JOINT HYPONORMALITY OF TOEPLITZ PAIRS

Raúl E. Curto
Woo Young Lee

CONTENTS

ABSTRACT

We characterize hyponormal "trigonometric" Toeplitz pairs, which are pairs of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$, with trigonometric polynomial symbols. Suppose φ and ψ are trigonometric polynomials of analytic degrees N and n ($n \le N$) and co-analytic degrees m and l, respectively. Then the hyponormality of the Toeplitz pair $\mathbf{T} = (T_\psi, T_\varphi)$ can be described as follows:

(i) If $l = m = 0$, then $\mathbf{T}$ is necessarily subnormal.
(ii) If $l = 0$ and $m \neq 0$, then $\mathbf{T}$ is hyponormal if and only if T_φ is hyponormal and $n \le N - m$.
(iii) If $l \neq 0$ and $m = 0$, then $\mathbf{T}$ is not hyponormal.
(iv) If $l \neq 0$ and $m \neq 0$, then $\mathbf{T}$ is hyponormal if and only if T_φ is hyponormal and $\varphi - c\,\psi = \sum_{j=0}^{N-m} d_j z^j$ for some $c, d_0, \dots, d_{N-m} \in \mathbb{C}$.

In the cases where $\mathbf{T}$ is hyponormal, the rank of the self-commutator of $\mathbf{T}$ equals the rank of the self-commutator of T_φ. Moreover weak hyponormality and hyponormality for $\mathbf{T}$ are equivalent properties. This characterization can be extended to trigonometric Toeplitz n-tuples. We also discuss the gap between 2-hyponormality and subnormality for Toeplitz operators, and we give applications to flatness of hyponormal Toeplitz pairs, Toeplitz extensions of positive moment matrices, and hyponormality of *single* Toeplitz operators.

1991 *Mathematics Subject Classification.* Primary 47B20, 47B35, 47A63; Secondary 47B37, 47B47.

Key words and phrases. Hyponormal, subnormal, jointly hyponormal, weakly hyponormal, k-hyponormal, trigonometric Toeplitz pairs, flatness, Toeplitz extensions, moment matrices.

For Inés, Carina, Roxanna and Vilsa

and

For Younhwa, Songeun and Kyusang

INTRODUCTION

The Bram-Halmos characterization of subnormality ([**At**], [**Br**], [**Con**]) states that a bounded linear operator T on a Hilbert space is subnormal if and only if the following holds:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad \text{for all } k \geq 1. \tag{0.1}$$

Condition (0.1) provides a measure of the gap between hyponormality and subnormality. In fact the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (0.1) for all k. Between those two extremes there exists a whole slew of increasingly stricter conditions, each expressible in terms of the joint hyponormality of the tuples $(I, T, T^2, \dots, T^k)$. In 1988, the notion of "joint hyponormality" (for the general case of n-tuples of operators) was first formally introduced by A. Athavale [**At**]. He conceived joint hyponormality as a notion at least as strong as requiring that the linear span of the operator coordinates consist of hyponormal operators, the latter notion being called weak joint hyponormality.

In the ensuing years, subnormality, joint hyponormality, and weak joint hyponormality were studied by A. Athavale [**At**], J. Conway and W. Szymanski [**CoS**], R. Douglas, V. Paulsen, and K. Yan [**DPY**], R. Douglas and K. Yan [**DY**], D. Farenick and R. McEachin [**FM**], S. McCullough and V. Paulsen [**McCP**], D. Xia [**Xi2**], the first named author, P. Muhly, and J. Xia [**CMX**], [**Cu2**], and others. Joint hyponormality originated from questions about commuting normal extensions of commuting operators, and it has also been considered with an aim at understanding the gap between hyponormality and subnormality for single operators. To date, much of the research on joint hyponormality has dealt with *commuting* tuples of hyponormal operators.

