Some Recent Developments in
TOEPLITZ OPERATOR THEORY*

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ABSTRACT

During the week of December 20-24, 2004, the author is one of two principal lecturers at the Winter School 2004 of Operator Theory and Operator Algebras. In this lecture I attempt to set forth some of the recent developments that had taken place in Toeplitz operator theory. In particular I focus on the hyponormlaity and subnormality of Toeplitz operators on the Hardy space of the unit circle.

CONTENTS

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1 Preliminaries

In this lecture all Hilbert spaces will be understood to be complex and $\mathcal{H}$ will be a separable Hilbert space. We write $\mathcal{L}(\mathcal{H})$ for the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ for the set of compact operators on $\mathcal{H}$. In this chapter we give basic notions and results which will be used in the sequel: spectra and essential spectra, weighted shifts, hyponormality and subnormality, Fourier transform and Beurling's theorem, Hardy spaces and elementary properties of Toeplitz operators on the Hardy space of the unit circle. We also present some proofs for the well-known results.

1.1 Spectra and Essential Spectra

If $T \in \mathcal{L}(\mathcal{H})$, then the *spectrum*, denoted $\sigma(T)$, and the *point spectrum*, denoted $\sigma_p(T)$, of T are defined by

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible}\};$$
$$\sigma_p(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not one-one}\}.$$

It is well-known that $\sigma(T)$ is a non-empty compact set in $\mathbb{C}$. However $\sigma_p(T)$ is liable to be empty. For example, if U is the unilateral shift on ℓ^2, i.e.,

$$U := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

then $\sigma_p(T) = \emptyset$. The *spectral radius*, denoted $r(T)$, of T is defined by

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda|.$$

By the Gelfand formula, we have

$$r(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *Fredholm* if T has closed range with finite dimensional null space and its range of finite co-dimension. The quotient map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ (=the Calkin algebra) is denoted by π. Then by the Atkinson's theorem,

$$T \text{ is Fredholm} \iff \pi(T) \text{ is invertible in } \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

The *index* of a Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is defined by the equality

$$\operatorname{ind}(T) := \dim T^{-1}(0) - \dim \mathcal{H}/\operatorname{cl} T(\mathcal{H}) = \dim T^{-1}(0) - \dim T^{*-1}(0).$$

The function $\operatorname{ind}(\cdot)$ satisfies the following:

1. (Index Product Theorem) $\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T$ for Fredholm operators S, T;
2. (Index Stability Theorem) $\operatorname{ind}(T + K) = \operatorname{ind} T$ if T is Fredholm and K is compact;
3. (Index Continuity Theorem) The map $\operatorname{ind}(\cdot)$ is continuous.

The *essential spectrum*, denoted $\sigma_e(T)$, of $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\sigma_e(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}.$$

By the Atkinson's theorem,

$$\sigma_e(T) = \sigma_{\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})}(\pi(T)).$$

Thus $\sigma_e(T)$ is compact. If $\dim \mathcal{H} = \infty$ then $\sigma_e(T) \neq \emptyset$, and if instead $\dim \mathcal{H} < \infty$ then $\sigma_e(T) = \emptyset$ because this case forces $\mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$. In particular,

$$\sigma_e(T + K) = \sigma_e(T) \quad \text{for all compact operators } K.$$

Write $\mathbb{D}$ for the open unit disk and let $\mathbb{T} \equiv \partial\mathbb{D}$.
If U is the unilateral shift on ℓ^2 then (cf. [Con1])

1. $\sigma(U) = \operatorname{cl} \mathbb{D}$;
2. $\sigma_p(U) = \emptyset$;
3. $|\lambda| < 1 \;\Rightarrow\; \dim \ker (U^* - \lambda) = 1$;
4. $\sigma_e(T) = \partial\, \mathbb{D}$;
5. $|\lambda| < 1 \;\Rightarrow \operatorname{ind}(U - \lambda) = -1$.

Two operators S and T in $\mathcal{L}(\mathcal{H})$ are said to be *unitarily equivalent* if there exists a unitary operator V such that $VTV^{-1} = S$, denoted by $T \cong S$.

An operator T is called *quasinilpotent* if $\sigma(T) = \{0\}$, or equivalently, $\lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = 0$. For example, if

$$T = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & \frac{1}{2} & 0 & & \\ & & \frac{1}{3} & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

then $\lim_{n\to\infty} ||T^n||^{\frac{1}{n}} = \lim_{n\to\infty} \left(1 \cdot \frac{1}{2} \cdots \frac{1}{n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0$, so that T is quasinilpotent.

Finally, $\sigma(T)$ and $\sigma_e(T)$ enjoy the spectral mapping theorem: i.e., if $f(z)$ is an analytic function in an open neighborhood of $\sigma(T)$ then

$$f(\sigma_*(T)) = \sigma_*(f(T)) \quad \text{where } \sigma_* = \sigma, \sigma_e.$$

1.2 Weighted Shifts

Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that

$$W_\alpha \text{ is compact } \iff \alpha_n \to 0.$$

Indeed, $W_\alpha = UD$, where U is the unilateral shift and D is the diagonal operator whose diagonal entries are α_n.

We observe:

Proposition 1.2.1. *If $T \equiv W_\alpha$ is a weighted shift and $\omega \in \partial\mathbb{D}$ then $T \cong \omega T$.*

Proof. If $Ve_n := \omega^n e_n$ for all n then $VTV^* = \omega T$. □

As a consequence of Proposition 1.2.1, we can see that the spectrum of a weighted shift must be a circular symmetry:

$$\sigma(W_\alpha) = \sigma(\omega W_\alpha) = \omega\sigma(W_\alpha).$$

Indeed we have:

Theorem 1.2.2. *If $T \equiv W_\alpha$ is a weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ such that $\alpha_n \to \alpha_+$ then*

(i) $\sigma_p(T) = \emptyset$;

(ii) $\sigma(T) = \{\lambda : |\lambda| \leq \alpha_+\}$;

(iii) $\sigma_e(T) = \{\lambda : |\lambda| = \alpha_+\}$;

(iv) $|\lambda| < \alpha_+ \Rightarrow \operatorname{ind}(T - \lambda) = -1$.

Proof. The assertion (i) is straightforward. For the other assertions, observe that if $\alpha_+ = 0$ then T is compact and quasinilpotent. If instead $\alpha_+ > 0$ then $T - \alpha_+ U$ (U :=the unilateral shift) is a weighted shift whose weight sequence converges to 0. Hence $T - \alpha_+ U$ is a compact and hence

$$\sigma_e(T) = \sigma_e(\alpha_+ U) = \alpha_+ \sigma_e(U) = \{\lambda : |\lambda| = \alpha_+\}.$$

If $|\lambda| < \alpha_+$ then $T - \lambda$ is Fredholm and

$$\operatorname{ind}(T - \lambda) = \operatorname{ind}(\alpha_+ U - \lambda) = -1.$$

In particular, $\{\lambda : |\lambda| \leq \alpha_+\} \subset \sigma(T)$. By the assertion (i), we can conclude that $\sigma(T) = \{\lambda : |\lambda| \leq \alpha_+\}$. □

Theorem 1.2.3. *If $T \equiv W_\alpha$ is a weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ then*

$$[T^*, T] = \begin{pmatrix} \alpha_0^2 & & & \\ & \alpha_1^2 - \alpha_0^2 & & \\ & & \alpha_2^2 - \alpha_1^2 & \\ & & & \ddots \end{pmatrix}$$

Proof. From a straightforward calculation. □

1.3 Hyponormality and Subnormality

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if the self-commutator $[T^*, T] \equiv T^*T - TT^* \geq 0$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$.

The following lemma is elementary:

Proposition 1.3.1. *subnormal* $\Rightarrow$ *hyponormal.*

Proof. If S is subnormal then there exists a normal operator $N = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix}$. Thus,

$$0 = N^*N - NN^* = \begin{pmatrix} [S^*, S] - AA^* & S^*A \\ A^*S & A^*A + [B^*, B] \end{pmatrix},$$

which implies $[S^*, S] = AA^* \geq 0$. $\square$

Definition 1.3.2. Let μ be a compactly supported measure on $\mathbb{C}$ and define N_μ on $L^2(\mu)$ by

$$N_\mu f = zf.$$

Then N_μ is normal since $N_\mu^* f = \bar{z}f$. If $P^2(\mu)$ denotes the closure in $L^2(\mu)$ of analytic polynomials, define S_μ on $P^2(\mu)$ by

$$S_\mu f = zf.$$

Then S_μ is subnormal and N_μ is a normal extension of S_μ.

Definition 1.3.3. A vector e_0 is called a *cyclic vector* for T if

$$\mathcal{H} = \text{cl}\{p(T)e_0 : p \text{ is a polynomial}\}$$

and a *star-cyclic vector* for T if

$$\mathcal{H} = \text{cl}\{Te_0 : T \in C^*(T)\},$$

where $C^*(T)$ denotes the C^*-algebra generated by T and 1. The operator $T \in \mathcal{L}(\mathcal{H})$ is called a *cyclic* [*star-cyclic*] operator if T has a cyclic [star-cyclic] vector.

It was known [Con1], [Con3] that if $T \in \mathcal{L}(\mathcal{H})$ then

1. T is a star-cyclic normal operator $\Longleftrightarrow T \cong N_\mu$;
2. T is a cyclic subnormal operator $\Longleftrightarrow T \cong S_\mu$;
3. If $\mu =$ Lebesgue measure on $\partial\mathbb{D}$ then N_μ is the bilateral shift on $L^2(\mathbb{T})$.

We here record basic properties of hyponormal operators which have been developed in the literature.

Proposition 1.3.4 (Basic Properties of Hyponormal Operators)**.** *Let $T \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator. Then we have:*

(a) *If $T \cong S$ then S is also hyponormal;*

(b) *$T-\lambda$ is hyponormal for every $\lambda \in \mathbb{C}$;*

(c) *If $T\mathcal{M} \subset \mathcal{M}$ then $T|_{\mathcal{M}}$ is hyponormal;*

(d) *$||T^*h|| \le ||Th||$ for all h, so that $\ker(T-\lambda) \subset \ker(T-\lambda)^*$;*

(e) *If f and g are eigenvectors corresponding to distinct eigenvalues of T then $f \perp g$;*

(f) *If $\lambda \in \sigma_p(T)$ then $\ker(T-\lambda)$ reduces T;*

(g) *If T is invertible then T^{-1} is hyponormal;*

(h) (Stampfli, 1962) *$||T^n|| = ||T||^n$, so that $||T|| = r(T)$ ($r(\cdot)$ denotes spectral radius);*

(i) *T is isoloid, i.e., $\operatorname{iso}\sigma(T) \subset \sigma_p(T)$;*

(j) *If $\lambda \notin \sigma(T)$ then $\operatorname{dist}(\lambda, \sigma(T)) = ||(T-\lambda)^{-1}||^{-1}$.*

(k) (Berger-Shaw theorem) *If T is cyclic then $\operatorname{tr}[T^*, T] \le \frac{1}{\pi}\mu(\sigma(T))$;*

(l) (Putnam's Inequality) *$||\,[T^*, T]\,|| \le \frac{1}{\pi}\mu(\sigma(T))$.*

Proof. (a)-(f) are straightforward.

(g) Note that if T is positive and invertible then

$$T \ge 1 \Leftrightarrow T^{-1} \le 1.$$

Since $T^*T \ge TT^*$ and T is invertible we have $T^{-1}(T^*T)(T^{-1})^* \ge T^{-1}(TT^*)(T^{-1})^* = 1$, so that $T^*T^{-1}(T^*)^{-1}T \le 1$, and hence

$$T^{-1}(T^*)^{-1} = (T^*)^{-1}(T^*T^{-1}(T^*)^{-1}T)T^{-1} \le (T^*)^{-1}T^{-1}.$$

(h) Observe

$$||T^n f||^2 = \langle T^n f, T^n f\rangle = \langle T^*T^n f, T^{n-1}f\rangle \le ||T^*T^n f|| \cdot ||T^{n-1}f|| \le ||T^{n+1}f|| \cdot ||T^{n-1}f||.$$

We use an induction. Clearly, it is true for $n = 1$. Suppose $||T^k|| = ||T||^k$ for $1 \le k \le n$. Then

$$||T||^{2n} = ||T^n||^2 \le ||T^{n+1}|| \cdot ||T^{n-1}|| = ||T^{n+1}|| \cdot ||T||^{n-1}, \text{ so that } ||T||^{n+1} \le ||T^{n+1}||.$$

(j) Observe that

$$\frac{1}{||(T-\lambda)^{-1}||} = \frac{1}{\max_{\mu \in \sigma((T-\lambda)^{-1})}|\mu|} = \min_{\mu \in \sigma(T-\lambda)} |\mu| = \operatorname{dist}(\lambda, \sigma(T)).$$

(l) Let $f \in \mathcal{H}$ with $||f|| = 1$ and $\mathcal{K} := \operatorname{cl}\{r(T)f : r \text{ is a rational function}\}$. If $S := T|_{\mathcal{K}}$ then S is a cyclic hyponormal operator and $||S^*f|| \le ||T^*f||$. By the Berger-Shaw theorem,

$$\begin{aligned}\langle [T^*, T]f, f\rangle = ||Tf||^2 - ||T^*f||^2 \le ||Sf||^2 - ||S^*f||^2 = \langle [S^*, S]f, f\rangle \\ \le \operatorname{tr}[S^*, S] \le \frac{1}{\pi}\mu(\sigma(S)) \le \frac{1}{\pi}\mu(\sigma(T)).\end{aligned}$$

Since f was arbitrary the result follows.

For (i) and (k), refer [Con2]. □

Theorem 1.3.5 (A Characterization of Subnormality)**.** *If $T \in \mathcal{L}(\mathcal{H})$ then the following are equivalent:*

(a) T *is subnormal;*

(b) (Bram-Halmos, 1955)

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\textit{all } k \geq 1).$$

(c)

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] & \dots & [T^{*k}, T] \\ [T^*, T^2] & [T^{*2}, T^2] & \dots & [T^{*k}, T^2] \\ \vdots & \vdots & \dots & \vdots \\ [T^*, T^k] & [T^{*2}, T^k] & \dots & [T^{*k}, T^k] \end{pmatrix} \geq 0 \qquad (\textit{all } k \geq 1).$$

(d) (Embry, 1973) *There is a positive operator-valued measure* Q *on some interval* $[0, a] \subset \mathbb{R}$ *such that*

$$T^{*n}T^n = \int t^{2n} dQ(t) \quad \textit{for all } n \geq 0.$$

Proof. See [Con2]. □

Condition (b) (or equivalently, condition (c)) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (b) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (b) for all k. So we define T to be *k-hyponormal* whenever the $(k+1) \times (k+1)$ operator matrix in (b) is positive semi-definite. Then the Bram-Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([CMX]).

Theorem 1.3.6 (Berger's Theorem). *Let* $T \equiv W_\alpha$ *be a weighted shift with weight sequence* $\alpha \equiv \{\alpha_n\}$ *and define the moment of* T *by*

$$\gamma_0 := 1 \quad \textit{and} \quad \gamma_n := \alpha_0^2 \alpha_1^2 \cdots \alpha_{n-1}^2 \ (n \geq 1).$$

Then T *is subnormal if and only if there exists a probability measure* ν *on* $[0, ||T||^2]$ *such that*

$$\gamma_n = \int_{[0,||T||^2]} t^n d\nu(t) \quad (n \geq 1). \tag{1.3.6.1}$$

Proof. ($\Rightarrow$) Note that T is cyclic. So if T is subnormal then $T \cong S_\mu$, i.e., there is an isomorphism $U : \ell^2 \to P^2(\mu)$ such that

$$Ue_0 = 1 \quad \text{and} \quad UTU^{-1} = S_\mu.$$

Observe $T^n e_0 = \sqrt{\gamma_n} e_n$ for all n. Also, $U(T^n e_0) = S_\mu^n U e_0 = S_\mu^n 1 = z^n$. So

$$\int |z|^{2n} d\mu = \int |UT^n e_0|^2 d\mu = \int |U(\sqrt{\gamma_n} e_n)|^2 d\mu = \gamma_n \int |Ue_n|^2 d\mu = \gamma_n ||Ue_n||^2 = \gamma_n.$$

If ν is defined on $[0, ||T||^2]$ by

$$\nu(\Delta) = \mu(\{z : |z|^2 \in \Delta\})$$

then ν is a probability measure and $\gamma_n = \int t^n d\nu(t)$.

($\Leftarrow$) If ν is the measure satisfying (1.3.6.1), define the measure μ by $d\mu(re^{i\theta}) = \frac{1}{2\pi} d\theta d\nu(r)$. Then we can see that $T \cong S_\mu$. □

Example 1.3.7. (a) The Bergman shift B_α is the weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}$ given by

$$\alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n \geq 0).$$

Then B_α is subnormal: indeed,

$$\gamma_n = \alpha_0^2 \cdots \alpha_{n-1}^2 = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1}$$

and if we define $\mu(t) = t$, i.e., $d\mu = dt$ then

$$\int_0^1 t^n d\mu(t) = \frac{1}{n+1} = \gamma_n.$$

(b) If $\alpha_n : \beta, 1, 1, 1, \cdots$ then W_α is subnormal: indeed $\gamma_n = \beta^2$ and if we define $d\mu = \beta^2\delta_1 + (1-\beta^2)\delta_0$ then $\int_0^1 t^n d\mu = \beta^2 = \gamma_n$.

