 & 0 & A & 0 \\ 0 & T_2 & 0 & C \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & D \end{pmatrix}$$

is a partially normal extension of $T_1 \oplus T_2$. But it is not clear whether $\mathcal{F}_{ws}$ is closed with respect to *arbitrary* direct sums.

Problem 3.46. *Is $\mathcal{F}_{ws}$ sot-closed?*

We can easily show that $\mathcal{S}$ is *sot*-closed (in fact, $\mathcal{S} = sot\text{-cl}\,\mathcal{N}$) and that the collection $\mathfrak{h}_k$ of k-hyponormal contractions is also *sot*-closed for each $k \geq 1$. On the other hand, we conjecture that

$$\mathfrak{h}_2 \subseteq \mathcal{F}_{ws} \subseteq \mathfrak{h}_1.$$

Thus an affirmative answer to Problem 3.46 would probably exhibit a *sot*-closed collection of operators between $\mathfrak{h}_2$ and $\mathfrak{h}_1$. More generally, we have:

Problem 3.47. *Is there a sot-closed collection of operators between $\mathfrak{h}_k$ and $\mathfrak{h}_{k+1}$ for each $k \geq 1$?*

On the other hand, if $\mathcal{F}_{ws}$ were *not sot*-closed, we would ask:

Problem 3.48. *Is every hyponormal operator a sot-limit of a sequence of weakly subnormal operators, i.e., $\mathfrak{h}_1 = sot\text{-cl}\,\mathcal{F}_{ws}$?*

4 Weyl theory in several variables

In this section we consider Weyl's theorem from multivariable operator theory. Let H be a complex Hilbert space and write $B(H)$ for the set of bounded linear operators acting on H. Let $T = (T_1, \cdots, T_n)$ be a commuting n-tuple of operators in $B(H)$, let $\Lambda[e] \equiv \{\Lambda^k[e_1, \cdots, e_n]\}_{k=0}^n$ be the exterior algebra on n generators ($e_i \wedge e_j = -e_j \wedge e_i$ for all $i, j = 1, \cdots, n$) and write $\Lambda(H) := \Lambda[e] \otimes H$. Let $\Lambda(T) : \Lambda(H) \to \Lambda(H)$ be defined by (cf. [Cu1], [Har1], [Har4], [Ta])

$$\Lambda(T)(\omega \otimes x) = \sum_{i=1}^{n} (e_i \wedge \omega) \otimes T_i x. \tag{184}$$

The operator $\Lambda(T)$ in (184) can be represented by the *Koszul complex* for T:

$$0 \longrightarrow \Lambda^0(H) \xrightarrow{\Lambda^0(T)} \Lambda^1(H) \xrightarrow{\Lambda^1(T)} \cdots \xrightarrow{\Lambda^{n-1}(T)} \Lambda(H) \longrightarrow 0\ , \tag{185}$$

where $\Lambda^k(H)$ is the collection of k-forms and $\Lambda^k(T) = \Lambda(T)|_{\Lambda^k(H)}$. For $n = 2$, the Koszul complex for $T = (T_1, T_2)$ is given by

$$0 \longrightarrow H \xrightarrow{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}} \begin{pmatrix} H \\ H \end{pmatrix} \xrightarrow{\begin{pmatrix} -T_2 & T_1 \end{pmatrix}} H \longrightarrow 0$$

Evidently, $\Lambda(T)^2 = 0$, so that $\operatorname{ran}\Lambda(T) \subseteq \ker\Lambda(T)$, or equivalently, $\operatorname{ran}\Lambda^{k-1}(T) \subseteq \ker\Lambda^k(T)$ for every $k = 0, \cdots, n$, where, for notational convenience, $\Lambda^{-1}(T) := 0$ and $\Lambda^n(T) = 0$. For the representation of $\Lambda(T)$, we may put together its odd and even parts, writing

$$\Lambda(T) = \begin{pmatrix} 0 & \Lambda^{\text{odd}}(T) \\ \Lambda^{\text{even}}(T) & 0 \end{pmatrix} : \begin{pmatrix} \Lambda^{\text{odd}}(H) \\ \Lambda^{\text{even}}(H) \end{pmatrix} \to \begin{pmatrix} \Lambda^{\text{odd}}(H) \\ \Lambda^{\text{even}}(H) \end{pmatrix},$$

where

$$\Lambda^*(H) = \bigoplus_{p \text{ is } *} \Lambda^p(H), \quad \Lambda^*(T) = \bigoplus_{p \text{ is } *} \Lambda^p(T) \qquad \text{with } * = \text{even, odd.}$$