In [**FM**], Farenick and McEachin studied operators that form hyponormal pairs in the presence of the unilateral shift. Since the unilateral shift is a Toeplitz operator on the Hardy space of the unit circle, one can ask whether the results in [**FM**] extend to Toeplitz pairs, that is, pairs whose coordinates are Toeplitz operators

Received by the editor June 30, 1998.

The work of the first author was partially supported by NSF research grant DMS-9401455 and DMS-9800931.

The work of the second author was partially supported by KOSEF through GARC at Seoul National University and research grant 971-0102-010-2.

on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T}$. One of our main results is a complete characterization of hyponormal trigonometric Toeplitz pairs, that is, pairs of Toeplitz operators with trigonometric polynomial symbols. As we will see in Chapter 2, this characterization can be extended to trigonometric Toeplitz n-tuples; in fact, we prove that the hyponormality of trigonometric Toeplitz n-tuples can be reduced to consideration of Toeplitz pairs.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For $A, B \in \mathcal{L}(\mathcal{H})$, we let $[A,B] := AB - BA$; $[A,B]$ is the *commutator* of A and B. Given an n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of operators on $\mathcal{H}$, we let $[\mathbf{T}^*, \mathbf{T}] \in \mathcal{L}(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ denote the *self-commutator* of $\mathbf{T}$, defined by $[\mathbf{T}^*, \mathbf{T}]_{ij} := [T_j^*, T_i]$ $(1 \le i, j \le n)$. (This definition of self-commutator for n-tuples of operators on a Hilbert space was introduced by A. Athavale in [**At**].) For instance, if $n = 2$,

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}.$$

By analogy with the case $n = 1$, we shall say ([**At**], [**CMX**]) that $\mathbf{T}$ is *jointly hyponormal* (or simply, *hyponormal*) if $[\mathbf{T}^*, \mathbf{T}]$ is a positive operator on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and every T_i is a normal operator, and *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, the normality, subnormality or hyponormality of an n-tuple requires as a necessary condition that every coordinate in the tuple be normal, subnormal or hyponormal, respectively. Normality and subnormality require that the coordinates commute, but hyponormality does not. The same is true of *weak hyponormality*, defined by asking that

$$LS(\mathbf{T}) = \left\{ \sum_{j=1}^{n} \alpha_j T_j : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n \right\}$$

consist entirely of hyponormal operators. In general, hyponormal $\Rightarrow$ weakly hyponormal, but the converse does not hold even under the assumption of commutativity ([**At**], [**CMX**]).

Related to this circle of ideas is the notion of k-hyponormality for single operators. For $k \ge 1$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *k-hyponormal* if $(T, T^2, \dots, T^k)$ is hyponormal. The classes of k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([**Cu1**], [**Cu2**], [**CF2**], [**CF3**], [**CMX**], [**DPY**], [**McCP**]).

Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \dots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If P denotes the projection operator $L^2(\mathbb{T}) \to H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operators T_φ and H_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi(g) := (I - P)(\varphi g) \qquad (g \in H^2(\mathbb{T}))$$

are called the *Toeplitz operator* and the *Hankel operator*, respectively, with symbol φ.

For U the unilateral shift on ℓ_2 we let $P_n := [U^{*n}, U^n]$; P_n is a rank-n projection. If φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero, then the nonnegative integers m and N denote the co-analytic and analytic degrees of φ, respectively. Pairs of Toeplitz operators will be called *Toeplitz pairs.* Finally, for an operator $T \in \mathcal{L}(\mathcal{H})$, we let $N(T)$ and $R(T)$ denote the kernel and range of T, respectively.

The organization of the paper is as follows. In Chapter 1 we characterize the hyponormality of Toeplitz pairs with *one* coordinate a Toeplitz operator with *analytic* polynomial symbol, and show that for these pairs hyponormality and weak hyponormality coincide.

In Chapter 2 we characterize the hyponormality of Toeplitz pairs in which *both* coordinates have *trigonometric* polynomial symbols, and we show that the hyponormality of a trigonometric Toeplitz n-tuple can be detected by checking the hyponormality of all of its sub-pairs. We also give an example to show that for non-Toeplitz n-tuples, the hyponormality of each sub-pair is not a sufficient condition. We do this by invoking the k-hyponormality of a unilateral weighted shift considered in [**Cu1**]; this shift is a close relative of the Bergman shift B_+.