Remark. Recall that the Bergman space $A(\mathbb{D})$ for $\mathbb{D}$ is defined by

$$A(\mathbb{D}) := \{f : \mathbb{D} \to \mathbb{C} : \ f \text{ is analytic with } \int_{\mathbb{D}} |f|^2 d\mu < \infty\}.$$

Then the orthonormal basis for $A(\mathbb{D})$ is given by $\{e_n \equiv \sqrt{n+1}\, z^n : \ n = 0, 1, 2, \cdots\}$ with $d\mu = \frac{1}{\pi} dA$. The Bergman operator $T : A(\mathbb{D}) \to A(\mathbb{D})$ is defined by

$$Tf = zf.$$

In this case the matrix (α_{ij}) of the Bergman operator T with respect to the basis $\{e_n \equiv \sqrt{n+1}\, z^n : n = 0, 1, 2, \cdots\}$ is given by

$$\begin{aligned}
\alpha_{ij} &= \langle Te_j,\, e_i \rangle \\
&= \langle T\sqrt{j+1}\, z^j,\, \sqrt{i+1}\, z^i \rangle \\
&= \langle \sqrt{j+1}\, z^{j+1},\, \sqrt{i+1}\, z^i \rangle \\
&= \sqrt{(j+1)(i+1)} \int_{\mathbb{D}} z^{j+1} \overline{z}^i d\mu \\
&= \sqrt{(j+1)(i+1)}\, \frac{1}{\pi} \int_0^{2\pi} \int_0^1 r^{j+1+i} e^{i(j+1-i)\theta} \cdot r dr\, d\theta \\
&= \begin{cases} \sqrt{\frac{j+1}{j+2}} & (i = j+1) \\ 0 & (i \neq j+1) : \end{cases}
\end{aligned}$$

therefore

$$T = \begin{pmatrix} 0 & & & & \\ \sqrt{\frac{1}{2}} & 0 & & & \\ & \sqrt{\frac{2}{3}} & 0 & & \\ & & \sqrt{\frac{3}{4}} & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Recall ([Ath],[CMX],[CoS]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS((T, T^2, \cdots, T^k)) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators. If $k = 2$ then T is said to be *quadratically hyponormal*. Similarly, T is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([Cu1], [Cu2], [CuF1], [CuF2], [CuF3], [CLL], [CuL1], [CuL2], [CuL3], [CMX], [DPY], [McCP]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle. For weighted shifts, positive results appear in [Cu1] and [CuF3], although no concrete example of a weighted shift which is polynomially hyponormal but not subnormal has yet been found (the existence of such weighted shifts was established in [CP1] and [CP2]).

1.4 Fourier Transform and Beurling's Theorem

A trigonometric polynomial is a function $p \in C(\mathbb{T})$ of the form $\sum_{k=-n}^{n} a_k z^k$. It was well-known that the set of trigonometric polynomials are uniformly dense in $C(\mathbb{T})$ and hence is dense in $\mathbf{L}^2(\mathbb{T})$. In fact, if $e_n := z^n$, $(n \in \mathbb{Z})$ then $\{e_n : n \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathbf{L}^2(\mathbb{T})$. The Hardy space $\mathbf{H}^2(\mathbb{T})$ is spanned by $\{e_n : n = 0, 1, 2, \cdots\}$. Write $\mathbf{H}^\infty(\mathbb{T}) := \mathbf{L}^\infty(\mathbb{T}) \cap \mathbf{H}^2(\mathbb{T})$. Then $\mathbf{H}^\infty$ is a subalgebra of $\mathbf{L}^\infty$.

Let m :=the normalized Lebesgue measure on $\mathbb{T}$ and write $\mathbf{L}^2 := \mathbf{L}^2(\mathbb{T})$. If $f \in \mathbf{L}^2$ then the Fourier transform of f, $\widehat{f} : \mathbb{Z} \to \mathbb{C}$, is defined by

$$\widehat{f}(n) \equiv \langle f, e_n \rangle = \int_{\mathbb{T}} f \overline{z}^n dm = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt,$$

which is called the n-th *Fourier coefficient* of f. By Parseval's identity,

$$f = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e_n,$$

which converges in the norm of $\mathbf{L}^2$. This series is called the *Fourier series* of f.

Proposition 1.4.1. (i) *$f \in \mathbf{L}^2 \Rightarrow \widehat{f} \in \ell^2(\mathbb{Z})$;*

(ii) *If $V : \mathbf{L}^2 \to \ell^2(\mathbb{Z})$ is defined by $Vf = \widehat{f}$ then V is an isomorphism.*

(iii) *If $W = N_m$ on $\mathbf{L}^2$ then VWV^{-1} is the bilateral shift on $\ell^2(\mathbb{Z})$.*

Proof. (i) Since by Parseval's identity, $\sum |\widehat{f}(n)|^2 = ||f||^2 < \infty$, it follows $\widehat{f} \in \ell^2(\mathbb{Z})$.

(ii) We claim that $||Vf|| = ||f||$: indeed, $||Vf||^2 = ||\widehat{f}||^2 = \sum |\widehat{f}(n)|^2 = ||f||^2$. If $f = z^n$ then

$$\widehat{f}(k) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n, \end{cases}$$

so that $\widehat{f}$ is the n-th basis vector in $\ell^2(\mathbb{Z})$. Thus $\operatorname{ran} V$ is dense and hence V is an isomorphism.

(iii) If $\{e_n\}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$ then by (ii), $Vz^n = e_n$. Thus $VWz^n = V(z^{n+1}) = e_{n+1} = UVz^n$. □

If $T \in \mathcal{L}(\mathcal{H})$, write $\operatorname{Lat} T$ for the set of all invariant subspaces for T, i.e.,

$$\operatorname{Lat} T := \{\mathcal{M} \subset \mathcal{H} : T\mathcal{M} \subset \mathcal{M}\}.$$

Theorem 1.4.2. *If μ is a compactly supported measure on $\mathbb{T}$ and $\mathcal{M} \in \operatorname{Lat} N_\mu$ then*

$$\mathcal{M} = \phi \mathbf{H}^2 \oplus \mathbf{L}^2(\mu|\Delta),$$

where $\phi \in \mathbf{L}^\infty(\mu)$ and Δ is a Borel set of $\mathbb{T}$ such that $\phi|\Delta = 0$ a.e. and $|\phi|^2 \mu = m$(:=the normalized Lebesgue measure).

Proof. See [Con3, p.121]. □

Now consider the case where $\mu = m$ (in this case, N_μ is the bilateral shift). Observe

$$\phi \in \mathbf{L}^2, \ |\phi|^2 m = m \implies |\phi| = 1 \text{ a.e.},$$

so that there is no Borel set Δ such that $\phi|\Delta = 0$ and $m(\Delta) \neq 0$. Therefore every invariant subspace for the bilateral shift must have one form or the other. We thus have:

Corollary 1.4.3. *If W is the bilateral shift on $\mathbf{L}^2$ and $\mathcal{M} \in \operatorname{Lat} W$ then*

$$\textit{either } \mathcal{M} = \mathbf{L}^2(m|\Delta) \textit{ or } \mathcal{M} = \phi\mathbf{H}^2$$

for a Borel set Δ and a function $\phi \in \mathbf{L}^\infty$ such that $|\phi| = 1$ a.e.

Definition 1.4.4. A function $\phi \in \mathbf{L}^\infty$ [$\phi \in \mathbf{H}^\infty$] is called a *unimodular* [*inner*] function if $|\phi| = 1$ a.e.

The following theorem has had an enormous influence on the development in operator theory and function theory.

Theorem 1.4.5 (Beurling's Theorem)**.** *If U is the unilateral shift on $\mathbf{H}^2$ then*

$$\operatorname{Lat} U = \{\phi\mathbf{H}^2 : \ \phi \textit{ is an inner function}\}.$$

Proof. Let W be the bilateral shift on $\mathbf{L}^2$. If $\mathcal{M} \in \operatorname{Lat} U$ then $\mathcal{M} \in \operatorname{Lat} W$. By Corollary 1.4.3, $\mathcal{M} = \mathbf{L}^2(m|\Delta)$ or $\mathcal{M} = \phi\mathbf{H}^2$, where ϕ is a unimodular function. Since U is a shift,

$$\bigcap U^n \mathcal{M} \subset \bigcap U^n \mathbf{H}^2 = \{0\},$$

so the first alternative is impossible. Hence $\phi\mathbf{H}^2 = \mathcal{M} \subset \mathbf{H}^2$. Since $\phi = \phi \cdot 1 \in \mathcal{M}$, it follows $\phi \in \mathbf{L}^\infty \cap \mathbf{H}^2 = \mathbf{H}^\infty$. $\square$

1.5 Hardy Spaces

If $f \in \mathbf{H}^2$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is its Fourier series expansion, this series converges uniformly on compact subsets of $\mathbb{D}$. Indeed, if $|z| \le r < 1$, then

$$\sum_{n=m}^{\infty} |a_n z^n| \le \left(\sum_{n=m}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=m}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} \le ||f||_2 \left(\sum_{n=m}^{\infty} r^{2n}\right)^{\frac{1}{2}}.$$

Therefore it is possible to identify $\mathbf{H}^2$ with the space of analytic functions on the unit disk whose Taylor coefficients are square summable.

Proposition 1.5.1. *If f is a real-valued function in $\mathbf{H}^1$ then f is constant.*

Proof. Let $\alpha = \int f dm$. By hypothesis, we have $\alpha \in \mathbb{R}$. Since $f \in \mathbf{H}^1$, we have $\int f z^n dm = 0$ for $n \ge 1$. So $\int (f - \alpha) z^n dm = 0$ for $n \ge 0$. Also,

$$0 = \overline{\int (f-\alpha) z^n dm} = \int (f-\alpha) z^{-n} dm \quad (n \ge 0),$$

so that $\int (f-\alpha) z^n dm = 0$ for all integers n. Thus $f - \alpha$ annihilates all the trigonometric polynomials. Therefore, $f - \alpha = 0$ in $\mathbf{L}^1$. □

Corollary 1.5.2. *If ϕ is inner such that $\overline{\phi} = \frac{1}{\phi} \in \mathbf{H}^2$ then ϕ is constant.*

Proof. By hypothesis, $\phi + \overline{\phi}$ and $\frac{\phi - \overline{\phi}}{i}$ are real-valued functions in $\mathbf{H}^2$. By Proposition 1.5.1, they are constant, so is ϕ. □

The proof of the following important theorem uses Beurling's theorem.

Theorem 1.5.3 (The F. and M. Riesz Theorem)**.** *If f is a nonzero function in $\mathbf{H}^2$, then $m\Big(\{z \in \partial\mathbb{D} : f(z) = 0\}\Big) = 0$. Hence, in particular, if $f, g \in \mathbf{H}^2$ and if $fg = 0$ a.e. then $f = 0$ a.e. or $g = 0$ a.e.*

Proof. Let $\triangle =$ a Borel set of $\partial\mathbb{D}$ and put

$$\mathcal{M} := \{h \in \mathbf{H}^2 : h(z) = 0 \text{ a.e. on } \triangle\}.$$

Then $\mathcal{M}$ is an invariant subspace for the unilateral shift. By Beurling's theorem, if $\mathcal{M} \ne \{0\}$, then there exists an inner function ϕ such that $\mathcal{M} = \phi \mathbf{H}^2$. Since $\phi = \phi \cdot 1 \in \mathcal{M}$, it follows $\phi = 0$ on $\triangle$. But $|\phi| = 1$ a.e., and hence $\mathcal{M} = \{0\}$. □

A function f in $\mathbf{H}^2$ is called an *outer function* if

$$\mathbf{H}^2 = \bigvee \{z^n f : n \ge 0\}.$$

So f is outer if and only if it is a cyclic vector for the unilateral shift.

Theorem 1.5.4 (Inner-Outer Factorization)**.** *If f is a nonzero function in $\mathbf{H}^2$, then*

$$\exists \text{ an inner function } \phi \text{ and an outer function } g \text{ in } \mathbf{H}^2 \ \ s.t. \ \ f = \phi g.$$

In particular, if $f \in \mathbf{H}^\infty$, then $g \in \mathbf{H}^\infty$.

Proof. Observe $\mathcal{M} \equiv \bigvee\{z^n f : n \geq 0\} \in \operatorname{Lat} U$. By Beurling's theorem,

$$\exists \text{ an inner function } \phi \text{ s.t. } \mathcal{M} = \phi\mathbf{H}^2.$$

Let $g \in \mathbf{H}^2$ be such that $f = \phi g$. We want to show that g is outer. Put $\mathcal{N} \equiv \bigvee\{z^n g : n \geq 0\}$. Again there exists an inner function ψ such that $\mathcal{N} = \psi\mathbf{H}^2$. Note that

$$\phi\mathbf{H}^2 := \bigvee\{z^n f : n \geq 0\} = \bigvee\{z^n \phi g : n \geq 0\} = \phi\psi\mathbf{H}^2.$$

Therefore there exists a function $h \in \mathbf{H}^2$ such that $\phi = \phi\psi h$ so that $\overline{\psi} = h \in \mathbf{H}^2$. Hence ψ is a constant by Corollary 1.5.2. So $\mathcal{N} = \mathbf{H}^2$ and g is outer. Assume $f \in \mathbf{H}^\infty$ with $f = \phi g$. Thus $|g| = |f|$ a.e. on $\partial\mathbb{D}$, so that g must be bounded, i.e., $g \in \mathbf{H}^\infty$. $\square$

1.6 Toeplitz Operators

Let P be the orthogonal projection of $\mathbf{L}^2(\mathbb{T})$ onto $\mathbf{H}^2(\mathbb{T})$. For $\varphi \in \mathbf{L}^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is defined by

$$T_\varphi f = P(\varphi f) \quad \text{for } f \in \mathbf{H}^2.$$

Remember that $\{z^n : n = 0, 1, 2, \cdots\}$ is an orthonormal basis for $\mathbf{H}^2$. Thus if $\varphi \in \mathbf{L}^\infty$ has the Fourier coefficients

$$\widehat{\varphi}(n) = \frac{1}{2\pi}\int_0^{2\pi} \varphi \overline{z}^n dt,$$

then the matrix (a_{ij}) for T_φ with respect to the basis $\{z^n : n = 0, 1, 2, \cdots\}$ is given by:

$$a_{ij} = (T_\varphi z^j, z^i) = \frac{1}{2\pi}\int_0^{2\pi} \varphi \overline{z}^{i-j} dt = \widehat{\varphi}(i-j).$$

Thus the matrix for T_φ is constant on diagonals:

$$(a_{ij}) = \begin{pmatrix} c_0 & c_{-1} & c_{-2} & c_{-3} & \cdots \\ c_1 & c_0 & c_{-1} & c_{-2} & \cdots \\ c_2 & c_1 & c_0 & c_{-1} & \cdots \\ c_3 & c_2 & c_1 & c_0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \text{where } c_j = \widehat{\varphi}(j):$$

Such a matrix is called a *Toeplitz matrix.*

Lemma 1.6.1. *Let $A \in \mathcal{L}(\mathbf{H}^2)$. The matrix A relative to the orthonormal basis $\{z^n : n = 0, 1, 2, \cdots\}$ is a Toeplitz matrix if and only if*

$$U^*AU = A, \text{ where } U \text{ is the unilateral shift.}$$

Proof. The hypothesis on the matrix entries $a_{ij} = \langle Az^j, z^i\rangle$ of A if and only if

(1.6.1.1) $$a_{i+1,j+1} = a_{ij} \ (i, j = 0, 1, 2, \cdots).$$

Noting $Uz^n = z^{n+1}$ for $n \geq 0$, we get

$$\begin{aligned}(1.6.1.1) &\iff \langle U^*AUz^j, z^i\rangle = \langle AUz^j, Uz^i\rangle = \langle Az^{j+1}, z^{i+1}\rangle = \langle Az^j, z^i\rangle, \quad \forall\, i, j \\ &\iff U^*AU = A.\end{aligned}$$

□

Remark. $AU = UA \Leftrightarrow A$ is an analytic Toeplitz operator (i.e., $A = T_\varphi$ with $\varphi \in \mathbf{H}^\infty$).

Consider the mapping $\xi : \mathbf{L}^\infty \to \mathcal{L}(\mathbf{H}^2)$ defined by $\xi(\varphi) = T_\varphi$. We have:

Proposition 1.6.2. *ξ is a contractive $*$-linear mapping from $\mathbf{L}^\infty$ to $\mathcal{L}(\mathbf{H}^2)$.*

Proof. It is obvious that ξ is contractive and linear. To show that $\xi(\varphi)^* = \xi(\overline{\varphi})$, let $f, g \in \mathbf{H}^2$. Then

$$\langle T_{\overline{\varphi}} f, g\rangle = \langle P(\overline{\varphi} f), g\rangle = \langle \overline{\varphi} f, g\rangle = \langle f, \varphi g\rangle = \langle f, P(\varphi g)\rangle = \langle f, T_\varphi g\rangle = \langle T_\varphi^* f, g\rangle,$$

so that $\xi(\varphi)^* = T_\varphi^* = T_{\overline{\varphi}} = \xi(\overline{\varphi})$. □

Remark. ξ is not multiplicative. For example, $T_zT_{\overline{z}} \neq I = T_1 = T_{|z|^2} = T_{z\overline{z}}$. Thus ξ is not a homomorphism.

In special cases, ξ is multiplicative.

Proposition 1.6.3. $T_\varphi T_\psi = T_{\varphi\psi} \iff$ *either* ψ *or* $\overline{\varphi}$ *is analytic.*

Proof. ($\Leftarrow$) Recall that if $f \in \mathbf{H}^2$ and $\psi \in \mathbf{H}^\infty$ then $\psi f \in \mathbf{H}^2$. Thus, $T_\psi f = P(\psi f) = \psi f$. So

$$T_\varphi T_\psi f = T_\varphi(\psi f) = P(\varphi\psi f) = T_{\varphi\psi} f, \text{ i.e., } T_\varphi T_\psi = T_{\varphi\psi}.$$

Taking adjoints reduces the second part to the first part.
($\Rightarrow$) From a straightforward calculation. □

Write M_φ for the multiplication operator on $\mathbf{L}^2$ with symbol $\varphi \in \mathbf{L}^\infty$. The *essential range* of $\varphi \in \mathbf{L}^\infty \equiv \mathfrak{R}(\varphi)$:=the set of all λ for which $\mu\Big(\{x : |f(x) - \lambda| < \epsilon\}\Big) > 0$ for any $\epsilon > 0$.

Lemma 1.6.4. *If* $\varphi \in \mathbf{L}^\infty(\mu)$ *then* $\sigma(M_\varphi) = \mathfrak{R}(\varphi)$.