Write

$$H^k(T) := \ker\Lambda^k(T)/\operatorname{ran}\Lambda^{k-1}(T) \quad (k = 0, \cdots, n),$$

which is called the *k-th cohomology* for the Koszul complex $\Lambda(T)$. We recall ([Cu1], [Har4], [Ta]) that T is said to be *Taylor invertible* if $\ker\Lambda(T) = \operatorname{ran}\Lambda(T)$ (in other words, the Koszul complex (185) is exact at every stage, i.e., $H^k(T) = \{0\}$ for every $k = 0, \cdots, n$) and is said to be *Taylor Fredholm* if $\ker\Lambda(T)/\operatorname{ran}\Lambda(T)$ is finite dimensional (in other words, all cohomologies of (185) are finite dimensional). If $T = (T_1, \cdots, T_n)$ is Taylor Fredholm, define the *index* of T by

$$\text{index}\,(T) \equiv \text{Euler}(0, \Lambda^{n-1}(T), \cdots, \Lambda^0(T), 0) := \sum_{k=0}^{n} (-1)^k \dim H^k(T),$$

where Euler$(\cdot)$ is the Euler characteristic of the Koszul complex for T. We shall write $\sigma_T(T)$ and $\sigma_{T_e}(T)$ for the *Taylor spectrum* and *Taylor essential spectrum* of T, respectively: namely,

$$\sigma_T(T) = \{\lambda \in \mathbf{C}^n : \ T - \lambda I \text{ is not Taylor invertible}\};$$
$$\sigma_{T_e}(T) = \{\lambda \in \mathbf{C}^n : \ T - \lambda I \text{ is not Taylor Fredholm}\}.$$

Following to R. Harte [Har4, Definition 11.10.5], we shall say that $T = (T_1, \cdots, T_n)$ is *Taylor Weyl* if T is Taylor Fredholm and index$(T) = 0$. The *Taylor Weyl spectrum*, $\sigma_{T_w}(T)$, of T is defined by

$$\sigma_{T_w}(T) = \{\lambda \in \mathbf{C}^n : \ T - \lambda I \text{ is not Taylor Weyl}\}.$$

It is known ([Har4, Theorem 10.6.4]) that $\sigma_{T_w}(T)$ is compact and evidently,

$$\sigma_{T_e}(T) \subset \sigma_{T_w}(T) \subset \sigma_T(T).$$

4.1 Weyl's theorem in several variables

"Weyl's theorem" for an operator on a Hilbert space is the statement that the complement in the spectrum of the Weyl spectrum coincides with the isolated eigenvalues of finite multiplicity. In this note we introduce the joint version of Weyl's theorem and then examine the classes of n-tuples of operators satisfying Weyl's theorem.

The spectral mapping theorem is liable to fail for $\sigma_{T_w}(T)$ even though $T = (T_1, \cdots, T_n)$ is a commuting n-tuple of hyponormal operators (remember [LeL] that if $n = 1$ then every hyponormal operator enjoys the spectral mapping theorem for the Weyl spectrum). For example, let U be the unilateral shift on ℓ^2 and $T := (U, U)$. Then a straightforward calculation shows that $\sigma_{T_w}(T) = \{(\lambda, \lambda) : |\lambda| = 1\}$. If $f : \mathbf{C}^2 \to \mathbf{C}^1$ is the map $f(z_1, z_2) = z_1 + z_2$ then $\sigma_{T_w} f(T) = \sigma_{T_w}(2U) = \{2\lambda : |\lambda| \le 1\} \not\subseteq f\sigma_{T_w}(T) = \{2\lambda : |\lambda| = 1\}$. If instead $f : \mathbf{C}^1 \to \mathbf{C}^2$ is the map $f(z) = (z, z)$ then $\sigma_{T_w} f(U) = \{(\lambda, \lambda) : |\lambda| = 1\} \not\supseteq f\sigma_{T_w}(U) = \{(\lambda, \lambda) : |\lambda| \le 1\}$. Therefore $\sigma_{T_w}(T)$ satisfies no way spectral mapping theorem in general.