In Chapter 3 we consider the following question: *Is every* 2*-hyponormal Toeplitz operator* T_φ *subnormal?* Since in general the pair (T_φ, T_φ^2) is not a Toeplitz pair (because the square of a Toeplitz operator need not be a Toeplitz operator), the 2-hyponormality of T_φ seems to be quite delicate. This question bears resemblance to the Halmos question [**Hal1**]: Is every subnormal Toeplitz operator either normal or analytic? Even though a negative answer to the Halmos question was given by C. Cowen and J. Long [**CoL**], in which a Toeplitz operator unitarily equivalent to a weighted shift was used, it remains still open to characterize subnormal Toeplitz operators T_φ in terms of their symbols φ. Our question seems to be at least as difficult as the Halmos question because, as we shall show in Corollary 3.3, unilateral weighted shifts cannot provide a negative answer. In other words, every 2-hyponormal Toeplitz operator which is unitarily equivalent to a unilateral weighted shift must be subnormal. Nevertheless, we provide an affirmative answer to the above question for *trigonometric* Toeplitz operators (Theorem 3.2): Every 2-hyponormal trigonometric Toeplitz operator is necessarily subnormal.

Chapter 4 contains applications of the results in previous chapters. Firstly, we introduce the notion of flatness for a Toeplitz pair and show that every hyponormal trigonometric Toeplitz pair is necessarily flat. Secondly, we discuss the *Toeplitz extension problem* of positive moment matrices, and then show that a solution of the associated quadratic moment problem exists ([**CF4**]) if and only if the moment matrix admits a Toeplitz extension. Lastly, we give an example which illustrates how the joint hyponormality of Toeplitz pairs can be applied to determine the hyponormality of *single* Toeplitz operators.

Chapter 5 is devoted to some concluding remarks and open problems.

Recall that if (T_1, T_2) is hyponormal, then so is $(T_1 - \lambda_1 I, T_2 - \lambda_2 I)$ for every $\lambda_1, \lambda_2 \in \mathbb{C}$. Thus if $\varphi, \psi \in L^\infty(\mathbb{T})$ have Fourier coefficients $\hat{\varphi}(n), \hat{\psi}(n)$ for every $n \in \mathbb{Z}$, respectively, then the hyponormality of (T_ψ, T_φ) is independent of the particular values of $\hat{\varphi}(0)$ and $\hat{\psi}(0)$. Therefore, throughout this paper, we will assume that the 0-th coefficient $\hat{\varphi}(0)$ of the given symbol φ of a Toeplitz operator is zero.

Acknowledgments. The work of the second author was undertaken during the summer and winter breaks of 1997 at The University of Iowa. He would like to take this opportunity to thank the Department of Mathematics for the hospitality received during his stay.

CHAPTER 1

HYPONORMALITY OF TOEPLITZ PAIRS WITH ONE COORDINATE A TOEPLITZ OPERATOR WITH ANALYTIC POLYNOMIAL SYMBOL

In [**FM**], Farenick and McEachin characterized hyponormality for Toeplitz pairs one of whose coordinates is the unilateral shift. We shall extend this result to arbitrary analytic polynomial symbols. We first give several lemmas that will be used extensively in the sequel.