Proof. If $\lambda \notin \mathfrak{R}(\varphi)$ then

$$\exists\, \varepsilon > 0 \text{ s.t. } \mu\Big(\{x : |\varphi(x) - \lambda| < \varepsilon\}\Big) = 0, \text{ i.e., } |\varphi(x) - \lambda| \geq \ \epsilon \text{ a.e. } [\mu].$$

So

$$g(x) := \frac{1}{\varphi(x) - \lambda} \ \in \ \mathbf{L}^\infty(X, \mu).$$

Hence M_g is the inverse of $M_\varphi - \lambda$, i.e., $\lambda \notin \sigma(M_\varphi)$. For the converse, suppose $\lambda \in \mathfrak{R}(\varphi)$. We will show that

$$\exists \text{ a sequence } \{g_n\} \text{ of unit vectors } \in \mathbf{L}^2 \text{ with the property } ||M_\varphi g_n - \lambda g_n|| \to 0,$$

showing that $M_\varphi - \lambda$ is not bounded below, and hence $\lambda \in \sigma(M_\varphi)$. By assumption, $\{x \in \mathbb{T} : |\varphi(x) - \lambda| \leq \frac{1}{n}\}$ has a positive measure. So we can find a subset

$$E_n \subseteq \left\{x \in \mathbb{T} : |\varphi(x) - \lambda| \leq \frac{1}{n}\right\}$$

satisfying $0 < \mu(E_n) < \infty$. Letting $g_n := \frac{\chi_{E_n}}{\sqrt{\mu(E_n)}}$, we have that

$$\big|(\varphi(x) - \lambda) g_n(x)\big| \leq \frac{1}{n} |g_n(x)|,$$

and hence $||(\varphi - \lambda) g_n||_{\mathbf{L}^2} \leq \frac{1}{n} \longrightarrow 0$. □

Proposition 1.6.5. *If* $\varphi \in \mathbf{L}^\infty$ *is such that* T_φ *is invertible, then* φ *is invertible in* $\mathbf{L}^\infty$.

Proof. In view of Lemma 1.6.4, it suffices to show that

$$T_\varphi \text{ is invertible} \implies M_\varphi \text{ is invertible.}$$

If T_φ is invertible then

$$\exists\, \varepsilon > 0 \ \text{ s.t. } \ ||T_\varphi f|| \geq \varepsilon ||f||, \ \ \forall f \in \mathbf{H}^2.$$

So for $n \in \mathbb{Z}$ and $f \in \mathbf{H}^2$,

$$||M_\varphi(z^n f)|| = ||\varphi z^n f|| = ||\varphi f|| \geq ||P(\varphi f)|| = ||T_\varphi f|| \geq \varepsilon ||f|| = \varepsilon ||z^n f||.$$

Since $\{z^n f : f \in \mathbf{H}^2, n \in \mathbb{Z}\}$ is dense in $\mathbf{L}^2$, it follows $||M_\varphi g|| \geq \varepsilon ||g||$ for $g \in \mathbf{L}^2$. Similarly, $||M_{\overline{\varphi}} f|| \geq \varepsilon ||f||$ since $T_\varphi^* = T_{\overline{\varphi}}$ is also invertible. Therefore M_φ is invertible. □

Theorem 1.6.6 (Hartman-Wintner). *If $\varphi \in \mathbf{L}^\infty$ then*

(i) $\mathfrak{R}(\varphi) = \sigma(M_\varphi) \subset \sigma(T_\varphi)$

(ii) $||T_\varphi|| = ||\varphi||_\infty$ *(i.e., ξ is an isometry).*

Proof. (i) From Lemma 1.6.4 and Proposition 1.6.5.

(ii) $||\varphi||_\infty = \sup_{\lambda \in \mathfrak{R}(\varphi)} |\lambda| \leq \sup_{\lambda \in \sigma(T_\varphi)} |\lambda| = r(T_\varphi) \leq ||T_\varphi|| \leq ||\varphi||_\infty$.

□

From Theorem 1.6.6 we can see that

(i) If T_φ is quasinilpotent then $T_\varphi = 0$ because $\mathfrak{R}(\varphi) \subseteq \sigma(T_\varphi) = \{0\} \;\Rightarrow\; \varphi = 0$.

(ii) If T_φ is self-adjoint then φ is real-valued because $\mathfrak{R}(\varphi) \subseteq \sigma(T_\varphi) \subseteq \mathbb{R}$.

If $\mathfrak{S} \subseteq \mathbf{L}^\infty$, write $\mathcal{T}(\mathfrak{S}) :=$ the smallest closed subalgebra of $\mathcal{L}(\mathbf{H}^2)$ containing $\{T_\varphi : \varphi \in \mathfrak{S}\}$.

If $\mathcal{A}$ is a C^*-algebra then its *commutator ideal* $\mathcal{C}$ is the closed ideal generated by the commutators $[a, b] := ab - ba$ $(a, b \in \mathcal{A})$. In particular, $\mathcal{C}$ is the smallest closed ideal in $\mathcal{A}$ such that $\mathcal{A}/\mathcal{C}$ is abelian.

Theorem 1.6.7. *If $\mathcal{C}$ is the commutator ideal in $\mathcal{T}(\mathbf{L}^\infty)$, then the mapping ξ_c induced from $\mathbf{L}^\infty$ to $\mathcal{T}(\mathbf{L}^\infty)/\mathcal{C}$ by ξ is a $*$-isometrical isomorphism. Thus there is a short exact sequence*

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{T}(\mathbf{L}^\infty) \longrightarrow \mathbf{L}^\infty \longrightarrow 0.$$

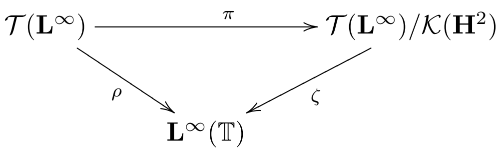
Proof. See [Do]. □

The commutator ideal $\mathcal{C}$ contains compact operators.

Proposition 1.6.8. *The commutator ideal in $\mathcal{T}(C(\mathbb{T})) = \mathcal{K}(\mathbf{H}^2)$. Hence the commutator ideal of $\mathcal{T}(\mathbf{L}^\infty)$ contains $\mathcal{K}(\mathbf{H}^2)$.*

Proof. Since T_z is the unilateral shift, we can see that the commutator ideal of $\mathcal{T}(C(\mathbb{T}))$ contains the rank one operator $T_z^* T_z - T_z T_z^*$. Moreover, $\mathcal{T}(C(\mathbb{T}))$ is irreducible since T_z has no proper reducing subspaces by Beurling's theorem. Therefore $\mathcal{T}(C(\mathbb{T}))$ contains $\mathcal{K}(\mathbf{H}^2)$. Since T_z is normal modulo a compact operator and generates the algebra $\mathcal{T}(C(\mathbb{T}))$, it follows that $\mathcal{T}(C(\mathbb{T}))/\mathcal{K}(\mathbf{H}^2)$ is commutative. Hence $\mathcal{K}(\mathbf{H}^2)$ contains the commutator ideal of $\mathcal{T}(C(\mathbb{T}))$. But since $\mathcal{K}(\mathbf{H}^2)$ is simple (i.e., it has no nontrivial closed ideal), we can conclude that $\mathcal{K}(\mathbf{H}^2)$ is the commutator ideal of $\mathcal{T}(C(\mathbb{T}))$. □

Corollary 1.6.9. *There exists a $*$-homomorphism $\zeta : \mathcal{T}(\mathbf{L}^\infty)/\mathcal{K}(\mathbf{H}^2) \longrightarrow \mathbf{L}^\infty$ such that the following diagram commutes:*



Corollary 1.6.10. *Let $\varphi \in \mathbf{L}^\infty$. If T_φ is Fredholm then φ is invertible in $\mathbf{L}^\infty$.*

Proof. If T_φ is Fredholm then $\pi(T_\varphi)$ is invertible in $\mathcal{T}(\mathbf{L}^\infty)/\mathcal{K}(\mathbf{H}^2)$, so $\varphi = \rho(T_\varphi) = (\zeta \circ \pi)(T_\varphi)$ is invertible in $\mathbf{L}^\infty$. □

From Corollary 1.6.10, we have:

(i) $||T_\varphi|| \le ||T_\varphi + K||$ for every compact operator K because $||T_\varphi|| = ||\varphi||_\infty = ||\zeta(T_\varphi + K)|| \le ||T_\varphi + K||$.

(ii) The only compact Toeplitz operator is 0 because $||K|| \le ||K + K|| \;\Rightarrow\; K = 0$.

Proposition 1.6.11. *If φ is invertible in $\mathbf{L}^\infty$ such that $\mathfrak{R}(\varphi) \subseteq$ the open right half-plane, then T_φ is invertible.*

Proof. If $\Delta \equiv \{z \in \mathbb{C} : |z-1| < 1\}$ then there exists $\epsilon > 0$ such that $\epsilon\,\mathfrak{R}(\varphi) \subseteq \Delta$. Hence $||\epsilon\varphi - 1|| < 1$, which implies $||I - T_{\epsilon\varphi}|| < 1$. Therefore $T_{\epsilon\varphi} = \epsilon T_\varphi$ is invertible. □

Corollary 1.6.12 (Bram-Halmos). *If $\varphi \in \mathbf{L}^\infty$, then $\sigma(T_\varphi) \subseteq \operatorname{conv}\mathfrak{R}(\varphi)$.*

Proof. It is sufficient to show that every open half-plane containing $\mathfrak{R}(\varphi)$ contains $\sigma(T_\varphi)$. This follow at once from Proposition 1.6.11 after a translation and rotation of the open half-plane to coincide with the open right half-plane. □

Proposition 1.6.13. *If $\varphi \in C(\mathbb{T})$ and $\psi \in \mathbf{L}^\infty$ then*

$$T_\varphi T_\psi - T_{\varphi\psi} \quad \textit{and} \quad T_\psi T_\varphi - T_{\psi\varphi} \quad \textit{are compact.}$$

Proof. If $\psi \in \mathbf{L}^\infty,\ f \in \mathbf{H}^2$ then

$$\begin{aligned} T_\psi T_{\overline{z}} f &= T_\psi P(\overline{z} f) = T_\psi(\overline{z} f - \widehat{f}(0)\overline{z}) \\ &= P M_\psi\Big(\overline{z} f - \widehat{f}(0)\overline{z}\Big) \\ &= P(\psi \overline{z} f) - \widehat{f}(0) P(\psi\overline{z}) \\ &= T_{\psi\overline{z}} f - \widehat{f}(0) P(\psi\overline{z}), \end{aligned}$$

which implies that $T_\psi T_{\overline{z}} - T_{\psi\overline{z}}$ is at most a rank one operator. Suppose $T_\psi T_{\overline{z}^n} - T_{\psi\overline{z}^n}$ is compact for every $\psi \in \mathbf{L}^\infty$ and $n = 1, \cdots, N$. Then

$$T_\psi T_{\overline{z}^{N+1}} - T_{\psi\overline{z}^{N+1}} = (T_\psi T_{\overline{z}^N} - T_{\psi\overline{z}^N})\, T_{\overline{z}} + (T_{\psi\overline{z}^N} T_{\overline{z}} - T_{(\psi\overline{z}^N)\overline{z}}),$$

which is compact. Also, since $T_\psi T_{z^n} = T_{\psi z^n}$ $(n \geq 0)$, it follows that $T_\psi T_p - T_{\psi p}$ is compact for every trigonometric polynomial p. But since the set of trigonometric polynomials is dense in $C(\mathbb{T})$ and ξ is isometric, we can conclude that $T_\psi T_\varphi - T_{\psi\varphi}$ is compact for $\psi \in \mathbf{L}^\infty$ and $\varphi \in C(\mathbb{T})$. $\square$

Theorem 1.6.14. *$\mathcal{T}(C(\mathbb{T}))$ contains $\mathcal{K}(\mathbf{H}^2)$ as its commutator and the sequence*

$$0 \longrightarrow \mathcal{K}(\mathbf{H}^2) \longrightarrow \mathcal{T}(C(\mathbb{T})) \longrightarrow C(\mathbb{T}) \longrightarrow 0$$

is a short exact sequence, i.e., $\mathcal{T}(C(\mathbb{T}))/\mathcal{K}(\mathbf{H}^2)$ is $*$-isometrically isomorphic to $C(\mathbb{T})$.

Proof. By Proposition 1.6.13 and Corollary 1.6.9. $\square$

Proposition 1.6.15 (Coburn)**.** *If $\varphi \neq 0$ a.e. in $\mathbf{L}^\infty$, then*

$$\textit{either } \ker T_\varphi = \{0\} \textit{ or } \ker T_\varphi^* = \{0\}.$$

Proof. If $f \in \ker T_\varphi$ and $g \in \ker T_\varphi^*$, i.e., $P(\varphi f) = 0$ and $P(\overline{\varphi} g)$=0, then

$$\overline{\varphi f} \in z\mathbf{H}^2 \quad \text{and} \quad \varphi\overline{g} \in z\mathbf{H}^2.$$

Thus $\overline{\varphi f}g,\ \varphi\overline{g}f \in z\mathbf{H}^1$ and therefore $\varphi f\overline{g} = 0$. If neither f nor g is 0, then by F. and M. Riesz theorem, $\varphi = 0$ a.e. on $\mathbb{T}$, a contradiction. $\square$

Corollary 1.6.16. *If $\varphi \in C(\mathbb{T})$ then T_φ is Fredholm if and only if φ vanishes nowhere.*

Proof. By Theorem 1.6.14, T_φ is Fredholm if and only if $\pi(T_\varphi)$ is invertible in $\mathcal{T}(C(\mathbb{T}))/\mathcal{K}(\mathbf{H}^2)$ if and only if φ is invertible in $C(\mathbb{T})$. $\square$

Corollary 1.6.17. *If $\varphi \in C(\mathbb{T})$, then $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$.*

Proof. $\sigma_e(T_\varphi) = \sigma(T_\varphi + \mathcal{K}(\mathbf{H}^2)) = \sigma(\varphi) = \varphi(\mathbb{T})$. $\square$

Theorem 1.6.18. *If $\varphi \in C(\mathbb{T})$ is such that T_φ is Fredholm, then*

$$\operatorname{ind}(T_\varphi) = -\operatorname{wind}(\varphi).$$

Proof. We claim that if φ and ψ determine homotopic curves in $\mathbb{C} \setminus \{0\}$, then

$$\operatorname{ind}(T_\varphi) = \operatorname{ind}(T_\psi).$$

To see this, let Φ be a constant map from $[0,1] \times \mathbb{T}$ to $\mathbb{C} \setminus \{0\}$ such that

$$\Phi(0, e^{it}) = \varphi(e^{it}) \quad \text{and} \quad \Phi(1, e^{it}) = \psi(e^{it}).$$

If we set $\Phi_\lambda(e^{it}) = \Phi(\lambda, e^{it})$, then the mapping $\lambda \mapsto T_{\Phi_\lambda}$ is norm continuous and each T_{Φ_λ} is a Fredholm operator. Since the map ind is continuous, $\operatorname{ind}(T_\varphi) = \operatorname{ind}(T_\psi)$. Now if $n = \operatorname{wind}(\varphi)$ then φ is homotopic in $\mathbb{C} \setminus \{0\}$ to z^n. Since $\operatorname{ind}(T_{z^n}) = -n$, we have that $\operatorname{ind}(T_\varphi) = -n$. $\square$

Theorem 1.6.19. *If U is the unilateral shift on $\mathbf{H}^2$ then $\operatorname{comm}(U) = \{T_\varphi : \varphi \in \mathbf{H}^\infty\}$.*

Proof. It is straightforward that $UT_\varphi = T_\varphi U$ for $\varphi \in \mathbf{H}^\infty$, i.e., $\{T_\varphi : \ \varphi \in \mathbf{H}^\infty\} \subset \text{comm}(U)$. For the reverse we suppose $T \in \text{comm}(U)$, i.e., $TU = UT$. Put $\varphi := T(1)$. So $\varphi \in \mathbf{H}^2$ and $T(p) = \varphi p$ for every polynomial p. If $f \in \mathbf{H}^2$, let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$ in $\mathbf{H}^2$. By passing to a subsequence, we can assume $p_n(z) \to f(z)$ a.e. $[m]$. Thus $\varphi p_n = T(p_n) \to T(f)$ in $\mathbf{H}^2$ and $\varphi p_n \to \varphi f$ a.e. $[m]$. Therefore $Tf = \varphi f$ for all $f \in \mathbf{H}^2$. We want to show that $\varphi \in \mathbf{L}^\infty$ and hence $\varphi \in \mathbf{H}^\infty$. We may assume, without loss of generality, that $||T|| = 1$. Observe

$$T^k f = \varphi^k f \quad \text{for } f \in \mathbf{H}^2,\ k \geq 1.$$

Hence $||\varphi^k f||_2 \leq ||f||_2$ for all $k \geq 1$. Taking $f = 1$ shows that $\int |\varphi|^{2k} dm \leq 1$ for all $k \geq 1$. If $\Delta := \{z \in \partial\mathbb{D} : \ |\varphi(z)| > 1\}$ then $\int_\Delta |\varphi|^{2k} dm \leq 1$ for all $k \geq 1$. If $m(\Delta) \neq 0$ then $\int_\Delta |\varphi|^{2k} dm \to \infty$ as $k \to \infty$, a contradiction. Therefore $m(\Delta) = 0$ and hence φ is bounded. Therefore $T = T_\varphi$ for $f \in \mathbf{H}^\infty$. □

D. Sarason [Sa] gave a generalization of Theorem 1.6.19.

Theorem 1.6.20 (Sarason's Interpolation Theorem)**.** *Let*

(i) U =*the unilateral shift on* $\mathbf{H}^2$*;*

(ii) $\mathcal{K} := \mathbf{H}^2 \ominus \psi\mathbf{H}^2$ *(*ψ *is an inner function);*

(iii) $S := PU|_{\mathcal{K}}$*, where* P *is the projection of* $\mathbf{H}^2$ *onto* $\mathcal{K}$*.*

If $T \in \text{comm}(S)$ *then there exists a function* $\varphi \in \mathbf{H}^\infty$ *such that* $T = T_\varphi|_{\mathcal{K}}$ *with* $||\varphi||_\infty = ||T||$*.*

Proof. See [Sa]. □

2 Hyponormality of Toeplitz operators

An elegant and useful theorem of C. Cowen [Cow3] characterizes the hyponormality of a Toeplitz operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} \subset \mathbb{C}$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. This result makes it possible to answer an algebraic question coming from operator theory – namely, is T_φ hyponormal ? - by studying the function φ itself. Normal Toeplitz operators were characterized by a property of their symbol in the early 1960's by A. Brown and P.R. Halmos [BH], and so it is somewhat of a surprise that 25 years passed before the exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the positivity of the selfcommutator $[T_\varphi^*, T_\varphi]$ was understood (via Cowen's theorem). As Cowen notes in his survey paper [Cow2], the intensive study of subnormal Toeplitz operators in the 1970's and early 80's is one explanation for the relatively late appearance of the sequel to the Brown-Halmos work. The characterization of hyponormality via Cowen's theorem requires one to solve a certain functional equation in the unit ball of H^∞. However the case of arbitrary trigonometric polynomials φ, though solved in principle by Cowen's theorem, is in practice very complicated. Indeed it may not even be possible to find tractable necessary and sufficient conditions for the hyponormality of T_φ in terms of the Fourier coefficients of φ unless certain assumptions are made about φ. In this chapter we present some recent development in this research.