The Taylor Weyl spectrum however satisfies a "subprojective" property.:

Lemma 4.1. *If $T = (T_1, \cdots, T_n)$ is a commuting n-tuple then $\sigma_{T_w}(T) \subset \prod_{j=1}^n \sigma_{T_e}(T_j)$.*

Proof. This follows at once from the fact (cf. [Cu1, p.144]) that every commuting n-tuple having a Fredholm coordinate has index zero. □

On the other hand, M. Cho and M. Takaguchi [ChT] have defined the *joint Weyl spectrum*, $\omega(T)$, of a commuting n-tuple $T = (T_1, \cdots, T_n)$ by

$$\omega(T) = \bigcap \big\{\sigma_T(T + K) : K = (K_1, \cdots, K_n) \text{ is an } n\text{-tuple of compact operators and } T + K = (T_1 + K_1, \cdots, T_n + K_n) \text{ is commutative.}\big\}$$

A question arises naturally: *For a commuting n-tuple T, does it follow that $\sigma_{T_w}(T) = \omega(T)$?* If $n = 1$ then $\sigma_{T_w}(T)$ and $\omega(T)$ coalesce: indeed, T is Weyl if and only if T is a sum of an invertible operator and a compact operator.

We first observe:

Lemma 4.2. *If $T = (T_1, \cdots, T_n)$ is a commuting n-tuple then*

$$\sigma_{T_w}(T) \subset \omega(T). \tag{186}$$

Proof. Instead of assembling the Koszul complex into the single operator $\Lambda(T)$, we put together its odd and even parts, writing

$$\Lambda(T) = \begin{pmatrix} 0 & \Lambda^{\text{odd}}(T) \\ \Lambda^{\text{even}}(T) & 0 \end{pmatrix} : \begin{pmatrix} \Lambda^{\text{odd}}(H) \\ \Lambda^{\text{even}}(H) \end{pmatrix} \longrightarrow \begin{pmatrix} \Lambda^{\text{odd}}(H) \\ \Lambda^{\text{even}}(H) \end{pmatrix},$$

where

$$\Lambda^{\text{even}}(H) = \bigoplus_{p \text{ even}} \Lambda^p(H), \quad \Lambda^{\text{odd}}(H) = \bigoplus_{p \text{ odd}} \Lambda^p(H),$$
$$\Lambda^{\text{even}}(T) = \bigoplus_{p \text{ even}} \Lambda^p(T), \quad \Lambda^{\text{odd}}(T) = \bigoplus_{p \text{ odd}} \Lambda^p(T).$$

Write $K_0(T) := \Lambda^{\text{odd}}(T) + \Lambda^{\text{even}}(T)^*$. Then it was known that (cf. [Cu1], [Har4], [Va])

$$T \text{ is Taylor invertible [Taylor Fredholm]} \iff K_0(T) \text{ is invertible [Fredholm]} \tag{187}$$

and moreover $\text{index}(T) = \text{index}(K_0(T))$. If $\lambda = (\lambda_1, \cdots, \lambda_n) \notin \omega(T)$ then there exists an n-tuple of compact operators $K = (K_1, \cdots, K_n)$ such that $T + K - \lambda I$ is commutative and Taylor invertible. By (187), $K_0(T + K - \lambda I)$ is invertible. But since $K_0(T + K - \lambda I) - K_0(T - \lambda I)$ is a compact operator it follows that $K_0(T - \lambda I)$ is Weyl, and hence, by (187), $T - \lambda I$ is Taylor Weyl, i.e., $\lambda \notin \sigma_{T_w}(T)$. □

The inclusion (186) cannot be strengthened by the equality. R. Gelca [Ge] showed that if S is a Fredholm operator with $\text{index}(S) \neq 0$ then there do not exist compact operators K_1 and K_2 such that $(T + K_1, K_2)$ is commutative and Taylor invertible. Thus for instance, if U is the unilateral shift then $\omega(U, 0) \nsubseteq \sigma_{T_w}(U, 0)$.

We introduce an interesting notion which commuting n-tuples may enjoy.