Lemma 1.1 ([CMX], [Xi1]). *Let* $\mathbf{T} = (T_1, T_2)$ *be a pair of operators on* $\mathcal{H}$*. Then* $\mathbf{T}$ *is hyponormal if and only if*

(i) T_1 *is hyponormal*
(ii) T_2 *is hyponormal*
(iii) $[T_2^*, T_1] = [T_1^*, T_1]^{\frac{1}{2}}\ D\ [T_2^*, T_2]^{\frac{1}{2}}$ *for some contraction* D*, or equivalently,*

$$|([T_2^*, T_1]y,\ x)|^2 \le ([T_1^*, T_1]x,\ x)([T_2^*, T_2]y,\ y) \quad \text{for all } x, y \in \mathcal{H}.$$

Lemma 1.2. *Let* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ *and let* $T = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} : \mathcal{H} \to \mathcal{H}$ *with* $A \ge 0$*,* A *invertible, and* C *self-adjoint. Then* $T \ge 0$ *if and only if* $B^* A^{-1} B \le C$*. More generally, if* $A \ge 0$*, if* C *is self-adjoint, and if* $R(A)$ *is closed (e.g., if* A *is of finite rank) then*

$$T \ge 0 \iff B^* A^{\#} B \le C \quad \text{and} \quad R(B) \subseteq R(A), \tag{1.2.1}$$

where, for an operator S*,* $S^{\#}$ *is the Moore-Penrose inverse of* S*, in the sense that* $SS^{\#}S = S$*,* $S^{\#}SS^{\#} = S^{\#}$*,* $(S^{\#}S)^* = S^{\#}S$*, and* $(SS^{\#})^* = SS^{\#}$*.*

Remark. It is known ([**Har**, Theorem 8.7.1]) that if an operator on a Hilbert space has closed range then it has a Moore-Penrose inverse. In fact, the converse is also true. Moreover the Moore-Penrose inverse is unique whenever it exists, and the Moore-Penrose inverse of a positive operator is also positive. In addition, observe that if the range condition is dropped in (1.2.1), then the backward implication may fail even though $A^{\#}$ exists: for example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $C = (\,1\,)$, then evidently $B^* A^{\#} B = C$, but $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is not positive.

Proof of Lemma 1.2. The first assertion is a well-known fact (cf. [**CMX**]). For the second assertion, observe that $R(\sqrt{A}) = R(A)$. Now suppose $T \ge 0$. Then we have

([**Smu**]) that $C \geq 0$ and $B = \sqrt{A}E\sqrt{C}$ for some contraction $E \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. Thus $R(B) \subseteq R(\sqrt{A}) = R(A)$. On the other hand, we can write

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : R(A) \oplus N(A) \longrightarrow R(A) \oplus N(A),$$

where A_0 is invertible. The Moore-Penrose inverse of A is then $A^{\#} := \left(\begin{smallmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since $R(B) \subseteq R(A)$, B can also be written as

$$B = \begin{pmatrix} B_0 \\ 0 \end{pmatrix} : \mathcal{H}_2 \longrightarrow R(A) \oplus N(A).$$

Thus by the first assertion, we have $B_0^* A_0^{-1} B_0 \leq C$, so $B^* A^{\#} B \leq C$. The argument for the converse is similar. This proves the second assertion. $\square$

An elegant theorem of C. Cowen [**Cow**] characterizes the hyponormality of a Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. We shall employ a variant of Cowen's Theorem that was first proposed by Nakazi and Takahashi in [**NT**].

Theorem 1.3 ([Cow], [NT]). *Suppose that $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and write*

$$\mathcal{E}(\varphi) := \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \leq 1 \quad \textit{and} \quad \varphi - k\bar{\varphi} \in H^\infty(\mathbb{T})\}.$$

Then T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty.

An abstract version of Cowen's criterion has been developed in [**Gu**]. We record here results on the hyponormality of single Toeplitz operators with trigonometric polynomial symbols, which have been developed recently ([**FL1**], [**FL2**]). The stability of the rank of the self-commutator in (iii) appears to be new.