2.1 Cowen's Theorem

In this section we present Cowen's theorem. Cowen's method is to recast the operator-theoretic problem of hyponormality of Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. This approach has been put to use in the works [CLL], [CuL1], [CuL2], [CuL3], [FL1], [FL2], [Gu1], [HKL1], [HKL2], [HL], [KL], [NaT], [Zhu] to study Toeplitz operators.

We begin with:

Lemma 2.1.1. *A necessary and sufficient condition that two Toeplitz operators commute is that either both be analytic or both be co-analytic or one be a linear function of the other.*

Proof. Let $\varphi = \sum_i \alpha_i z^i$ and $\psi = \sum_j \beta_j z^j$. Then a straightforward calculation shows that

$$T_\varphi T_\psi = T_\psi T_\varphi \iff \alpha_{i+1}\beta_{-j-1} = \beta_{i+1}\alpha_{-j-1} \quad (i, j \geq 0).$$

Thus either $\alpha_{-j-1} = \beta_{-j-1} = 0$ for $j \geq 0$, i.e., φ and ψ are both analytic, or $\alpha_{i+1} = \beta_{i+1} = 0$ for $i \geq 0$, i.e., φ and ψ are both co-analytic, or there exist i_0, j_0 such that $\alpha_{i_0+1} \neq 0$ and $\alpha_{-j_0-1} \neq 0$. So for the last case, if the common value of $\beta_{-j_0-1}/\alpha_{-j_0-1}$ and $\beta_{i_0+1}/\alpha_{i_0+1}$ is denoted by λ, then

$$\beta_{i+1} = \lambda\, \alpha_{i+1} \ \ (i \geq 0) \quad \text{and} \quad \beta_{-j-1} = \lambda\, \alpha_{-j-1} \ \ (j \geq 0).$$

Therefore, $\beta_k = \lambda\, \alpha_k$ $(k \neq 0)$. □

Theorem 2.1.2 (Brown-Halmos). *Normal Toeplitz operators are translations and rotations of hermitian Toeplitz operators i.e.,*

$$T_\varphi \text{ normal} \iff \exists\, \alpha, \beta \in \mathbb{C},\ \text{a real valued } \psi \in \mathbf{L}^\infty \text{ s.t. } T_\varphi = \alpha T_\psi + \beta 1.$$

Proof. If $\varphi = \sum_i \alpha_i z^i$, then

$$\overline{\varphi} = \sum_i \overline{\alpha_i}\, \overline{z}^i = \sum_i \overline{\alpha_{-i}} z^i.$$

So if φ is real, then $\alpha_i = \overline{\alpha_{-i}}$. Thus no real φ can be analytic or co-analytic unless φ is a constant. Write $T_\varphi = T_{\varphi_1 + i\varphi_2}$, where φ_1, φ_2 are real-valued. Then by Lemma 2.1.1, $T_\varphi T_{\overline{\varphi}} = T_{\overline{\varphi}} T_\varphi$ iff $T_{\varphi_1} T_{\varphi_2} = T_{\varphi_2} T_{\varphi_1}$ iff either φ_1 and φ_2 are both analytic or φ_1 and φ_2 are both co-analytic or $\varphi_1 = \alpha\varphi_2 + \beta$ $(\alpha, \beta \in \mathbb{C})$. So if $\varphi \neq$ a constant, then $\varphi = \alpha\varphi_2 + \beta + i\varphi_2 = (\alpha + i)\varphi_2 + \beta$. □

For $\psi \in \mathbf{L}^\infty$, the Hankel operator H_ψ is the operator on $\mathbf{H}^2$ defined by

$$H_\psi f = J(I - P)(\psi f) \quad (f \in \mathbf{H}^2),$$

where J is the unitary operator from $(\mathbf{H}^2)^\perp$ onto $\mathbf{H}^2$:

$$J(z^{-n}) = z^{n-1}\ (n \geq 1).$$

Denoting $v^*(z) := \overline{v(\overline{z})}$, another way to put this is that H_ψ is the operator on $\mathbf{H}^2$ defined by

(2.1.2.1) $$< zuv, \overline{\psi} > = < H_\psi u, v^* > \ \text{ for all } v \in \mathbf{H}^\infty.$$

If ψ has the Fourier series expansion $\psi := \sum_{n=-\infty}^{\infty} a_n z^n$, then the matrix of H_ψ is given by

$$H_\psi \equiv \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & & \\ a_{-3} & & \ddots & \\ \vdots & & & \ddots \end{pmatrix}.$$

The following are basic properties of Hankel operators.

1. $H_\psi^* = H_{\psi^*}$;
2. $H_\psi U = U^* H_\psi$ (U is the unilateral shift);
3. $\text{Ker} H_\psi = \{0\}$ or $\theta\mathbf{H}^2$ for some inner function θ (by Beurling's theorem);
4. $T_{\varphi\psi} - T_\varphi T_\psi = H_{\overline{\varphi}}^* H_\psi$;
5. $H_\varphi T_h = H_{\varphi h} = T_{h^*}^* H_\varphi$ $(h \in \mathbf{H}^\infty)$.

We are ready for:

Theorem 2.1.3 (Cowen's Theorem). *If $\varphi \in \mathbf{L}^\infty$ is such that $\varphi = \overline{g} + f$ $(f, g \in \mathbf{H}^2)$, then*

$$T_\varphi \text{ is hyponormal} \iff g = c + T_{\overline{h}} f$$

for some constant c and some $h \in \mathbf{H}^\infty(\mathbb{D})$ with $||h||_\infty \leq 1$.

Proof. Let $\varphi = f + \overline{g}$ ($f, g \in \mathbf{H}^2$). For every polynomial $p \in \mathbf{H}^2$,

$$\begin{aligned}\langle (T_\varphi^* T_\varphi - T_\varphi T_\varphi^*)p,\, p\rangle &= \langle T_\varphi p, T_\varphi p\rangle - \langle T_\varphi^* p, T_\varphi^* p\rangle \\ &= \langle fp + P\overline{g}p, fp + P\overline{g}p\rangle - \langle P\overline{f}p + gp, P\overline{f}p + gp\rangle \\ &= \langle \overline{f}p, \overline{f}p\rangle - \langle P\overline{f}p, P\overline{f}p\rangle - \langle \overline{g}p, \overline{g}p\rangle + \langle P\overline{g}p, P\overline{g}p\rangle \\ &= \langle \overline{f}p, (I-P)\overline{f}p\rangle - \langle \overline{g}p, (I-p)\overline{g}p\rangle \\ &= \langle (I-P)\overline{f}p, (I-P)\overline{f}p\rangle - \langle (I-P)\overline{g}p, (I-P)\overline{g}p\rangle \\ &= ||H_{\overline{f}}p||^2 - ||H_{\overline{g}}p||^2.\end{aligned}$$

Since polynomials are dense in $\mathbf{H}^2$,

$$\text{(2.1.3.1)} \qquad T_\varphi \text{ hyponormal} \iff ||H_{\overline{g}}u|| \le ||H_{\overline{f}}u||, \quad \forall u \in \mathbf{H}^2$$

Write $\mathcal{K} := \operatorname{cl}\operatorname{ran}(H_{\overline{f}})$ and let S be the compression of the unilateral shift U to $\mathcal{K}$. Since $\mathcal{K}$ is invariant for U^* (why: $H_{\overline{f}}U = U^* H_{\overline{f}}$), we have $S^* = U^*|_{\mathcal{K}}$. Suppose T_φ is hyponormal. Define A on $\operatorname{ran}(H_{\overline{f}})$ by

$$\text{(2.1.3.2)} \qquad A(H_{\overline{f}}u) = H_{\overline{g}}u.$$

Then A is well defined because by (2.1.3.1)

$$H_{\overline{f}}u_1 = H_{\overline{f}}u_2 \implies H_{\overline{f}}(u_1 - u_2) = 0 \implies H_{\overline{g}}(u_1 - u_2) = 0.$$

By (2.1.3.1), $||A|| \le 1$, so A has an extension to $\mathcal{K}$, which will also be denoted A. Observe that

$$H_{\overline{g}}U = AH_{\overline{f}}U = AU^*H_{\overline{f}} = AS^*H_{\overline{f}} \quad \text{and} \quad H_{\overline{g}}U = U^*H_{\overline{g}} = U^*AH_{\overline{f}} = S^*AH_{\overline{f}}.$$

Thus $AS^* = S^*A$ on $\mathcal{K}$ since $\operatorname{ran}H_{\overline{f}}$ is dense in $\mathcal{K}$, and hence $SA^* = A^*S$. By Sarason's interpolation theorem,

$$\exists\ k \in \mathbf{H}^\infty(\mathbb{D}) \text{ with } ||k||_\infty = ||A^*|| = ||A|| \text{ s.t. } A^* = \text{the compression of } T_k \text{ to } \mathcal{K}.$$

Since $T_k^* H_{\overline{f}} = H_{\overline{f}} T_{k^*}$, we have that $\mathcal{K}$ is invariant for $T_k^* = T_{\overline{k}}$, which means that A is the compression of $T_{\overline{k}}$ to $\mathcal{K}$ and

$$\text{(2.1.3.3)} \qquad H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}} \qquad \text{(by (2.1.3.2))}.$$

Conversely, if (2.1.3.3) holds for some $k \in \mathbf{H}^\infty(\mathbb{D})$ with $||k||_\infty \le 1$, then (2.1.3.1) holds for all u, and hence T_φ is hyponormal. Consequently,

$$T_\varphi \text{ hyponormal} \iff H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}}.$$

But $H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}}$ if and only if $\forall\ u, v \in \mathbf{H}^\infty$,

$$\begin{aligned}\langle zuv, g\rangle = \langle H_{\overline{g}}u, v^*\rangle &= \langle T_{\overline{k}} H_{\overline{f}}u, v^*\rangle = \langle H_{\overline{f}}u, kv^*\rangle \\ &= \langle zuk^*v, f\rangle = \langle zuv, \overline{k^*}f\rangle = \langle zuv, T_{\overline{k^*}}f\rangle.\end{aligned}$$

Since $\bigvee\{zuv : u, v \in \mathbf{H}^\infty\} = z\mathbf{H}^2$, it follows that

$$H_{\overline{g}} = T_{\overline{k}} H_{\overline{f}} \iff g = c + T_{\overline{h}} f \text{ for } h = k^*.$$

$\square$

Theorem 2.1.4 (Nakazi-Takahashi Variation of Cowen's Theorem)**.** *For* $\varphi \in \mathbf{L}^\infty$*, put*

$$\mathcal{E}(\varphi) := \{k \in \mathbf{H}^\infty : ||k||_\infty \le 1 \textit{ and } \varphi - k\overline{\varphi} \in \mathbf{H}^\infty\}.$$

Then T_φ *is hyponormal if and only if* $\mathcal{E}(\varphi) \neq \varnothing$*.*

Proof. Let $\varphi = f + \overline{g} \in \mathbf{L}^\infty$ $(f, g \in \mathbf{H}^2)$. By Cowen's theorem,

$$T_\varphi \text{ is hyponormal } \iff g = c + T_{\overline{k}} f$$

for some constant c and some $k \in \mathbf{H}^\infty$ with $||k||_\infty \le 1$. If $\varphi = k\overline{\varphi} + h$ $(h \in H^\infty)$ then $\varphi - k\overline{\varphi} = \overline{g} - k\overline{f} + f - kg \in H^\infty$. Thus $\overline{g} - k\overline{f} \in \mathbf{H}^2$, so that $P(g - \overline{k}f) = c$ ($c =$ a constant), and hence $g = c + T_{\overline{k}} f$ for some constant c. Thus T_φ is hyponormal. The argument is reversible. □

2.2 Trigonometric Polynomial Symbols Cases

In this section we consider the hyponormality of Toeplitz operators with trigonometric polynomial symbols. To do this we first review the dilation theory.

If $B = \begin{pmatrix} A & * \\ * & * \end{pmatrix}$, then B is called a *dilation* of A and A is called a *compression* of B. It was well-known that every contraction has a unitary dilation: indeed if $||A|| \leq 1$, then

$$B \equiv \begin{pmatrix} A & (I - AA^*)^{\frac{1}{2}} \\ (I - A^*A)^{\frac{1}{2}} & -A^* \end{pmatrix}$$

is unitary.

On the other hand, an operator B is called a *power* (or *strong*) *dilation* of A if B^n is a dilation of A^n for all $n = 1, 2, 3, \cdots$. So if B is a (power) dilation of A then B should be of the form $B = \begin{pmatrix} A & 0 \\ * & * \end{pmatrix}$. Sometimes, B is called a lifting of A and A is said to be lifted to B. It was also well-known that every contraction has a isometric (power) dilation. In fact, the minimal isometric dilation of a contraction A is given by

$$B \equiv \begin{pmatrix} A & 0 & 0 & 0 & \cdots \\ (I - A^*A)^{\frac{1}{2}} & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix}.$$

We then have:

Theorem 2.2.1 (Commutant Lifting Theorem)**.** *Let A be a contraction and T be a minimal isometric dilation of A. If $BA = AB$ then there exists a dilation S of B such that*

$$S = \begin{pmatrix} B & 0 \\ * & * \end{pmatrix}, \quad ST = TS, \quad \text{and} \quad ||S|| = ||B||.$$

Proof. See [GGK, p.658]. □

We next consider the following interpolation problem, called the Carathéodory-Schur Interpolation Problem (CSIP).

Given $c_0, \cdots, c_{N-1}$ in $\mathbb{C}$, find an analytic function φ on $\mathbb{D}$ such that

(i) $\widehat{\varphi}(j) = c_j$ $(j = 0, \cdots, N-1)$;

(ii) $||\varphi||_\infty \leq 1$.

The following is a solution of CSIP.

Theorem 2.2.2.

$$\text{CSIP is solvable} \iff C \equiv \begin{pmatrix} c_0 & & & & \\ c_1 & c_0 & & \text{O} & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix} \text{ is a contraction.}$$

Moreover, if φ is a solution if and only if T_φ is a contractive lifting of C which commutes with the unilateral shift.

Proof. (⇒) Assume that we have a solution φ. Then the condition (ii) implies

$$T_\varphi = \begin{pmatrix} \varphi_0 & & & \\ \varphi_1 & \varphi_0 & & \text{O} \\ \varphi_2 & \varphi_1 & \varphi_0 & \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \qquad (\varphi_j := \widehat{\varphi}(j))$$

is a contraction because $||T_\varphi|| = ||\varphi||_\infty \le 1$. So the compression of T_φ is also contractive. In particular,

$$\begin{pmatrix} \varphi_0 & & & \\ \varphi_1 & \varphi_0 & \text{O} & \\ \vdots & \vdots & \ddots & \\ \varphi_{n-1} & \varphi_{n-2} & \cdots & \varphi_0 \end{pmatrix}$$

must have norm less than or equal to 1 for all n. Therefore if CSIP is solvable, then $||C|| \le 1$.

(⇐) Let

$$C \equiv \begin{pmatrix} c_0 & & & & \\ c_1 & c_0 & & \text{O} & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix} \text{ with } ||C|| \le 1$$

and let

$$A := \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} : \mathbb{C}^N \to \mathbb{C}^N.$$

Then A and C are contractions and $AC = CA$. Observe that the unilateral shift U is the minimal isometric dilation of A (please check it!). By the Commutant Lifting Theorem, C can be lifted to a contraction S such that $SU = US$. But then S is an analytic Toeplitz operator, i.e., $S = T_\varphi$ with $\varphi \in \mathbf{H}^\infty$. Since S is a lifting of C we must have

$$\widehat{\varphi}(j) = c_j \ (j = 0, 1, \cdots, N-1).$$

Since S is a contraction, it follows that $||\varphi||_\infty = ||T_\varphi|| \le 1$. □

Now suppose φ is a trigonometric polynomial of the form

$$\varphi(z) = \sum_{n=-N}^{N} a_n z^n \ (a_N \neq 0).$$

If a function $k \in \mathbf{H}^\infty(\mathbb{T})$ satisfies $\varphi - k\overline{\varphi} \in \mathbf{H}^\infty$ then k necessarily satisfies

(2.2.2.1) $$k \sum_{n=1}^{N} \overline{a_n} z^{-n} - \sum_{n=1}^{N} a_{-n} z^{-n} \in \mathbf{H}^\infty.$$

From (2.2.2.1) one computes the Fourier coefficients $\widehat{k}(0), \cdots, \widehat{k}(N-1)$ to be $\widehat{k}(n) = c_n$ ($n = 0, 1, \cdots, N-1$), where $c_0, c_1, \cdots, c_{N-1}$ are determined uniquely from the coefficients of φ by the following relation

$$
\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_{N-1} \end{pmatrix} = \begin{pmatrix} \overline{a_1} & \overline{a_2} & \overline{a_3} & \cdots & \overline{a_N} \\ \overline{a_2} & \overline{a_3} & \dots & \cdot & \\ \overline{a_3} & \dots & \cdots & & \\ \vdots & \dots & & O & \\ \overline{a_N} & & & & \end{pmatrix}^{-1} \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-N} \end{pmatrix}. \tag{2.2.2.2}
$$

Thus if $k(z) = \sum_{j=0}^{\infty} c_j z^j$ is a function in $\mathbf{H}^\infty$ then

$$
\varphi - k\overline{\varphi} \in \mathbf{H}^\infty \iff c_0, c_1, \cdots, c_{N-1} \text{ are given by (2.2.2.2).}
$$

Thus by Cowen's theorem, if $c_0, c_1, \cdots, c_{N-1}$ are given by (2.6) then the hyponormality of T_φ is equivalent to the existence of a function $k \in \mathbf{H}^\infty$ such that

$$
\begin{cases} \widehat{k}(j) = c_j \ (j = 0, \cdots, N-1) \\ ||k||_\infty \le 1, \end{cases}
$$

which is precisely the formulation of CSIP. Therefore we have:

Theorem 2.2.3. *If $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where $a_N \neq 0$ and if $c_0, c_1, \cdots, c_{N-1}$ are given by (2.2.2.2) then*

$$
T_\varphi \text{ is hyponormal} \iff C \equiv \begin{pmatrix} c_0 & & & & \\ c_1 & c_0 & & O & \\ c_2 & c_1 & c_0 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{N-1} & c_{N-2} & \cdots & c_1 & c_0 \end{pmatrix} \text{ is a contraction.}
$$

2.3 Bounded Type Symbols Cases

A function $\varphi \in \mathbf{L}^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions ψ_1, ψ_2 in $\mathbf{H}^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all z in $\mathbb{T}$. Evidently, rational functions in $\mathbf{L}^\infty$ are of bounded type.