A commuting n-tuple $T = (T_1, \cdots, T_n)$ is said to have the *quasitriangular property* if the dimension of the left cohomology for the Koszul complex $\Lambda(T - \lambda I)$ is greater than or equal to the dimension of the right cohomology for $\Lambda(T - \lambda I)$ for all $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$, i.e.,

$$\dim H^n(T - \lambda I) \le \dim H^0(T - \lambda I) \quad \text{for all } \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n. \tag{188}$$

Since $H^0(T - \lambda I) = \ker \Lambda^0(T - \lambda I) = \bigcap_{i=1}^n \ker(T_i - \lambda_i I)$ and $H^n(T - \lambda I) = \ker \Lambda^n(T - \lambda I)/\text{ran}\Lambda^{n-1}(T - \lambda I) \cong \left(\text{ran}\Lambda^{n-1}(T - \lambda I)\right)^\perp \cong \bigcap_{i=1}^n \ker(T_i - \lambda_i I)^*$, the condition (188) is equivalent to the condition

$$\dim \bigcap_{i=1}^n \ker(T_i - \lambda_i I)^* \le \dim \bigcap_{i=1}^n \ker(T_i - \lambda_i I).$$

If $n = 1$, the condition (188) is equivalent to the condition $\dim(T-\lambda I)^{*-1}(0) \le \dim(T-\lambda I)^{-1}(0)$ for all $\lambda \in \mathbf{C}$, or equivalently, the spectral picture of T contains no holes or pseudoholes associated with a negative index, which, by the celebrated theorem due to Apostol, Foias and Voiculescu, is equivalent to the fact that T is quasitriangular (cf. [Pe, Theorem 1.31]). Evidently, every commuting n-tuple of quasitriangular operators has the quasitriangular property. Also if a commuting n-tuple $T = (T_1, \cdots, T_n)$ has a coordinate whose adjoint has no eigenvalues then T has the quasitriangular property.

As we have seen in the above, the inclusion (186) cannot be reversible even though $T = (T_1, \cdots, T_n)$ is a doubly commuting n-tuple (i.e., $[T_i, T_j^*] \equiv T_iT_j^* - T_j^*T_i = 0$ for all $i \ne j$) of hyponormal operators. On the other hand, R. Curto [Cu1, Corollary 3.8] showed that if $T = (T_1, \cdots, T_n)$ is a doubly commuting n-tuple of hyponormal operators then

$$T \text{ is Taylor invertible [Taylor Fredholm]} \Longleftrightarrow \sum_{i=1}^{n} T_iT_i^* \text{ is invertible [Fredholm].} \tag{189}$$

On the other hand, many authors have considered the joint version of the Browder spectrum. We recall ([BDW], [CuD], [Da1], [Da2], [Har4], [JeL], [Sn]) that a commuting n-tuple $T = (T_1, \cdots, T_n)$ is called *Taylor Browder* if T is Taylor Fredholm and there exists a deleted open neighborhood N_0 of $0 \in \mathbf{C}^n$ such that $T - \lambda I$ is Taylor invertible for all $\lambda \in N_0$. The *Taylor Browder spectrum*, $\sigma_{T_b}(T)$, is defined by

$$\sigma_{T_b}(T) = \{\lambda \in \mathbf{C}^n : \ T - \lambda I \text{ is not Taylor Browder}\}.$$

Note that $\sigma_{T_b}(T) = \sigma_{T_e}(T) \cup \operatorname{acc} \sigma_T(T)$, where $\operatorname{acc}(\cdot)$ denotes the set of accumulation points. We can easily show that

$$\sigma_{T_w}(T) \subset \sigma_{T_b}(T). \tag{190}$$

Indeed, if $\lambda \notin \sigma_{T_b}(T)$ then $T - \lambda I$ is Taylor Fredholm and there there exists $\delta > 0$ such that $T - (\lambda + \mu)I$ is Taylor invertible for $0 < |\mu| < \delta$. Since the index is continuous it follows that $\operatorname{index}(T - \lambda) = 0$, which says that $\lambda \notin \sigma_{T_w}(T)$, giving (190).

If $T = (T_1, \cdots, T_n)$ is a commuting n-tuple, we write $\pi_{00}(T)$ for the set of all isolated points of $\sigma_T(T)$ which are joint eigenvalues of finite multiplicity and write $\mathcal{R}(T) \equiv \operatorname{iso} \sigma_T(T) \setminus \sigma_{T_e}(T)$ for the *Riesz points* of $\sigma_T(T)$. By the continuity of the index, we can see that $\mathcal{R}(T) = \operatorname{iso} \sigma_T(T) \setminus \sigma_{T_w}(T)$.