Lemma 1.4. *Suppose that φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$.*

(i) *If T_φ is a hyponormal operator then $m \leq N$ and $|a_{-m}| \leq |a_N|$.*
(ii) *If T_φ is a hyponormal operator then $N - m \leq \operatorname{rank}[T_\varphi^*, T_\varphi] \leq N$.*
(iii) *The hyponormality of T_φ is independent of the particular values of the Fourier coefficients $a_0, a_1, \ldots, a_{N-m}$ of φ. Moreover, for T_φ hyponormal, the rank of the self-commutator of T_φ is independent of those coefficients.*
(iv) *If $m \leq N$ and $|a_{-m}| = |a_N| \neq 0$, then T_φ is hyponormal if and only if the following equation holds:*

$$\bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \bar{a}_{N-m+1} \\ \bar{a}_{N-m+2} \\ \vdots \\ \vdots \\ \bar{a}_N \end{pmatrix}. \tag{1.4.1}$$

In this case, the rank of $[T_\varphi^*, T_\varphi]$ is $N - m$.
(v) *T_φ is normal if and only if $m = N$, $|a_{-N}| = |a_N|$, and (1.4.1) holds with $m = N$.*

Proof. By the results in [**FL1**] and [**FL2**], it suffices to focus on the second assertion of (iii). To prove this, suppose T_φ is hyponormal. Then by Theorem 10 in [**NT**], there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$ such that the degree of b equals the rank of $[T_\varphi^*, T_\varphi]$. The function b is of the form

$$b(z) = e^{i\theta}\, z^l \prod_{j=1}^{n} \left(\frac{z - \zeta_j}{1 - \bar{\zeta}_j\, z} \right),$$

where $0 < |\zeta_j| < 1$ for $j = 1, \ldots, n$. However, if $b \in \mathcal{E}(\varphi)$ is of the form $b(z) = \sum_{j=0}^{\infty} b_j z^j$, then since $\varphi - b\,\bar{\varphi} \in H^\infty$ we must have $b_0 = b_1 = \cdots = b_{N-m-1} = 0$. Therefore the Blaschke product b must be of the form

$$b(z) = e^{i\theta} z^{N-m} \prod_{j=1}^{n} \left(\frac{z - \zeta_j}{1 - \bar{\zeta}_j\, z} \right) \quad (n \le m).$$

This shows that every Blaschke product in $\mathcal{E}(\varphi)$ is independent of the values of $a_0, a_1, \ldots, a_{N-m}$ and therefore so is rank$\,[T_\varphi^*, T_\varphi]$. □

The following lemma provides a strengthened version of Lemma 1.4(iii).

Lemma 1.5. *Suppose φ is a trigonometric polynomial such that $\varphi = \bar{f} + g$, where f and g are analytic polynomials of degrees m and N $(m \le N)$, respectively. If $\psi := \bar{f} + T_{\bar{z}^r} g$ $(r \le N - m)$, then T_φ is hyponormal if and only if T_ψ is hyponormal.*

Proof. Let $r \le N - m$ and define

$$g_0 := \sum_{n=r}^{N} a_n z^n, \quad \varphi_0 := \bar{f} + g_0, \quad \text{and} \quad h := T_{\bar{z}^r} g.$$

Then $h = \bar{z}^r g_0$ and $\psi = \bar{f} + h$. Observe, by Lemma 1.4(iii), that T_φ is hyponormal if and only if T_{φ_0} is hyponormal. Suppose T_φ is hyponormal and hence so is T_{φ_0}. By Theorem 1.3, there exists a function $k \in H^\infty(\mathbb{T})$ such that $||k||_\infty \le 1$ and $\varphi_0 - k\,\bar{\varphi}_0 \in H^\infty(\mathbb{T})$. Then $\bar{f} - k\,\bar{g}_0 \in H^\infty(\mathbb{T})$, which forces k to be of the form $k(z) = \sum_{n=N-m}^{\infty} c_n z^n$. Thus we have $\bar{f} - k\,\bar{z}^r \bar{h} \in H^\infty(\mathbb{T})$. If we now define $\tilde{k} := k\,\bar{z}^r$, then since $r \le N - m$ it follows that $\tilde{k} \in H^\infty(\mathbb{T})$ and $||\tilde{k}||_\infty = ||k||_\infty \le 1$. Hence $\bar{f} - \tilde{k}\,\bar{h} \in H^\infty(\mathbb{T})$ and therefore $\psi - \tilde{k}\,\bar{\psi} \in H^\infty(\mathbb{T})$, which implies that T_ψ is hyponormal. The argument for the converse is similar. □

The following is a generalization of [**FM**, Theorem].