If θ is an inner function, the degree of θ, denoted by $\deg(\theta)$, is defined by the number of zeros of θ lying in the open unit disk $\mathbb{D}$ if θ is a finite Blaschke product of the form

$$\theta(z) = e^{i\xi} \prod_{j=1}^{n} \frac{z - \beta_j}{1 - \overline{\beta_j} z} \qquad (|\beta_j| < 1 \text{ for } j = 1, \cdots, n),$$

otherwise the degree of θ is infinite. For an inner function θ, write

$$\mathcal{H}(\theta) := \mathbf{H}^2 \ominus \theta \mathbf{H}^2.$$

Note that for $f \in \mathbf{H}^2$,

$$\begin{aligned}\langle [T_\varphi^*, T_\varphi] f, f\rangle = ||T_\varphi f||^2 - ||T_{\overline{\varphi}} f||^2 &= ||\varphi f||^2 - ||H_\varphi f||^2 - (||\overline{\varphi} f||^2 - ||H_{\overline{\varphi}} f||^2) \\ &= ||H_{\overline{\varphi}} f||^2 - ||H_\varphi f||^2.\end{aligned}$$

Thus we have

$$T_\varphi \text{ hyponormal} \iff ||H_{\overline{\varphi}} f|| \geq ||H_\varphi f|| \quad (f \in H^2).$$

Now let $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where f and g are in $\mathbf{H}^2$. Since $H_\varphi U = U^* H_\varphi$ (U =the unilateral shift), it follows from the Beurling's theorem that

$$\ker H_{\overline{f}} = \theta_0 \mathbf{H}^2 \quad \text{and} \quad \ker H_{\overline{g}} = \theta_1 \mathbf{H}^2 \quad \text{for some inner functions } \theta_0, \theta_1.$$

Thus if T_φ is hyponormal then since $||H_{\overline{f}} h|| \geq ||H_{\overline{g}} h||$ ($h \in \mathbf{H}^2$), we have

(2.3.0.1) $$\theta_0 \mathbf{H}^2 = \ker H_{\overline{f}} \subset \ker H_{\overline{g}} = \theta_1 \mathbf{H}^2,$$

which implies that θ_1 divides θ_0, so that $\theta_0 = \theta_1 \theta_2$ for some inner function θ_2.

On the other hand, note that if $f \in \mathbf{H}^2$ and $\overline{f}$ is of bounded type, i.e., $\overline{f} = \psi_2/\psi_1$ ($\psi_i \in \mathbf{H}^\infty$), then dividing the outer part of ψ_1 into ψ_2 one obtain $\overline{f} = \psi/\theta$ with θ inner and $\psi \in \mathbf{H}^\infty$, and hence $f = \theta \overline{\psi}$. But since $f \in \mathbf{H}^2$ we must have $\psi \in \mathcal{H}(\theta)$. Thus if $f \in \mathbf{H}^2$ and $\overline{f}$ is of bounded type then we can write

(2.3.0.2) $$f = \theta \overline{\psi} \quad (\theta \text{ inner}, \ \psi \in \mathcal{H}(\theta)).$$

Therefore if $\varphi = \overline{g} + f$ is of bounded type and T_φ is hyponormal then by (2.3.0.1) and (2.3.0.2), we can write

$$f = \theta_1 \theta_2 \overline{a} \quad \text{and} \quad g = \theta_1 \overline{b},$$

where $a \in \mathcal{H}(\theta_1 \theta_2)$ and $b \in \mathcal{H}(\theta_1)$.

We now have:

Lemma 2.3.1. *Let $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where f and g are in $\mathbf{H}^2$. Assume that*

(2.3.1.1) $$f = \theta_1 \theta_2 \overline{a} \quad \text{and} \quad g = \theta_1 \overline{b}$$

for $a \in \mathcal{H}(\theta_1 \theta_2)$ and $b \in \mathcal{H}(\theta_1)$. Let $\psi := \theta_1 \overline{P_{\mathcal{H}(\theta_1)}(a)} + \overline{g}$. Then T_φ is hyponormal if and only if T_ψ is.

Proof. This assertion follows at once from [Gu2, Corollary 3.5]. □

In view of Lemma 2.3.1, when we study the hyponormality of Toeplitz operators with bounded type symbols φ, we may assume that the symbol $\varphi = \overline{g} + f \in \mathbf{L}^\infty$ is of the form

$$f = \theta\overline{a} \quad \text{and} \quad g = \theta\overline{b}, \tag{2.3.1.2}$$

where θ is an inner function and $a, b \in \mathcal{H}(\theta)$ such that the inner parts of a, b and θ are coprime.

On the other hand, let $f \in \mathbf{H}^\infty$ be a rational function. Then we may write

$$f = p_m(z) + \sum_{i=1}^{n}\sum_{j=0}^{l_i-1} \frac{a_{ij}}{(1-\overline{\alpha_i}z)^{l_i-j}} \qquad (0 < |\alpha_i| < 1),$$

where $p_m(z)$ denotes a polynomial of degree m. Let θ be a finite Blaschke product of the form

$$\theta = z^m \prod_{i=1}^{n} \left(\frac{z-\alpha_i}{1-\overline{\alpha_i}z}\right)^{l_i}.$$

Observe that

$$\frac{a_{ij}}{1-\overline{\alpha_i}z} = \frac{\overline{\alpha_i}a_{ij}}{1-|\alpha_i|^2}\Big(\frac{z-\alpha_i}{1-\overline{\alpha_i}z} + \frac{1}{\overline{\alpha_i}}\Big).$$

Thus $f \in \mathcal{H}(z\theta)$. Letting $a := \theta\overline{f}$, we can see that $a \in \mathcal{H}(z\theta)$ and $f = \theta\overline{a}$. Thus if $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where f and g are rational functions and if T_φ is hyponormal, then we can write

$$f = \theta\overline{a} \quad \text{and} \quad g = \theta\overline{b}$$

for a finite Blaschke product θ with $\theta(0) = 0$ and $a, b \in \mathcal{H}(\theta)$.

Now let θ be a finite Blaschke product of degree d. We can write

$$\theta = e^{i\xi} \prod_{i=1}^{n} B_i^{n_i}, \tag{2.3.1.3}$$

where $B_i(z) := \frac{z-\alpha_i}{1-\overline{\alpha_i}z}$, $(|\alpha_i| < 1)$, $n_i \geq 1$ and $\sum_{i=1}^n n_i = d$. Let $\theta = e^{i\xi}\prod_{j=1}^d B_j$ and each zero of θ be repeated according to its multiplicity. Note that this Blaschke product is precisely the same Blaschke product in (2.3.1.3). Let

$$\phi_j := \frac{d_j}{1-\overline{\alpha_j}z} B_{j-1}B_{j-2}\cdots B_1 \quad (1 \leq j \leq d), \tag{2.3.1.4}$$

where $\phi_1 := d_1(1-\overline{\alpha_1}z)^{-1}$ and $d_j := (1-|\alpha_j|^2)^{\frac{1}{2}}$. It is well known that $\{\phi_j\}_1^d$ is an orthonormal basis for $\mathcal{H}(\theta)$ (cf. [FF,Theorem X.1.5]). Let $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where $g = \theta\overline{b}$ and $f = \theta\overline{a}$ for $a, b \in \mathcal{H}(\theta)$ and write

$$\mathcal{C}(\varphi) := \{k \in \mathbf{H}^\infty : \varphi - k\overline{\varphi} \in \mathbf{H}^\infty\}.$$

Then k is in $\mathcal{C}(\varphi)$ if and only if $\overline{\theta}b - k\overline{\theta}a \in \mathbf{H}^2$, or equivalently,

$$b - ka \in \theta\mathbf{H}^2. \tag{2.3.1.5}$$

Note that $\theta^{(n)}(\alpha_i) = 0$ for all $0 \le n < n_i$. Thus the condition (2.3.1.5) is equivalent to the following equation: for all $1 \le i \le n$,

$$\text{(2.3.1.6)} \qquad \begin{pmatrix} k_{i,0} \\ k_{i,1} \\ k_{i,2} \\ \vdots \\ k_{i,n_i-2} \\ k_{i,n_i-1} \end{pmatrix} = \begin{pmatrix} a_{i,0} & 0 & 0 & 0 & \cdots & 0 \\ a_{i,1} & a_{i,0} & 0 & 0 & \cdots & 0 \\ a_{i,2} & a_{i,1} & a_{i,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{i,n_i-2} & a_{i,n_i-3} & \ddots & \ddots & a_{i,0} & 0 \\ a_{i,n_i-1} & a_{i,n_i-2} & \cdots & a_{i,2} & a_{i,1} & a_{i,0} \end{pmatrix}^{-1} \begin{pmatrix} b_{i,0} \\ b_{i,1} \\ b_{i,2} \\ \vdots \\ b_{i,n_i-2} \\ b_{i,n_i-1} \end{pmatrix},$$

where

$$k_{i,j} := \frac{k^{(j)}(\alpha_i)}{j!}, \quad a_{i,j} := \frac{a^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad b_{i,j} := \frac{b^{(j)}(\alpha_i)}{j!}.$$

Conversely, if $k \in \mathbf{H}^\infty$ satisfies the equality (2.3.1.6) then k must be in $\mathcal{C}(\varphi)$. Thus k belongs to $\mathcal{C}(\varphi)$ if and only if k is a function in $\mathbf{H}^\infty$ for which

$$\text{(2.3.1.7)} \qquad \frac{k^{(j)}(\alpha_i)}{j!} = k_{i,j} \qquad (1 \le i \le n,\ 0 \le j < n_i),$$

where the $k_{i,j}$ are determined by the equation (2.3.1.6). If in addition $||k||_\infty \le 1$ is required then this is exactly the classical Hermite-Fejér Interpolation Problem (HFIP). Therefore we have:

Theorem 2.3.2. *Let $\varphi = \overline{g} + f \in \mathbf{L}^\infty$, where f and g are rational functions. Then T_φ is hyponormal if and only if the corresponding HFIP (2.3.1.7) is solvable.*

Now we can summarize that tractable criteria for the hyponormality of Toeplitz operators T_φ are accomplished for the cases where the symbol φ is a trigonometric polynomial or a rational function via solutions of some interpolation problems.

We conclude this chapter with:

PROBLEM A. *Let $\varphi \in \mathbf{L}^\infty$ be arbitrary. Find necessary and sufficient conditions, in terms of the coefficients of φ, for T_φ to be hyponormal. In particular, for the cases where φ is of bounded type.*

3 Subnormality of Toeplitz operators

The present chapter concerns the question: *Which Toeplitz operators are subnormal?* Recall that a Toeplitz operator T_φ is called analytic if φ is in $\mathbf{H}^\infty$, that is, φ is a bounded analytic function on $\mathbb{D}$. These are easily seen to be subnormal: $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in \mathbf{H}^2$, where M_φ is the normal operator of multiplication by φ on $\mathbf{L}^2$. P.R. Halmos raised the following problem, so-called the *Halmos's Problem 5* in his 1970 lectures "Ten Problems in Hilbert Space" [Ha1], [Ha2]:

Is every subnormal Toeplitz operator either normal or analytic ?

The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are obviously subnormal.

3.1 Halmos's Problem 5

We begin with a brief survey of research related to P.R. Halmos's Problem 5.

In 1976, M. Abrahamse [Ab] gave a general sufficient condition for the answer to the Halmos's Problem 5 to be affirmative.

Theorem 3.1.1 (Abrahamse's Theorem)**.** *If*

(i) T_φ *is hyponormal;*

(ii) φ *or* $\overline{\varphi}$ *is of bounded type;*

(iii) $\ker[T_\varphi^*, T_\varphi]$ *is invariant for* T_φ*,*

then T_φ *is normal or analytic.*

Proof. See [Ab]. □

On the other hand, observe that if S is a subnormal operator on $\mathcal{H}$ and if $N := \text{mne}\,(S)$ then

$$\ker[S^*, S] = \{f : < f, [S^*, S]f >= 0\} = \{f : \ ||S^*f|| = ||Sf||\} = \{f : \ N^*f \in \mathcal{H}\}.$$

Therefore, $S(\ker[S^*, S]) \subseteq \ker[S^*, S]$.

By Theorem 3.1.1 and the preceding remark we get:

Corollary 3.1.2. *If* T_φ *is subnormal and if* φ *or* $\overline{\varphi}$ *is of bounded type, then* T_φ *is normal or analytic.*

Lemma 3.1.3. *A function φ is of bounded type if and only if* $\ker H_\varphi \neq \{0\}$.

Proof. If $\ker H_\varphi \neq \{0\}$ then since $H_\varphi f = 0 \Rightarrow (1-P)\varphi f = 0 \Rightarrow \varphi f = P\varphi f := g$, we have

$$\exists\ f, g \in \mathbf{H}^2 \text{ s.t. } \varphi f = g.$$

Hence $\varphi = \frac{g}{f}$. Remembering that if $\frac{1}{\varphi} \in \mathbf{L}^\infty$ then φ is outer if and only if $\frac{1}{\varphi} \in \mathbf{H}^\infty$ and dividing the outer part of f into g gives

$$\varphi = \frac{\psi}{\theta} \quad (\psi \in \mathbf{H}^\infty, \theta \text{ inner}).$$

Conversely, if $\varphi = \frac{\psi}{\theta}$ ($\psi \in \mathbf{H}^\infty$, θ inner), then $\theta \in \ker H_\varphi$ because $\varphi\theta = \psi \in \mathbf{H}^\infty \Rightarrow (1-P)\varphi\theta = 0 \Rightarrow H_\varphi \theta = 0$. □

From Theorem 3.1.1 we can see that

(3.1.3.1) $$\varphi = \frac{\psi}{\theta}\ (\theta, \psi \text{ inner}),\ T_\varphi \text{ subnormal } \Rightarrow\ T_\varphi \text{ normal or analytic}$$

The following proposition strengthen the conclusion of (3.1.3.1), whereas weakens the hypothesis of (3.1.3.1).

Proposition 3.1.4. *If* $\varphi = \frac{\psi}{\theta}$ (θ, ψ inner) *and if* T_φ *is hyponormal, then* T_φ *is analytic.*

Proof. Observe that

$$\begin{aligned} 1 = ||\theta|| = ||P(\theta)|| = ||P(\overline{\varphi}\theta\varphi)|| = ||P(\overline{\varphi}\psi)|| \\ = ||T_{\overline{\varphi}}(\psi)|| \leq ||T_\varphi(\psi)|| = ||P(\frac{\psi^2}{\theta})|| \leq ||\frac{\psi^2}{\theta}|| = 1, \end{aligned}$$

which implies that $\frac{\psi^2}{\theta} \in \mathbf{H}^2$, so θ divides ψ^2. Thus if one choose ψ and θ to be relatively prime (i.e., if $\varphi = \frac{\psi}{\theta}$ is in lowest terms), then θ is constant. Therefore T_φ is analytic. □

Proposition 3.1.5. *If A is a weighted shift with weights $a_0, a_1, a_2, \cdots$ such that*

$$0 \leq a_0 \leq a_1 \leq \cdots < a_N = a_{N+1} = \cdots = 1,$$

then A is not unitarily equivalent to any Toeplitz operator.

Proof. Note that A is hyponormal, $||A|| = 1$ and A attains its norm. If A is unitarily equivalent to T_φ then by a result of Brown and Douglas [BD], T_φ is hyponormal and $\varphi = \frac{\psi}{\theta}$ (θ, ψ inner). By Proposition 3.1.4, $T_\varphi \equiv T_\psi$ is an isometry, so $a_0 = 1$, a contradiction. □

Recall that the Bergman shift (whose weights are given by $\sqrt{\frac{n+1}{n+2}}$) is subnormal. The following question arises naturally:

(3.1.5.1) *Is the Bergman shift unitarily equivalent to a Toeplitz operator?*

An affirmative answer to the question (3.1.5.1) gives a negative answer to Halmos's Problem 5. To see this, assume that the Bergman shift S is unitarily equivalent to T_φ, then

$$\Re(\varphi) \subseteq\ \sigma_e(T_\varphi) = \sigma_e(S) = \text{the unit circle } \mathbb{T}.$$

Thus φ is unimodular. Since S is not an isometry it follows that φ is not inner. Therefore T_φ is not an analytic Toeplitz operator.

To the question (3.1.5.1) we need an auxiliary lemma:

Lemma 3.1.6. *If a Toeplitz operator T_φ is a weighted shift with weights $\{a_n\}_{n=0}^\infty$ with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$, i.e.,*

$$T_\varphi e_n = a_n e_{n+1} \ (n \geq 0) \tag{3.1.6.1}$$

then $e_0(z)$ is an outer function.