Lemma 4.3. *If $T = (T_1, \cdots, T_n)$ is a commuting n-tuple then $\omega(T) \subset \sigma_{T_b}(T)$.*

Proof. Suppose without loss of generality that $0 \notin \sigma_{T_b}(T)$. Then T is Taylor invertible and $0 \in \operatorname{iso}\sigma_T(T)$. So there exists a projection $P \in B(H)$ satisfying that

(i) P commutes with T_i $(i = 1, \cdots, n)$;

(ii) $\sigma_T(T|_{P(H)}) = \{0\}$ and $\sigma_T(T|_{(I-P)(H)}) = \sigma_T(T) \setminus \{0\}$;

(iii) P is of finite rank

(see [Ta2, Theorem 4.9]). Put $Q = (P, \cdots, P)$. Evidently, $0 \notin \sigma_T((T+Q)|_{(I-P)(H)})$. Since a commuting quasinilpotent perturbation of an invertible operator is also invertible, it follows that $0 \notin \sigma_T((T + Q)|_{P(H)})$. But since $\sigma_T(T) = \sigma_T((T + Q)|_{(I-P)(H)}) \bigcup \sigma_T((T + Q)|_{P(H)})$, we can conclude that $T + Q$ is Taylor invertible. So $0 \notin \omega(T)$. $\square$

"Weyl's theorem" for an operator on a Hilbert space is the statement that the complement in the spectrum of the Weyl spectrum coincides with the isolated eigenvalues of finite multiplicity. There are two versions of Weyl's theorem in several variables (cf. [Ch1], [Ch2], [ChT]).

If $T = (T_1, \cdots, T_n)$ is a commuting n-tuple then we say that *Weyl's theorem (I) holds for T* if

$$\sigma_T(T) \setminus \pi_{00}(T) = \sigma_{T_w}(T) \tag{191}$$

and that *Weyl's theorem (II) holds for T* if

$$\sigma_T(T) \setminus \pi_{00}(T) = \omega(T). \tag{192}$$

We note that

$$\text{Weyl's theorem (I)} \implies \text{Weyl's theorem (II)}. \tag{193}$$

Indeed, since $\sigma_{T_w}(T) \subset \omega(T)$, it follows that if $\sigma_T(T) \setminus \pi_{00}(T) \subset \sigma_{T_w}(T)$, then $\sigma_T(T) \setminus \pi_{00}(T) \subset \omega(T)$. Now suppose $\sigma_{T_w}(T) \subset \sigma_T(T) \setminus \pi_{00}(T)$. So if $\lambda \in \pi_{00}(T)$ then $T - \lambda I$ is Taylor Weyl, and hence Taylor Browder. By Lemma 4.3, $\lambda \notin \omega(T)$. Therefore $\omega(T) \subset \sigma_T(T) \setminus \pi_{00}(T)$, and so Weyl's theorem (II) holds for T, which gives (193).

But the converse of (193) is not true in general. To see this, let $T := (U, 0)$, where U is the unilateral shift on ℓ^2. Then
(a) $\sigma_T(T) = \text{cl}\,\mathbf{D} \times \{0\}$;
(b) $\sigma_{Tw}(T) = \partial\,\mathbf{D} \times \{0\}$;
(c) $\omega(T) = \text{cl}\,\mathbf{D} \times \{0\}$;
(d) $\pi_{00}(T) = \emptyset$,
where $\mathbf{D}$ is the open unit disk. So Weyl's theorem (II) holds for T while Weyl's theorem (I) fails even though T is a doubly commuting n-tuple of hyponormal operators.

M. Cho [Ch2] showed that Weyl's theorem (II) holds for a commuting n-tuple of normal operators. The following theorem is an extension of this result.

Theorem 4.4. *Let $T = (T_1, \cdots, T_n)$ be a doubly commuting n-tuple of hyponormal operators. If T has the quasitriangular property then Weyl's theorem (I) holds for T.*

Proof. In [Ch2] it was shown that if T is a doubly commuting n-tuple of hyponormal operators then $\omega(T) \subset \sigma_T(T) \setminus \pi_{00}(T)$. Then by Lemma 4.2, $\sigma_{Tw}(T) \subset \sigma_T(T) \setminus \pi_{00}(T)$. For the reverse inclusion, we first claim that

$$\sigma_{Te}(T) = \sigma_{Tw}(T) = \omega(T). \tag{194}$$

In view of Lemma 4.2, we need to show that $\omega(T) \subset \sigma_{Te}(T)$. Suppose without loss of generality that $0 \notin \sigma_{Te}(T)$. Thus by (189) we have that $\sum_{i=1}^n T_i T_i^*$ is Fredholm (and hence Weyl since it is self-adjoint). Let P denote the orthogonal projection onto $\ker \sum_{i=1}^n T_i T_i^*$. Since P is of finite rank and Weyl-ness is stable under compact perturbations, we have that $\sum_{i=1}^n T_i T_i^* + nP$ is Weyl. In particular, a straightforward calculation shows that $\sum_{i=1}^n T_i T_i^* +$

nP is one-one and therefore $\sum_{i=1}^n T_iT_i^* + nP$ is invertible. Since each T_i is a hyponormal operator, we have that

$$\text{ran}\, P = \ker\, \begin{pmatrix} T_1, \cdots, T_n \end{pmatrix} \begin{pmatrix} T_1^* \\ \vdots \\ T_n^* \end{pmatrix} = \bigcap_{i=1}^n \ker T_i^* \supset \bigcap_{i=1}^n \ker T_i.$$

So if T has the quasitriangular property then since $\text{ran}\, P$ is finite dimensional, it follows that

$$\text{ran}\, P = \bigcap_{i=1}^n \ker T_i = \bigcap_{i=1}^n \ker T_i^*.$$

So $T_iP = PT_i = 0$ for all $i = 1, \cdots, n$. Hence we can see that $(T_1 + P, \cdots, T_n + P)$ is a doubly commuting n-tuple of hyponormal operators. Thus $(T_1 + P, \cdots, T_n + P)$ is Taylor invertible if and only if $\sum T_iT_i^* + nP$ is invertible. Therefore $(T_1, \cdots, T_n) + (P, \cdots, P)$ is Taylor invertible, and hence $0 \notin \omega(T)$, which proves (194). So in view of (194), it now suffices to show that $\sigma_T(T) \setminus \pi_{00}(T) \subset \sigma_{Te}(T)$. To see this we need to prove that

$$\text{acc}\, \sigma_T(T) \ \subset \ \sigma_{Te}(T). \tag{195}$$

Suppose $\lambda = \lim \lambda_k$ with distinct $\lambda_k \in \sigma_T(T)$. Write $\lambda := (\lambda_1, \cdots, \lambda_n)$ and $\lambda_k := (\lambda_{k_1}, \cdots, \lambda_{k_n})$. If $\lambda_k \in \sigma_{Te}(T)$ then clearly, $\lambda \in \sigma_{Te}(T)$ since $\sigma_{Te}(T)$ is a closed set. So we assume $\lambda_k \in \sigma_T(T) \setminus \sigma_{Te}(T)$. Then by (189), $\sum_{i=1}^n (T_i - \lambda_{k_i} I)(T_i - \lambda_{k_i} I)^*$ is Fredholm but not invertible. So there exists a unit vector x_k such that $(T_i - \lambda_{k_i})^* x_k = 0$ for all $i = 1, \cdots, n$. If T has the quasitriangular property, it follows that $(T_i - \lambda_{k_i} I)x_k = 0$. In particular, since the T_i are hyponormal, $\{x_k\}$ forms an orthonormal sequence. Further, we have

$$\begin{aligned} \sum_{i=1}^n ||(T_i - \lambda_i I)x_k|| &\le \sum_{i=1}^n \left(||(T_i - \lambda_{k_i} I)x_k|| + ||(\lambda_{k_i} - \lambda_k)x_k|| \right) \\ &= \sum_{i=1}^n |\lambda_{k_i} - \lambda_i| \longrightarrow 0 \quad \text{as } k \to \infty. \end{aligned}$$

Therefore $\lambda \in \sigma_{Te}(T)$ (see [Da1, Theorem 2.6] or [Ch2, Theorem 1]), which proves (195) and completes the proof. □