Proof. By Coburn's theorem, $\ker T_\varphi = \{0\}$ or $\ker T_\varphi^* = \{0\}$. The expression (3.1.6.1) gives $e_0 \in \ker T_\varphi^*$, and hence $\ker T_\varphi = \{0\}$. Thus $a_n > 0$ $(n \geq 0)$. Write

$$e_0 := gF, \text{ where } g \text{ is inner and } F \text{ is outer.}$$

Because $T_\varphi^* e_0 = 0$, we get

$$T_\varphi^* F = T_{\overline{\varphi}}(\overline{g} e_0) = T_{\overline{g}} T_{\overline{\varphi}} e_0 = T_{\overline{g}} T_\varphi^* e_0 = 0.$$

Note that $\dim \ker T_\varphi^* = 1$. So we have $F = ce_0$ ($c =$a constant), so that g is a constant, and hence e_0 is an outer function. □

Theorem 3.1.7 (Sun's Theorem)**.** *Let T be a weighted shift with a strictly increasing weight sequence $\{a_n\}_{n=0}^\infty$. If $T \cong T_\varphi$ then*

$$a_n = \sqrt{1-\alpha^{2n+2}}\, ||T_\varphi|| \quad (0 < \alpha < 1).$$

Proof. Assume $T \cong T_\varphi$. We assume, without loss of generality, that $||T|| = 1$ (so $a_n < 1$). Since T is a weighted shift, $\sigma_e(T) = \{z : |z| = 1\}$. Since $\mathfrak{R}(\varphi) \subset \sigma_e(T_\varphi)$, it follows that $|\varphi| = 1$, i.e., φ is unimodular. By Lemma 3.1.6,

$$\exists \text{ an orthonormal basis } \{e_n\}_{n=0}^\infty \text{ s.t. (3.1.6.1) holds.}$$

Expression (3.1.6.1) can be written as follows:

$$\begin{cases} \varphi e_n = a_n e_{n+1} + \sqrt{1-a_n^2}\, \eta_n \\ \overline{\varphi} e_{n+1} = a_n e_n + \sqrt{1-a_n^2}\, \xi_n \end{cases} \tag{3.1.7.1}$$

where $\eta_n, \xi_n \in (\mathbf{H}^2)^\perp$ and $||\eta_n|| = ||\xi_n|| = 1$. Since $\{\varphi e_n\}_{n=0}^\infty$ is an orthonomal system and $a_n < 1$, we have

$$< \eta_\ell, \eta_k >=< \xi_\ell, \xi_k >= \begin{cases} 0, & \ell \neq k \\ 1, & \ell = k \end{cases} \tag{3.1.7.2}$$

From (3.1.7.1) we have

$$e_n = \overline{\varphi}\left(a_n e_{n+1} + \sqrt{1-a_n^2}\ \eta_n\right) = a_n^2 e_n + a_n \sqrt{1-a_n^2}\ \xi_n + \sqrt{1-a_n^2}\ \overline{\varphi}\, \eta_n. \tag{3.1.7.3}$$

Then (3.1.7.3) is equivalent to

$$\varphi \overline{\eta_n} = -a_n \xi_n + \sqrt{1-a_n^2}\ \overline{e_n}. \tag{3.1.7.4}$$

Set $d_n := \frac{\overline{\eta_n}}{t}$ and $\rho_n := \frac{\overline{\xi_n}}{t}$ ($|t| = 1$). Then (3.1.7.4) is equivalent to

(3.1.7.5) $$\varphi d_n = -a_n \rho_n + \sqrt{1 - a_n^2}\ \frac{\overline{e_n}}{t}.$$

Since $\frac{\overline{e_n}}{t} \in (\mathbf{H}^2)^\perp$ and $\{d_n\}_{n=0}^\infty$ is an orthonormal basis for $\mathbf{H}^2$, we can see that

(3.1.7.6) $$\begin{cases} ||T_\varphi d_0|| = a_0 = \inf_{||x||=1} ||T_\varphi x|| = ||T_\varphi e_0|| \\ ||T_\varphi d_\ell|| = a_\ell = ||T_\varphi e_\ell||\ . \end{cases}$$

Then (3.1.6.1)+(3.1.7.6) implies

(3.1.7.7) $$d_n = r_n e_n \quad (|r_n| = 1).$$

Substituting (3.1.7.7.) into (3.1.7.6) and comparing it with (3.1.7.1) gives

$$a_n e_{n+1} + \sqrt{1 - a_n^2}\, \eta_n = \varphi e_n = -\frac{a_n}{r_n}\, \rho_n + \frac{\sqrt{1 - a_n^2}}{r_n}\ \frac{\overline{e_n}}{t},$$

which implies

(3.1.7.8) $$\begin{cases} -\overline{r_n} \rho_n = e_{n+1} \\ \overline{r_n} \frac{\overline{e_n}}{t} = \eta_n\ . \end{cases}$$

Therefore (3.1.7.1) is reduced to:

(3.1.7.9) $$\begin{cases} \varphi e_n = a_n e_{n+1} + \sqrt{1 - a_n^2}\, \overline{r_n}\, \frac{\overline{e_n}}{t} \\ \overline{\varphi} e_{n+1} = a_n e_n - \sqrt{1 - a_n^2}\, \overline{r_n}\, \frac{\overline{e_{n+1}}}{t} \end{cases}$$

Put $e_{-(n+1)} := \frac{\overline{e_n}}{t} \in (\mathbf{H}^2)^\perp$ ($n \geq 0$). We now claim that

(3.1.7.10) $$\overline{\varphi} e_0 = r e_{-1}\ (|r| = 1)\ :$$

indeed, $T_{\overline{\varphi}}\left(\frac{\varphi \overline{e_0}}{t}\right) = P(\frac{\overline{e_0}}{t}) = 0$, so $e_0 = r \frac{\varphi \overline{e_0}}{t}$ for $|r| = 1$, and hence $\overline{\varphi} e_0 = r e_{-1}$. From (3.1.7.9) we have

(3.1.7.11) $$\varphi e_0 = a_0 e_1 + \overline{r_0}\, \sqrt{1 - a_0^2}\, e_{-1} = a_0 e_1 + \overline{r_0}\, \overline{r}\, \sqrt{1 - a_0^2}\, \overline{\varphi} e_0,$$

or, equivalently,

(3.1.7.12) $$\left(\varphi - \overline{r_0}\, \overline{r} \sqrt{1 - a_0^2}\, \overline{\varphi}\right) e_0 = a_0 e_1.$$

Write

(3.1.7.13) $$\psi \equiv \varphi - \overline{r_0}\, \overline{r} \sqrt{1 - a_0^2}\, \overline{\varphi}.$$

Evidently,

$$V := \{x \in \mathbf{H}^2 : \psi x \in \mathbf{H}^2\}$$

is not empty. Moreover, since V is invariant for U, it follows from Beurling's theorem that

$$V = \chi \mathbf{H}^2 \text{ for an inner function } \chi.$$

Since $e_0 \in V$ and e_0 is an outer function, we must have $\chi = 1$. This means that $\psi = \psi \cdot 1 \in \mathbf{H}^2$. Therefore $\psi e_1 = T_\psi e_1 \in \mathbf{H}^2$. On the other hand, by (3.1.7.9),

$$\begin{aligned}
\psi e_1 &= \left(\varphi - \overline{r_0}\,\overline{r}\sqrt{1-a_0^2}\,\overline{\varphi}\right) e_1 \\
&= a_1 e_2 + \overline{r_1}\sqrt{1-a_1^2}e_{-2} - \overline{r_0}\,\overline{r}\sqrt{1-a_0^2}\left(a_0 e_0 - \sqrt{1-a_0^2}\,\overline{r_0}\,e_{-2}\right) \\
&= a_1 e_2 - \overline{r_0}\,\overline{r} a_0 \sqrt{1-a_1^2}\,e_0 + \left(\overline{r_1}\sqrt{1-a_1^2} + \overline{r}\,\overline{r_0}^2\,(1-a_1^2)\right) e_{-2}.
\end{aligned}$$

Thus we have

$$\overline{r_1}\sqrt{1-a_1^2} + \overline{r}\,\overline{r_0}^2(1-a_0^2) = 0$$

So, $\sqrt{1-a_1^2} = 1-a_0^2$, i.e., $a_1 = \sqrt{1-(1-a_0^2)^2}$. If we put $\alpha^2 \equiv 1-a_0^2$, i.e., $a_0 = (1-\alpha^2)^{\frac{1}{2}}$ then $a_1 = (1-\alpha^4)^{\frac{1}{2}}$. Inductively, we get $a_n = (1-\alpha^{2n+2})^{\frac{1}{2}}$. □

Corollary 3.1.8. *The Bergman shift is not unitarily equivalent to any Toeplitz operator.*

Proof. $\frac{n+1}{n+2} \neq 1-\alpha^{2n+2}$ for any $\alpha > 0$. □

Lemma 3.1.9. *The weighted shift $T \equiv W_\alpha$ with weights $\alpha_n \equiv (1-\alpha^{2n+2})^{\frac{1}{2}}$ $(0 < \alpha < 1)$ is subnormal.*

Proof. Write $r_n := \alpha_0^2\alpha_1^2\cdots\alpha_{n-1}^2$ for the moment of W. Define a discrete measure μ on $[0,1]$ by

$$\mu(z) = \begin{cases} \Pi_{j=1}^{\infty}(1-\alpha^{2j}) & (z=0) \\ \Pi_{j=1}^{\infty}(1-\alpha^{2j})\frac{\alpha^{2k}}{(1-\alpha^2)\cdots(1-\alpha^{2k})} & (z=\alpha^k; k=1,2,\cdots). \end{cases}$$

Then $r_n = \int_0^1 t^n d\mu$. By Berger's theorem, T is subnormal. □

Corollary 3.1.10. *If $T_\varphi \cong$ a weighted shift, then T_φ is subnormal.*

Remark 3.1.11. If $T_\varphi \cong$ a weighted shift, what is the form of φ ? A careful analysis of the proof of Theorem 3.1.7 shows that

$$\psi = \varphi - \alpha\overline{\varphi} \in \mathbf{H}^\infty.$$

But

$$\begin{aligned}
T_\psi = T_\varphi - \alpha T_\varphi^* &= \begin{pmatrix} 0 & -\alpha a_0 & & & \\ a_0 & 0 & -\alpha a_1 & & \\ & a_1 & 0 & -\alpha a_2 & \\ & & a_2 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\alpha & & & \\ 1 & 0 & -\alpha & & \\ & 1 & 0 & -\alpha & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} + K \quad (K \text{ compact}) \\
&\cong T_{z-\alpha\overline{z}} + K.
\end{aligned}$$

Thus $\text{ran}\,(\psi) = \sigma_e(T_\psi) = \sigma_e(T_{z-\alpha\bar{z}}) = \text{ran}(z - \alpha\bar{z})$. Thus ψ is a conformal mapping of $\mathbb{D}$ onto the interior of the ellipse with vertices $\pm i(1+\alpha)$ and passing through $\pm(1-\alpha)$. On the other hand, $\psi = \varphi - \alpha\overline{\varphi}$. So $\alpha\overline{\psi} = \alpha\overline{\varphi} - \alpha^2\varphi$, which implies

$$\varphi = \frac{1}{1-\alpha^2}(\psi + \alpha\overline{\psi}).$$

We now have:

Theorem 3.1.12 (Cowen and Long Theorem)**.** *For $0 < \alpha < 1$, let ψ be a conformal map of $\mathbb{D}$ onto the interior of the ellipse with vertices $\pm i(1-\alpha)^{-1}$ and passing through $\pm(1+\alpha)^{-1}$. Then $T_{\psi+\alpha\overline{\psi}}$ is a subnormal weighted shift that is neither analytic nor normal.*

Proof. Let $\varphi = \psi + \alpha\overline{\psi}$. Then φ is a continuous map of $\mathbb{D}$ onto $\mathbb{D}$ with $\text{wind}(\varphi) = 1$. Let

$$K := 1 - T_{\overline{\varphi}}T_\varphi = T_{\overline{\varphi}\varphi} - T_{\overline{\varphi}}T_\varphi = H_\varphi^* H_\varphi,$$

which is compact since φ is continuous. Now $\varphi - \alpha\overline{\varphi} = (1-\alpha^2)\psi \in \mathbf{H}^\infty$, so $H_\psi = 0$ and hence, $H_\varphi = \alpha H_{\overline{\varphi}}$. Thus

$$K = H_\varphi^* H_\varphi = \alpha^2 H_{\overline{\varphi}}^* H_{\overline{\varphi}} = \alpha^2(1 - T_\varphi T_{\overline{\varphi}}),$$

so that

$$KT_\varphi = \alpha^2(1 - T_\varphi T_{\overline{\varphi}})T_\varphi = \alpha^2 T_\varphi(1 - T_{\overline{\varphi}}T_\varphi) = \alpha^2 T_\varphi K.$$

By Coburn's theorem, $\ker T_\varphi = \{0\}$ or $\ker T_{\overline{\varphi}} = \{0\}$. But since

$$\text{ind}(T_\varphi) = -\text{wind}(\varphi) = -1,$$

it follows

$$\ker T_\varphi = \{0\} \text{ and } \dim\ker T_{\overline{\varphi}} = 1.$$

Let $e_0 \in \ker T_{\overline{\varphi}}$ and $||e_0|| = 1$. Write

$$e_{n+1} := \frac{T_\varphi e_n}{||T_\varphi e_n||}.$$

We claim that $Ke_n = \alpha^{2n+2}e_n$: indeed, $Ke_0 = \alpha^2(1 - T_\varphi T_{\overline{\varphi}})e_0 = \alpha^2 e_0$ and if we assume $Ke_j = \alpha^{2j+2}e_j$ then

$$Ke_{j+1} = ||T_\varphi e_j||^{-1}(KT_\varphi e_j) = ||T_\varphi e_j||^{-1}(\alpha^2 T_\varphi K e_j) = ||T_\varphi e_j||^{-1}(\alpha^{2j+4}T_\varphi e_j) = \alpha^{2j+4}e_{j+1}.$$

Thus we can see that

$$\begin{cases} \alpha^2, \alpha^4, \alpha^6, \cdots \text{ are eigenvalues of } K\,; \\ \{e_n\}_{n=0}^\infty \text{ is an orthonormal set since } K \text{ is self-adjoint.} \end{cases}$$

We will then prove that $\{e_n\}$ forms an orthonormal basis for $\mathbf{H}^2$. Observe

$$\text{tr}(H_\varphi^* H_\varphi) = \text{the sum of its eigenvalues.}$$

Thus

$$\sum_{n=0}^\infty \alpha^{2n+2} \le \text{tr}(H_\varphi^* H_\varphi) = ||H_\varphi||_2^2 \quad (||\cdot||_2 \text{ denotes the Hilbert-Schmidt norm}). \tag{3.1.12.1}$$

Since $\psi \in \mathbf{H}^\infty$, we have

$$\begin{aligned} ||H_\varphi||_2^2 = ||H_\psi + \alpha H_{\overline{\psi}}||_2^2 = \alpha^2 ||H_{\overline{\psi}}||_2^2 = \alpha^2 \text{tr}(H_{\overline{\psi}}^* H_{\overline{\psi}}) = \alpha^2 \text{tr}\,[T_{\overline{\psi}}, T_\psi] \\ \le \frac{\alpha^2}{\pi}\mu(\sigma(T_\psi)) = \frac{\alpha^2}{\pi}\mu(\psi(\mathbb{D})) = \frac{\alpha^2}{1-\alpha^2}, \end{aligned}$$

which together with (3.1.12.1) implies that

$$\sum \alpha^{2n+2} \le ||H_\varphi||_2^2 \le \frac{\alpha^2}{1-\alpha^2} = \sum_{n=0}^{\infty} \alpha^{2n+2},$$

so $\text{tr}(H_\varphi^* H_\varphi) = \sum_{n=0}^{\infty} \alpha^{2n+2}$, which say that $\{\alpha^{2n+2}\}_{n=0}^{\infty}$ is a complete set of non-zero eigenvalues for $K \equiv H_\varphi^* H_\varphi$ and each has multiplicity one. Now, by Beurling's theorem,

$$\ker K = \ker H_\varphi^* H_\varphi = \ker H_\varphi = b\mathbf{H}^2, \text{ where } b \text{ is inner or } b = 0.$$

Since $KT_\varphi = \alpha^2 T_\varphi K$, we see that

$$f \in \ker K \;\Rightarrow\; T_\varphi f \in \ker K$$

So, since $b \in \ker K$, it follows

$$T_\varphi b = b\varphi - H_\varphi b = b\varphi \in \ker K,$$

which means that $b\varphi = bh$ for some $h \in \mathbf{H}^2$. Since $\varphi \notin \mathbf{H}^2$ it follows that $b = 0$ and $\ker K = 0$. Thus 0 is not an eigenvalue. Therefore $\{e_n\}_{n=0}^{\infty}$ is an onthonormal basis for $\mathbf{H}^2$. Remember that $T_\varphi e_n = ||T_\varphi e_n|| e_{n+1}$. So we can see that T_φ is a weighted shift with weights $\{||T_\varphi e_n||\}$. Since

$$\alpha^{2n+2} e_n = K e_n = (1 - T_{\overline{\varphi}} T_\varphi) e_n,$$

we have

$$(1 - \alpha^{2n+2}) e_n = T_{\overline{\varphi}} T_\varphi e_n,$$

so that

$$1 - \alpha^{2n+2} = \langle (1-\alpha^{2n+2}) e_n, e_n \rangle = \langle T_{\overline{\varphi}} T_\varphi e_n,\ e_n \rangle = ||T_\varphi e_n||^2.$$

Thus the weights are $(1-\alpha^{2n+2})^{\frac{1}{2}}$. By Lemma 3.1.9, T_φ is subnormal. Evidently, $\varphi \notin \mathbf{H}^\infty$ and T_φ is not normal since $\text{ran}(\varphi)$ is not contained in a line segment. □

Corollary 3.1.13. *If $\varphi = \psi + \alpha\overline{\psi}$ is as in Theorem 3.1.12, then neither φ nor $\overline{\varphi}$ is bounded type.*

Proof. From Abrahamse's theorem and Theorem 3.1.12. □

We will present a couple of open problems which are related to the subnormality of Toeplitz operators. They are of particular interest in operator theory.