Corollary 4.5. *A commuting n-tuple of normal operators satisfies Weyl's theorem (I) and hence Weyl's theorem (II).*

Proof. Immediate from (193) and Theorem 4.4. □

Corollary 4.6. (Riesz-Schauder theorem in several variables) *Let $T = (T_1, \cdots, T_n)$ be a doubly commuting n-tuple of hyponormal operators. If T has the quasitriangular property then*

$$\omega(T) = \sigma_{T_b}(T).$$

Proof. In view of Lemma 4.3, we need to show that $\sigma_{T_b}(T) \subset \omega(T)$. Indeed if $\lambda \in \sigma_T(T) \setminus \omega(T)$ then by (195), $\lambda \in \text{iso}\, \sigma_T(T)$, and hence $T - \lambda I$ is Taylor-Browder. □

4.2 Browder's theorem in several variables

We give a several-variables version of Browder's theorem.

Definition 4.7. If $T = (T_1, \cdots, T_n)$ is a commuting n-tuple then we say that *Browder's theorem holds for* T if $\sigma_T(T) \setminus \sigma_{Tw}(T) = \mathcal{R}(T)$.

We then have:

Theorem 4.8. *Let* $T = (T_1, \cdots, T_n)$ *be a commuting n-tuple. Then we have:*

(i) *Weyl's theorem (I) implies Browder's theorem.*

(ii) *Each of the following conditions is equivalent to Browder's theorem:*

(a) $\sigma_T(T) = \sigma_{Tw}(T) \cup \pi_{00}(T)$;

(b) $\sigma_{Tw}(T) = \sigma_{Tb}(T)$;

(c) $\sigma_T(T) \setminus \sigma_{Tw}(T) \subset iso\, \sigma_T(T)$.

(iii) *Necessary and sufficient for Weyl's theorem (I) is Browder's theorem together with either of the following:*

(d) $\sigma_{Tw}(T) \cap \pi_{00}(T) = \emptyset$;

(e) $\pi_{00}(T) \subset \mathcal{R}(T)$.

Proof. (i) Evident.

(ii) Implication Browder's theorem $\Rightarrow$ (a) comes from the fact $\mathcal{R}(T) \subset \pi_{00}(T)$. If (a) holds then $\sigma_T(T) \setminus \sigma_{Tw}(T) = \pi_{00}(T) \setminus \sigma_{Tw}(T) \subset \mathcal{R}(T)$. The remaining part is evident.

(iii) Note that (e) implies (d) with no assumption. Browder's theorem says that $\sigma_T(T) \setminus \sigma_{Tw}(T)$ is a subset of $\pi_{00}(T)$, while (e) ensures that $\pi_{00}(T)$ is a subset of this complement. Also the inclusion $\pi_{00}(T) \subset \operatorname{iso} \sigma_T(T)$ and Weyl's theorem (I) gives (e). □

The disjoint condition (d) of Theorem 4.8 can fail whether or not Browder's theorem holds. For example, let

$$V : (x_1, x_2, \cdots) \mapsto (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots) \quad \text{on } \ell^2.$$

If $T := (V, V)$ then $\sigma_T(T) = \sigma_{Te}(T) = \sigma_{Tw}(T) = \sigma_{Tb}(T) = \{(0,0)\}$. Since

$$\begin{pmatrix} V \\ V \end{pmatrix}^{-1} (0) = V^{-1}(0) = \mathbb{C}\delta_1 = \{(\lambda, 0, 0, \cdots) : \lambda \in \mathbb{C}\}$$

is of dimension 1, we have that $\pi_{00}(T) = \{(0,0)\}$. Note that Browder's theorem holds for T, while Weyl's theorem mark I does not.

Theorem 4.9. *Let* $T = (T_1, \cdots, T_n)$ *be a commuting n-tuple. Necessary and sufficient for Browder's theorem to hold for* T *is that*

$$\operatorname{acc} \sigma_T(T) \subset \sigma_{Tw}(T). \tag{196}$$

Hence, in particular, Browder's theorem holds for n-tuples of operators with finite spectrum.

Proof. If (196) holds then $\sigma_T(T) \setminus \sigma_{Tw}(T) \subset \operatorname{iso}\sigma_T(T)$, giving Browder's theorem. The converse is evident. If $\sigma_T(T)$ consists of finite elements then T satisfies (196). □

For the single-variable case, Browder's theorem holds for a compact operator K because

$$\operatorname{acc}\sigma(K) \subset \{0\} \subset \sigma_e(T).$$

However, if $n \geq 2$ this is not the case. For example, let

$$K : (x_1, x_2, \cdots) \mapsto (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \cdots) \quad \text{on } \ell^2.$$

Then $\sigma(K) = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Put $T := (K, K)$. Then

$$\operatorname{acc}\sigma_T(T) = \{(0,0), (0, \frac{1}{n}), (\frac{1}{n}, 0) : n \in \mathbb{N}\}.$$

However, $(0, \frac{1}{n}) \notin \sigma_{Tw}(T)$: indeed, by Lemma 4.2, $(K, K - \frac{1}{n})$ is Taylor Weyl since $K - \frac{1}{n}$ is Fredholm.

4.3 Concluding remarks and open problems

It was known that

(i) If $(A_1, \cdots, A_n)$ and $\left(\left(\begin{smallmatrix} A_1 & B_1 \\ 0 & C_1 \end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix} A_n & B_n \\ 0 & C_n \end{smallmatrix}\right)\right)$ are invertible then $(C_1, \cdots, C_n)$ is invertible.

(ii) If $(A_1, \cdots, A_n)$ and $(C_1, \cdots, C_n)$ are invertible then $\left(\left(\begin{smallmatrix} A_1 & B_1 \\ 0 & C_1 \end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix} A_n & B_n \\ 0 & C_n \end{smallmatrix}\right)\right)$ is invertible.

Problem 4.10. *If* $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) \equiv \left(\left(\begin{smallmatrix} A_1 & B_1 \\ 0 & C_1 \end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix} A_n & B_n \\ 0 & C_n \end{smallmatrix}\right)\right)$*, find a necessary and sufficient condition for* $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ *to be invertible for some* B*.*

If $n = 1$ then it was known that $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$ is invertible for some B if and only if

(i) A is left invertible;

(ii) C is right invertible;

(iii) $\operatorname{ran}(A)^\perp \cong \ker(C)$.

Problem 4.11. *What is a kind of several variable version of the punctured neighborhood theorem ?*

The punctured neighborhood theorem says that $\partial\sigma(T) \setminus \sigma_e(T) \subset \operatorname{iso}\sigma(T)$. Our question is that if $T = (T_1, \cdots, T_n)$ then

$$\partial\sigma_T(T) \setminus \sigma_{Te}(T) \subset \ (\ ?\ ) \text{ of } \sigma_T(T).$$

Problem 4.12. *rm (Deformation Problem) Given two Fredholm n-tuples $A = (A_1, \cdots, A_n)$ and $B = (B_1, \cdots, B_n) \in \mathcal{F}$ with the same index, is it always possible to find a continuous path $\gamma : [0, 1] \to \mathcal{F}$ such that $\gamma(0) = A$ and $\gamma(1) = B$?*

The answer for $n = 1$ is yes. Also if $\dim H < \infty$ then the answer is affirmative (cf. [Cu1], [Cu2]).

R. Curto and M. Putinar [CP] showed that

$$\sigma_T(N) \subset \sigma_T(S) \subset \eta\sigma_T(N).$$

If $n = 1$ then $\sigma(S)$ is obtained from $\sigma(N)$ by "filling in some holes".

Problem 4.13. *If $T = (T_1, \cdots, T_n)$ is commutative then*

(i) $\sigma_T(T) \subset \prod_{j=1}^{n} \sigma(T_j)$;

(ii) *If $p \in poly_n^m$ then $\sigma_T(p(T)) = p(\sigma_T(T))$.*

Let $T = (T_1, \cdots, T_n)$ be a hyponormal n-tuple of commuting operators and $p \in poly_n^m$. Does it follow

$$\sigma_T(p(T)) = 0 \implies p(T) = 0\,?$$

If $n = 1$ then the answer is yes: indeed, if $\sigma(p(T)) = 0$ and hence $p(\sigma(T)) = 0$ then $\sigma(T)$ is finite, so that T should be normal, which implies that $p(T)$ is normal and quasinilpotent then $p(T) = 0$ (cf. [Cu2]).

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