PROBLEM B. *For which $f \in \mathbf{H}^\infty$, is there λ $(0 < \lambda < 1)$ with $T_{f+\lambda\overline{f}}$ subnormal ?*

PROBLEM C. *Suppose ψ is as in Theorem 3.1.12 (i.e., the ellipse map). Are there $g \in \mathbf{H}^\infty$, $g \neq \lambda\psi + c$, such that $T_{\psi+\overline{g}}$ is subnormal ?*

PROBLEM D. *More generally, if $\psi \in \mathbf{H}^\infty$, define*

$$\mathcal{S}(\psi) := \{g \in \mathbf{H}^\infty :\ T_{\psi+\overline{g}} \text{ is subnormal } \}.$$

Describe $\mathcal{S}(\psi)$. For example, for which $\psi \in \mathbf{H}^\infty$, is it balanced?, or is it convex?, or is it weakly closed? What is $\text{ext}\,\mathcal{S}(\psi)$? For which $\psi \in \mathbf{H}^\infty$, is it strictly convex ?, i.e., $\partial\mathcal{S}(\psi) \subset \text{ext}\,\mathcal{S}(\psi)$?

In general, $\mathcal{S}(\psi)$ is not convex. In the below (Theorem 3.2.14), we will show that if ψ is as in Theorem 3.1.12 then $\{\lambda :\ T_{\psi+\lambda\overline{\psi}} \text{ is subnormal}\}$ is a non-convex set.

C. Cowen gave an interesting remark with no demonstration in [Cow3]: *If T_φ is subnormal then $\mathcal{E}(\varphi) = \{\lambda\}$ with $|\lambda| < 1$.* However we were unable to decide whether or not it is true. By comparison, if T_φ is normal then $\mathcal{E}(\varphi) = \{e^{i\theta}\}$.

PROBLEM E. *Is the above Cowen's remark true ? That is, if T_φ is subnormal, does it follow that $\mathcal{E}(\varphi) = \{\lambda\}$ with $|\lambda| < 1$?*

If the answer to Problem E is affirmative, i.e., the Cowen's remark is true then for $\varphi = \overline{g} + f$,

$$T_\varphi \text{ is subnormal } \Longrightarrow \overline{g} - \lambda\overline{f} \in \mathbf{H}^2 \text{ with } |\lambda| < 1 \Longrightarrow g = \overline{\lambda} f + c \ (c \text{ a constant}),$$

which says that the answer to Problem C is negative.

When ψ is as in Theorem 3.1.12, we examine the question: For which λ, is $T_{\psi+\lambda\psi}$ subnormal ? We then have:

Theorem 3.1.14. *Let $\lambda \in \mathbb{C}$ and $0 < \alpha < 1$. Let ψ be the conformal map of the disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ passing through $\pm(1-\alpha)$. For $\varphi = \psi + \lambda\overline{\psi}$, T_φ is subnormal if and only if $\lambda = \alpha$ or $\lambda = \frac{\alpha^k e^{i\theta} + \alpha}{1+\alpha^{k+1}e^{i\theta}}$ $(-\pi < \theta \le \pi)$.*

To prove Theorem 3.1.14, we need an auxiliary lemma:

Proposition 3.1.15. *Let T be the weighted shift with weights*

$$w_n^2 = \sum_{j=0}^{n} \alpha^{2j}.$$

Then $T + \mu T^*$ is subnormal if and only if $\mu = 0$ or $|\mu| = \alpha^k$ $(k = 0, 1, 2, \cdots)$.

Proof. See [CoL]. □

Proof of Theorem 3.1.14. By Theorem 3.1.12, $T_{\psi+\alpha\overline{\psi}} \cong (1-\alpha^2)^{\frac{3}{2}} T$, where T is a weighted shift of Proposition 3.1.15. Thus $T_\psi \cong (1-\alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so

$$T_\varphi = T_\psi + \lambda T_\psi^* \cong (1-\alpha^2)^{\frac{1}{2}}(1-\lambda\alpha)\left(T + \frac{\lambda-\alpha}{1-\lambda\alpha}T^*\right).$$

Applying Proposition 3.1.15 with $\frac{\lambda-\alpha}{1-\lambda\alpha}$ in place of μ gives that for $k = 0, 1, 2, \cdots$,

$$\begin{aligned}
\left|\frac{\lambda-\alpha}{1-\lambda\alpha}\right| = \alpha^k &\iff \frac{\lambda-\alpha}{1-\lambda\alpha} = \alpha^k e^{i\theta} \\
&\iff \lambda - \alpha = \alpha^k e^{i\theta} - \lambda\alpha^{k+1}e^{i\theta} \\
&\iff \lambda(1+\alpha^{k+1}e^{i\theta}) = \alpha + \alpha^k e^{i\theta} \\
&\iff \lambda = \frac{\alpha + \alpha^k e^{i\theta}}{1+\alpha^{k+1}e^{i\theta}} \ (-\pi < \theta \le \pi)
\end{aligned}$$

□

However we find that, surprisingly, some analytic Toeplitz operators are unitarily equivalent to some non-analytic Toeplitz operators. So C. Cowen noted that subnormality of Toeplitz operators may not be the *wrong* question to be studying.

Example 3.1.16. *Let ψ be the ellipse map as in the example of Cowen and Long. Then*

$$T_\psi \cong T_\varphi \text{ with } \varphi = \frac{i\,e^{-\frac{i\theta}{2}}(1+\alpha^2 e^{i\theta})}{1-\alpha^2}\left(\psi \;+\; \frac{\alpha e^{i\theta}+\alpha}{1+\alpha^2 e^{i\theta}}\;\overline{\psi}\right) \quad (-\pi < \theta \le \pi)$$

Proof. Note that

$$T \cong e^{\frac{i\theta}{2}} T \quad \text{and} \quad T + \lambda T^* \cong e^{\frac{i\theta}{2}} T + \lambda e^{-\frac{i\theta}{2}} T^*.$$

Thus we have

$$\begin{aligned}
T_\psi &\cong (1-\alpha^2)^{\frac{1}{2}}(T-\alpha T^*)\\
&\cong (1-\alpha^2)^{\frac{1}{2}} i(T+\alpha T^*)\\
&\cong (1-\alpha^2)^{\frac{1}{2}} ie^{-\frac{i\theta}{2}}(T+\alpha e^{i\theta}T^*)\\
&\cong (1-\alpha^2)^{-1} ie^{-\frac{i\theta}{2}}\left(T_\psi + \alpha T_{\overline{\psi}} + \alpha e^{i\theta}(T_{\overline{\psi}}+\alpha T_\psi)\right)\\
&\cong (1-\alpha^2)^{-1} i\,e^{-\frac{i\theta}{2}} T_{(1+\alpha^2 e^{i\theta})\psi + \alpha(1+e^{i\theta})\overline{\psi}} \quad (-\pi < \theta < \pi)\\
&\cong \frac{i\,e^{-\frac{i\theta}{2}}(1+\alpha^2 e^{i\theta})}{1-\alpha^2} T_{\psi + \frac{\alpha e^{i\theta}+\alpha}{1+\alpha^2 e^{i\theta}}\overline{\psi}} \quad (-\pi < \theta \le \pi).
\end{aligned}$$

□

PROBLEM F. *Let ψ be the ellipse map as in the example of Cowen and Long. Is $T_{\psi+\alpha\overline{\psi}} \cong T_\zeta$ for some $\zeta \in \mathbf{H}^\infty$?*

If the answer to Problem F would affirmative then we could say that Halmos's Problem 5 remains still open. In this case we have a reformulation of Halmos's Problem 5:

If T_φ is a non-normal subnormal Toeplitz operator, does it follow that

$$T_\varphi \cong T_\psi \quad \textit{for some } \psi \in \mathbf{H}^\infty\,?$$

(*Answer* (2012 Updated)): Problem F was answered in the negative in : R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. **263(8)**(2012), 2333-2354.

3.2 Weak Subnormality

Now it seems to be interesting to understand the gap between k-hyponormality and subnormality for Toeplitz operators. As a candidate for the first question in this line we posed the following ([CuL1]):

Question A. Is every 2-hyponormal Toeplitz operator subnormal?

In [CuL1], the following was shown:

Theorem 3.2.1 ([CuL1]). *Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.*

It is well known ([Cu1]) that there is a gap between hyponormality and 2–hyponormality for weighted shifts. Theorem 3.2.1 also shows that there is a big gap between hyponormality and 2-hyponormality for Toeplitz operators. For example, if

$$\varphi(z) = \sum_{n=-m}^{N} a_n z^n \quad (m < N)$$

is such that T_φ is hyponormal then by Theorem 3.2.1, T_φ is never 2-hyponormal because T_φ is neither analytic nor normal (recall that if $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is such that T_φ is normal then $m = N$ (cf. [FL1])).

We can extend Theorem 3.2.1 First of all we observe:

Proposition 3.2.2 ([CuL2]). *If $T \in \mathcal{L}(\mathcal{H})$ is 2-hyponormal then*

$$T\big(ker[T^*, T]\big) \subseteq ker[T^*, T]. \tag{3.2.2.1}$$

Proof. Suppose that $[T^*, T]f = 0$. Since T is 2-hyponormal, it follows that (cf. [CMX, Lemma 1.4])

$$|\langle [T^{*2}, T]g, f\rangle|^2 \le \langle [T^*, T]f, f\rangle \langle [T^{*2}, T^2]g, g\rangle \quad \text{for all } g \in \mathcal{H}.$$

By assumption, we have that for all $g \in \mathcal{H}$, $0 = \langle [T^{*2}, T]g, f\rangle = \langle g, [T^{*2}, T]^* f\rangle$, so that $[T^{*2}, T]^* f = 0$, i.e., $T^*T^2 f = T^2 T^* f$. Therefore,

$$[T^*, T]Tf = (T^*T^2 - TT^*T)f = (T^2T^* - TT^*T)f = T[T^*, T]f = 0,$$

which proves (3.2.2.1). □

Corollary 3.2.3. *If T_φ is 2-hyponormal and if φ or $\bar{\varphi}$ is of bounded type then T_φ is normal or analytic, so that T_φ is subnormal.*

Proof. This follows at once from Abrahamse's theorem and Proposition 3.2.2. □

Corollary 3.2.4. *If T_φ is a 2-hyponormal operator such that $\mathcal{E}(\varphi)$ contains at least two elements then T_φ is normal or analytic, so that T_φ is subnormal.*

Proof. This follows from Corollary 3.2.3 and the fact ([NaT, Proposition 8]) that if $\mathcal{E}(\varphi)$ contains at least two elements then φ is of bounded type. □

From Corollaries 3.2.3 and 3.2.4, we can see that if T_φ is 2-hyponormal but not subnormal then φ is not of bounded type and $\mathcal{E}(\varphi)$ consists of exactly one element.

For a strategy to answer Question A we will introduce the notion of "weak subnormality," which was introduced by R. Curto and W.Y. Lee [CuL2]. Recall that the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases} \tag{3.2.4.1}$$

We now introduce:

Definition 3.2.5 ([CuL2])**.** An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (2.4.1) hold: $[T^*, T] = AA^*$ and $A^*T = BA^*$. The operator $\widehat{T}$ is said to be a *partially normal extension* of T.

Clearly,

$$\text{subnormal} \implies \text{weakly subnormal} \implies \text{hyponormal}. \tag{3.2.5.1}$$

The converses of both implications in (3.2.5.1) are not true in general. Moreover, we can easily see that the following statements are equivalent for $T \in \mathcal{L}(\mathcal{H})$:

(a) T is weakly subnormal;

(b) There is an extension $\widehat{T}$ of T such that $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ for all $f \in \mathcal{H}$;

(c) There is an extension $\widehat{T}$ of T such that $\mathcal{H} \subseteq \ker\,[\widehat{T}^*, \widehat{T}]$.

Weakly subnormal operators possess the following invariance properties:

(i) (Unitary equivalence) if T is weakly subnormal with a partially normal extension $\left(\begin{smallmatrix} T & A \\ 0 & B \end{smallmatrix}\right)$ then for every unitary U, $\left(\begin{smallmatrix} U^*TU & U^*A \\ 0 & B \end{smallmatrix}\right)$ $\left(= \left(\begin{smallmatrix} U^* & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} T & A \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} U & 0 \\ 0 & I \end{smallmatrix}\right)\right)$ is a partially normal extension of U^*TU, i.e., U^*TU is also weakly subnormal.

(ii) (Translation) if $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal then $T - \lambda$ is also weakly subnormal for every $\lambda \in \mathbb{C}$: indeed if T has a partially normal extension $\widehat{T}$ then $\widehat{T - \lambda} := \widehat{T} - \lambda$ satisfies the properties in Definition 3.2.5.

(iii) (Restriction) if $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal and if $\mathfrak{M} \in \operatorname{Lat} T$ then $T|_{\mathfrak{M}}$ is also weakly subnormal because for a partially normal extension $\widehat{T}$ of T, $\widehat{T|_{\mathfrak{M}}} := \widehat{T}$ still satisfies the required properties.

How does one find partially normal extensions of weakly subnormal operators ? Since weakly subnormal operators are hyponormal, one possible solution of the equation $AA^* = [T^*, T]$ is $A := [T^*, T]^{\frac{1}{2}}$. Indeed this is the case.

Theorem 3.2.6 ([CuL2])**.** *If $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal then T has a partially normal extension $\widehat{T}$ on $\mathcal{K}$ of the form*

$$\widehat{T} = \begin{pmatrix} T & [T^*, T]^{\frac{1}{2}} \\ 0 & B \end{pmatrix} \quad \text{on } \mathcal{K} := \mathcal{H} \oplus \mathcal{H}. \tag{3.2.6.1}$$

The proof of Theorem 3.2.6 will make use of the following elementary fact.

Lemma 3.2.7. *If T is weakly subnormal then*

$$T(ker[T^*,T]) \subseteq ker[T^*,T].$$

Proof. By definition, there exist operators A and B such that $[T^*,T] = AA^*$ and $A^*T = BA^*$. If $[T^*,T]f = 0$ then $AA^*f = 0$ and hence $A^*f = 0$. Therefore

$$[T^*,T]Tf = AA^*Tf = ABA^*f = 0,$$

as desired. □

Definition 3.2.8. Let T be a weakly subnormal operator on $\mathcal{H}$ and let $\widehat{T}$ be a partially normal extension of T on $\mathcal{K}$. We shall say that $\widehat{T}$ is a *minimal partially normal extension* of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. We write $\widehat{T} :=$ m.p.n.e.(T).

Lemma 3.2.9. *Let T be a weakly subnormal operator on $\mathcal{H}$ and let $\widehat{T}$ be a partially normal extension of T on $\mathcal{K}$. Then $\widehat{T} = m.p.n.e.(T)$ if and only if*

$$\mathcal{K} = \bigvee \big\{ \widehat{T}^{*n} h : \; h \in \mathcal{H}, \; n = 0, 1 \big\}. \tag{3.2.9.1}$$

Proof. See [CuL2]. □

It is well known (cf. [Con2, Proposition II.2.4]) that if T is a subnormal operator on $\mathcal{H}$ and N is a normal extension of T then N is a minimal normal extension of T if and only if

$$\mathcal{K} = \bigvee \{ \widehat{T}^{*n} h : \; h \in \mathcal{H}, \; n \geq 0 \}.$$

Thus if T is a subnormal operator then T may have a partially normal extension different from a normal extension. For, consider the unilateral (unweighted) shift U_+ acting on $\ell^2(\mathbb{Z}_+)$. Then m.n.e. $(U_+) = U$, the bilateral shift acting on $\ell^2(\mathbb{Z})$, with orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$. It is easy to verify that m.p.n.e. $(U_+) = U|_{\mathcal{L}}$, where $\mathcal{L} := < e_{-1} > \oplus \, \ell^2(\mathbb{Z}_+)$.

Theorem 3.2.10. *Let $T \in \mathcal{L}(\mathcal{H})$.*

(i) *If T is 2-hyponormal then $[T^*,T]^{\frac{1}{2}} T [T^*,T]^{-\frac{1}{2}}|_{\mathrm{Ran}[T^*,T]}$ is bounded;*

(ii) *T is $(k+1)$-hyponormal if and only if T is weakly subnormal and $\widehat{T} :=$ m.p.n.e.(T) is k-hyponormal.*

Proof. See [CJP, Theorems 2.7 and 3.2]. □

In 1966, Stampfli [Sta] explicitly exhibited for a subnormal weighted shift A_0 its minimal normal extension

$$N := \begin{pmatrix} A_0 & B_1 & & 0 \\ & A_1 & B_2 & \\ & & A_2 & \ddots \\ 0 & & & \ddots \end{pmatrix}, \tag{3.2.10.1}$$

where A_n is a weighted shift with weights $\{a_0^{(n)}, a_1^{(n)}, \cdots\}$, $B_n := \mathrm{diag}\{b_0^{(n)}, b_1^{(n)}, \cdots\}$, and these entries satisfy:

(I) $(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2 \geq 0$ ($b_j^{(0)} = 0$ for all j);

(II) $b_j^{(n)} = 0 \Longrightarrow b_{j+1}^{(n)} = 0$;

(III) there exists a constant M such that $|a_j^{(n)}| \leq M$ and $|b_j^{(n)}| \leq M$ for $n = 0, 1, \cdots$ and $j = 0, 1, \cdots$, where

$$b_j^{(n+1)} := [(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2]^{\frac{1}{2}} \quad \text{and} \quad a_j^{(n+1)} := a_j^{(n)} \frac{b_{j+1}^{(n+1)}}{b_j^{(n+1)}}$$

(if $b_{j_0}^{(n)} = 0$, then $a_{j_0}^{(n)}$ is taken to be 0).

We will now discuss analogues of the preceding results for k-hyponormal operators. Our criterion on k-hyponormality follows:

Theorem 3.2.11. *An operator $A_0 \in \mathcal{L}(\mathcal{H}_0)$ is k-hyponormal if and only if the following three conditions hold for all n such that $0 \leq n \leq k-1$:*

(I_n) $D_n \geq 0$;

(II_n) $A_{n-1}(\mathrm{Ker}\, D_{n-1}) \subseteq \mathrm{Ker}\, D_{n-1}$ $(n \geq 1)$;

(III_n) $D_{n-1}^{\frac{1}{2}} A_{n-1} D_{n-1}^{-\frac{1}{2}}|_{\mathrm{Ran}\,(D_{n-1})}$ $(n \geq 1)$ *is bounded,*

where

$$D_0 := [A_0^*, A_0], \quad D_{n+1} := D_n|_{\mathcal{H}_{n+1}} + [A_{n+1}^*, A_{n+1}], \quad \mathcal{H}_{n+1} := \overline{\mathrm{Ran}\,(D_n)}$$

and A_{n+1} denotes the bounded extension of $D_n^{\frac{1}{2}} A_n D_n^{-\frac{1}{2}}$ to $\overline{\mathrm{Ran}\,(D_n)}(= \mathcal{H}_{n+1})$ from $\mathrm{Ran}\,(D_n)$.

Proof. Suppose A_0 is k-hyponormal. We now use induction on k. If $k = 2$ then A_0 is 2-hyponormal, and so $D_0 := [A_0^*, A_0] \geq 0$. By Theorem 3.2.10 (i), $D_0^{\frac{1}{2}} A_0 D_0^{-\frac{1}{2}}|_{\mathrm{Ran}\,(D_0)}$ is bounded. Let A_1 be the bounded extension of $D_0^{\frac{1}{2}} A_0 D_0^{-\frac{1}{2}}$ from $\mathrm{Ran}\,(D_0)$ to $\mathcal{H}_1 := \overline{\mathrm{Ran}\,(D_0)}$ and $D_1 := D_0|_{\mathcal{H}_1} + [A_1^*, A_1]$. Writing $\widehat{A_0} := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} \\ 0 & A_1 \end{pmatrix}$, we have $\widehat{A_0} = \text{m.p.n.e.}\,(A_0)$, which is hyponormal by Theorem 3.2.10(ii). Thus

$$[\widehat{A_0}^*, \widehat{A_0}] = \begin{pmatrix} 0 & 0 \\ 0 & D_0|_{\mathcal{H}_1} + [A_1^*, A_1] \end{pmatrix} \geq 0.$$

and hence $D_1 \geq 0$. Also by [CuL2, Lemma 2.2], $A_0(\mathrm{Ker}\, D_0) \subseteq \mathrm{Ker}\, D_0$ whenever A_0 is 2-hyponormal. Thus (I_n), (II_n), and (III_n) hold for $n = 0, 1$. Assume now that if A_0 is k-hyponormal then (I_n),(II_n) and (III_n) hold for all $0 \leq n \leq k-1$. Suppose A_0 is $(k+1)$-hyponormal. We must show that (I_n),(II_n) and (III_n) hold for $n = k$. Define

$$S := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{k-2}^{\frac{1}{2}} \\ 0 & & & & A_{k-1} \end{pmatrix} : \bigoplus_{i=0}^{k-1} \mathcal{H}_i \longrightarrow \bigoplus_{i=0}^{k-1} \mathcal{H}_i.$$

By our inductive assumption, $D_{k-1} \geq 0$. Writing $\widehat{T}^{(n)} := \text{m.p.n.e.}(\widehat{T}^{(n-1)})$ when it exists, we can see by our assumption that $S = \widehat{A_0}^{(k-1)}$: indeed, if

$$S_l := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{l-2}^{\frac{1}{2}} \\ 0 & & & & A_{l-1} \end{pmatrix}$$

then since by assumption $[S_l^*, S_l] = 0 \oplus D_l$ and $A_l = D_{l-1}^{\frac{1}{2}} A_{l-1} D_{l-1}^{-\frac{1}{2}}|_{\text{Ran}\,(D_{l-1})}$, it follows that S_l is the minimal partially normal extension of S_{l-1} $(1 \leq l \leq k-1)$. But since by our assumption A_0 is $(k+1)$-hyponormal, it follows from Lemma 3.2.10(ii) that S is 2-hyponormal. Thus by Theorem 3.2.10(i), $[S^*, S]^{\frac{1}{2}} S [S^*, S]^{-\frac{1}{2}}|_{\text{Ran}\,([S^*,S])}$ is bounded, which says that $D_{k-1}^{\frac{1}{2}} A_{k-1} D_{k-1}^{-\frac{1}{2}}|_{\text{Ran}\,(D_{k-1})}$ is bounded, proving (III$_n$) for $n = k$. Observe that A_k, $\mathcal{H}_k$ and D_k are well-defined. Writing $\widehat{S} := \begin{pmatrix} S & D_{k-1}^{\frac{1}{2}} \\ 0 & A_k \end{pmatrix}$, we can see that $\widehat{S} = \text{m.p.n.e.}(S)$, which is hyponormal, again by Theorem 3.2.10(ii). Thus, since $[\widehat{S}^*, \widehat{S}] = \begin{pmatrix} 0 & 0 \\ 0 & D_k \end{pmatrix} \geq 0$, we have $D_k \geq 0$, proving (I$_n$) for $n = k$. On the other hand, since S is 2-hyponormal, it follows that $S(\text{Ker}[S^*, S]) \subseteq \text{Ker}[S^*, S]$. Since $[S^*, S] = \begin{pmatrix} 0 & 0 \\ 0 & D_{k-1} \end{pmatrix}$, we have $\text{Ker}\,[S^*, S] = \bigoplus_{i=0}^{k-2} \mathcal{H}_i \bigoplus \text{Ker}\,(D_{k-1})$. Thus, since

$$\begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{k-2}^{\frac{1}{2}} \\ 0 & & & & A_{k-1} \end{pmatrix} \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{k-2} \\ \text{Ker}\,(D_{k-1}) \end{pmatrix} \subseteq \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{k-2} \\ \text{Ker}\,(D_{k-1}) \end{pmatrix},$$

we must have that $A_{k-1}(\text{Ker}\,(D_{k-1})) \subseteq \text{Ker}\,(D_{k-1})$, proving (II$_n$) for $n = k$. This proves the necessity condition.

Toward sufficiency, suppose that conditions (I$_n$), (II$_n$) and (III$_n$) hold for all n such that $0 \leq n \leq k-1$. Define

$$S_n := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{n-2}^{\frac{1}{2}} \\ 0 & & & & A_{n-1} \end{pmatrix} \quad (1 \leq n \leq k-1).$$

Then S_{k-2} is weakly subnormal and $S_{k-1} = \text{m.p.n.e.}\,(S_{k-2})$. Since, by assumption, $D_{k-1} \geq 0$, we have $[S_{k-1}^*, S_{k-1}] = \begin{pmatrix} 0 & 0 \\ 0 & D_{k-1} \end{pmatrix} \geq 0$. It thus follows from Theorem 3.2.10(ii) that S_{k-2} is 2-hyponormal. Note that $S_n = \text{m.p.n.e.}\,(S_{n-1})$ for $n = 1, \cdots, k-1$ $(S_0 := A_0)$. Thus, again by Theorem 3.2.10(ii), S_{k-3} is 3-hyponormal. Now repeating this argument, we can conclude that $S_0 \equiv A_0$ is k-hyponormal. This completes the proof. □

Corollary 3.2.12. *An operator $A_0 \in \mathcal{L}(\mathcal{H}_0)$ is subnormal if and only if the conditions* (I_n), (II_n), *and* (III_n) *hold for all $n \geq 0$. In this case, the minimal normal extension N of A_0 is given by*

$$N = \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & \\ & & A_2 & \ddots \\ 0 & & & \ddots \end{pmatrix} : \bigoplus_{i=0}^{\infty} \mathcal{H}_i \to \bigoplus_{i=0}^{\infty} \mathcal{H}_i.$$

3.3 Gaps between k-Hyponormality and Subnormality

We find gaps between subnormality and k-hyponormality for Toeplitz operators.

Theorem 3.3.1 ([Gu2],[CLL]). *Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi = \psi + \lambda\bar{\psi}$ and let T_φ be the corresponding Toeplitz operator on H^2. Then T_φ is k-hyponormal if and only if λ is in the circle $\left|z - \frac{\alpha(1-\alpha^{2j})}{1-\alpha^{2j+2}}\right| = \frac{\alpha^j(1-\alpha^2)}{1-\alpha^{2j+2}}$ for $j = 0, 1, \cdots, k-2$ or in the closed disk $\left|z - \frac{\alpha(1-\alpha^{2(k-1)})}{1-\alpha^{2k}}\right| \le \frac{\alpha^{k-1}(1-\alpha^2)}{1-\alpha^{2k}}$.*

For $0 < \alpha < 1$, let $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$, where (cf. [Cow2, Proposition 9])

$$\beta_n := (\sum_{j=0}^{n} \alpha^{2j})^{\frac{1}{2}} \quad \text{for } n = 0, 1, \cdots. \tag{3.3.1.1}$$

Let D be the diagonal operator, $D = \operatorname{diag}(\alpha^n)$, and let $S_\lambda \equiv T + \lambda T^*$ ($\lambda \in \mathbb{C}$). Then we have that

$$[T^*, T] = D^2 = \operatorname{diag}(\alpha^{2n}) \quad \text{and} \quad [S_\lambda^*, S_\lambda] = (1-|\lambda|^2)[T^*, T] = (1-|\lambda|^2)D^2.$$

Define

$$A_l := \alpha^l T + \frac{\lambda}{\alpha^l} T^* \quad (l = 0, \pm 1, \pm 2, \cdots).$$

It follows that $A_0 = S_\lambda$ and

$$DA_l = A_{l+1}D \quad \text{and} \quad A_l^* D = DA_{l+1}^* \quad (l = 0, \pm 1, \pm 2, \cdots). \tag{3.3.1.2}$$

Theorem 3.3.2. *Let $0 < \alpha < 1$ and $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$, where*

$$\beta_n = (\sum_{j=0}^{n} \alpha^{2j})^{\frac{1}{2}} \quad \textit{for } n = 0, 1, \cdots.$$

Then $A_0 := T + \lambda T^*$ is k-hyponormal if and only if $|\lambda| \le \alpha^{k-1}$ or $|\lambda| = \alpha^j$ for some $j = 0, 1, \cdots, k-2$.

Proof. Observe that

$$\begin{aligned} [A_l^*, A_l] &= [\alpha^l T^* + \tfrac{\bar{\lambda}}{\alpha^l} T,\ \alpha^l T + \tfrac{\lambda}{\alpha^l} T^*] \\ &= \alpha^{2l}[T^*, T] - \tfrac{|\lambda|^2}{\alpha^{2l}}[T^*, T] = \left(\alpha^{2l} - \tfrac{|\lambda|^2}{\alpha^{2l}}\right) D^2. \end{aligned} \tag{3.3.2.1}$$

Since $\operatorname{Ker} D = \{0\}$ and $DA_n = A_{n+1}D$, it follows that $\mathcal{H}_n = \mathcal{H}$ for all n; if we use A_l for the operator A_n in Theorem 3.2.11 then we have, by (3.3.2.1) and the definition of D_j, that

$$\begin{aligned} D_j &= D_{j-1} + [A_j^*, A_j] = D_{j-2} + [A_{j-1}^*, A_{j-1}] + [A_j^*, A_j] = \cdots \\ &= [A_0^*, A_0] + [A_1^*, A_1] + \cdots + [A_j^*, A_j] = (1-|\lambda|^2)D^2 + \cdots + \left(\alpha^{2j} - \frac{|\lambda|^2}{\alpha^{2j}}\right) D^2 \\ &= \left(\frac{1-\alpha^{2(j+1)}}{1-\alpha^2}\right)\left(1 - \frac{|\lambda|^2}{\alpha^{2j}}\right) D^2. \end{aligned}$$

By Theorem 3.2.11, A_0 is k-hyponormal if and only if $D_{k-1} \geq 0$ or $D_j = 0$ for some j such that $0 \leq j \leq k-2$ (in this case A_0 is subnormal). Note that $D_j = 0$ if and only if $|\lambda| = \alpha^j$. On the other hand, if $D_j > 0$ for $j = 0, 1, \cdots, k-2$, then

$$D_{k-1} = \left(\frac{1-\alpha^{2k}}{1-\alpha^2}\right)\left(1 - \frac{|\lambda|^2}{\alpha^{2(k-1)}}\right) D^2 \geq 0$$

if and only if $|\lambda| \leq \alpha^{k-1}$. Therefore A_0 is k-hyponormal if and only if $|\lambda| \leq \alpha^{k-1}$ or $|\lambda| = \alpha^j$ for some j, $j = 0, 1, \cdots, k-2$. □

We are ready for:

Proof. of Theorem 3.3.1 It was shown in [CoL] that $T_{\psi+\alpha\bar{\psi}}$ is unitarily equivalent to $(1-\alpha^2)^{\frac{3}{2}}T$, where T is the weighted shift in Theorem 3.3.2. Thus T_ψ is unitarily equivalent to $(1-\alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so T_φ is unitarily equivalent to

$$(1-\alpha^2)^{\frac{1}{2}}(1-\lambda\,\alpha)(T + \frac{\lambda-\alpha}{1-\lambda\,\alpha}T^*) \quad \text{(cf. [Cow1, Theorem 2.4]).}$$

Applying Theorem 3.3.2 with $\frac{\lambda-\alpha}{1-\lambda\,\alpha}$ in place of λ, we have that for $k = 0, 1, 2, \cdots$,

$$\begin{aligned}
\left|\frac{\lambda-\alpha}{1-\lambda\,\alpha}\right| \leq \alpha^k &\Longleftrightarrow |\lambda-\alpha|^2 \leq \alpha^{2k}|1-\lambda\,\alpha|^2 \\
&\Longleftrightarrow |\lambda|^2 - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}(\lambda+\bar{\lambda}) + \frac{\alpha^2-\alpha^{2k}}{1-\alpha^{2k+2}} \leq 0 \\
&\Longleftrightarrow \left|\lambda - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}\right| \leq \frac{\alpha^k(1-\alpha^2)}{1-\alpha^{2k+2}}.
\end{aligned}$$

This completes the proof. □

3.4 Miscellany

From Corollary 3.2.3 we can see that if T_φ is a 2-hyponormal operator such that φ or $\bar{\varphi}$ is of bounded type then T_φ has a nontrivial invariant subspace. The following question is naturally raised:

Problem G. Does every 2-hyponormal Toeplitz operator have a nontrivial invariant subspace? More generally, does every 2-hyponormal operator have a nontrivial invariant subspace?

It is well known ([Bro]) that if T is a hyponormal operator such that $R(\sigma(T)) \neq C(\sigma(T))$ then T has a nontrivial invariant subspace. But it remains still open whether every hyponormal operator with $R(\sigma(T)) = C(\sigma(T))$ (i.e., a *thin* spectrum) has a nontrivial invariant subspace. Recall that $T \in \mathcal{L}(\mathcal{H})$ is called a *von-Neumann operator* if $\sigma(T)$ is a spectral set for T, or equivalently, $f(T)$ is normaloid (i.e., norm equals spectral radius) for every rational function f with poles off $\sigma(T)$. Recently, B. Prunaru [Pru] has proved that polynomially hyponormal operators have nontrivial invariant subspaces. It was also known ([Ag]) that von-Neumann operators enjoy the same property. The following is a sub-question of Problem G.

Problem H. Is every 2-hyponormal operator with thin spectrum a von-Neumann operator?

Although the existence of a non-subnormal polynomially hyponormal weighted shift was established in [CP1] and [CP2], it is still an open question whether the implication "polynomially hyponormal $\Rightarrow$ subnormal" can be disproved with a Toeplitz operator.

Problem I. Does there exist a Toeplitz operator which is polynomially hyponormal but not subnormal?

In [CuL2] it was shown that every pure 2-hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. McCarthy and Yang [McCYa] classified all rationally cyclic subnormal operators with finite rank self-commutators. However it remains still open what are the pure subnormal operators with finite rank self-commutators.

Now the following question comes up at once:

Problem J. If T_φ is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic?

For affirmativeness to Problem J we shall give a partial answer. To do this we recall Theorem 15 in [NaT] which states that if T_φ is subnormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic. But from a careful examination of the proof of the theorem we can see that its proof uses subnormality assumption only for the fact that $\ker [T_\varphi^*, T_\varphi]$ is invariant under T_φ. Thus in view of Proposition 3.2.2, the theorem is still valid for "2–hyponormal" in place of "subnormal". We thus have:

Theorem 3.4.1. *If T_φ is 2–hyponormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic.*

We now give a partial answer to Problem J.

Theorem 3.4.2. *Suppose $\log|\varphi|$ is not integrable. If T_φ is a 2–hyponormal operator with nonzero finite rank self-commutator then T_φ is analytic.*

Proof. If T_φ is hyponormal such that $\log|\varphi|$ is not integrable then by an argument of [NaT, Theorem 4], $\varphi = q\bar{\varphi}$ for some inner function q. Also if T_φ has a finite rank self-commutator then by [NaT, Theorem 10], there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$. If $q \neq b$, so that $\mathcal{E}(\varphi)$ contains at least two elements, then by Corollary 3.2.4, T_φ is normal or analytic. If instead $q = b$ then by Theorem 3.4.1, T_φ is also normal or analytic. $\square$

Theorem 3.4.2 reduces Problem J to the class of Toeplitz operators such that $\log|\varphi|$ is integrable. If $\log|\varphi|$ is integrable then there exists an outer function e such that $|\varphi| = |e|$. Thus we may write $\varphi = ue$, where u is a unimodular function. Since by the Douglas-Rudin theorem (cf. [Ga, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if $\log|\varphi|$ is integrable then φ can be approximated by functions of bounded type. Therefore if we could obtain such a sequence ψ_n converging to φ such that T_{ψ_n} is 2–hyponormal with finite rank self-commutator for each n, then we would answer Problem J affirmatively. On the other hand, if T_φ attains its norm then by a result of Brown and Douglas [BD], φ is of the form $\varphi = \lambda\frac{\psi}{\theta}$ with $\lambda > 0$, ψ and θ inner. Thus φ is of bounded type. Therefore by Corollary 3.2.4, if T_φ is 2–hyponormal and attains its norm then T_φ is normal or analytic. However we were not able to decide that if T_φ is a 2–hyponormal operator with finite rank self-commutator then T_φ attains its norm.

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