Proposition 1.6. *Suppose $\varphi \in L^\infty(\mathbb{T})$.*

(i) *If p is an analytic polynomial of the form $p(z) = \sum_{j=0}^{n} \alpha_j z^j$, where α_n is nonzero, then a pair (T_p, T_φ) is hyponormal if and only if*

$$\left(\sum_{j=1}^{n} \alpha_j H_{\bar{z}^j} P_j H_{\bar{\varphi}} \right)^* R C_p^{-1} R^* \left(\sum_{j=1}^{n} \alpha_j H_{\bar{z}^j} P_j H_{\bar{\varphi}} \right) \le [T_\varphi^*, T_\varphi], \tag{1.6.1}$$

where C_p is the compression of $[T_p^*, T_p]$ to $\bigvee\{1, z, \dots, z^{n-1}\}$ and R is the restriction of the identity I_{H^2} to $\bigvee\{1, z, \dots, z^{n-1}\}$; that is,

$$R = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{1.6.2}$$

(here $\bigvee \mathbf{G}$ denotes the closed linear span of $\mathbf{G}$).

(ii) *For every $n \in \mathbb{Z}^+$, we have*

$$(U^n, T_\varphi) \text{ is hyponormal } \iff \big(P_n H_{\bar\varphi}\big)^* \big(P_n H_{\bar\varphi}\big) \le [T_\varphi^*, T_\varphi]. \tag{1.6.3}$$

(iii) *Moreover, if $\varphi \in L^\infty(\mathbb{T})$ is such that $\varphi = f + g$ for some $f \in L^2 \ominus H^2$ and $g \in H^2$, and if $\psi := f + T_{\bar z^n} g \in L^\infty(\mathbb{T})$, then*

$$(U^n, T_\varphi) \text{ is hyponormal } \iff T_\psi \text{ is hyponormal.} \tag{1.6.4}$$

(iv) *Also we have*

$$(U^n, T_\varphi) \text{ is subnormal } \iff \varphi \text{ is analytic.} \tag{1.6.5}$$

Proof. Let $\mathbf{T} := (T_p, T_\varphi)$, and recall that the self-commutator of $\mathbf{T}$ is given by

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_p^*, T_p] & [T_\varphi^*, T_p] \\ [T_p^*, T_\varphi] & [T_\varphi^*, T_\varphi] \end{pmatrix}.$$

Suppose $\varphi \in L^\infty$ has Fourier coefficients $\hat\varphi(n) = a_n$ for every $n \in \mathbb{Z}$. Note that $[T_p^*, T_p]$ has a matrix representation of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where A is an $n \times n$ (finite) matrix. Since, by Lemma 1.4(ii), the rank of the self-commutator of a Toeplitz operator with (analytic) polynomial symbol is just the degree of the polynomial, we have $n = \operatorname{rank}[T_p^*, T_p] = \operatorname{rank} A$, which implies that A is invertible. On the other hand,

$$[T_p^*, T_\varphi] = T_p^* T_\varphi - T_\varphi T_p^* = \sum_{j=1}^n \bar\alpha_j\, [U^{*j}, T_\varphi].$$

A straightforward calculation shows that with respect to the decomposition of $H^2 \oplus H^2$ as $P_n(H^2) \oplus \big(H^2 \ominus P_n(H^2)\big) \oplus H^2$, the self-commutator of $\mathbf{T}$ has the form

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B^* & 0 & [T_\varphi^*, T_\varphi] \end{pmatrix},$$

where

$$B^* = \bar\alpha_1 \begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \bar\alpha_2 \begin{pmatrix} a_2 & a_1 & 0 & \dots & 0 \\ a_3 & a_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} + \dots + \bar\alpha_n \begin{pmatrix} a_n & a_{n-1} & \dots & a_1 \\ a_{n+1} & a_n & \dots & a_2 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

On the other hand, by Lemma 1.2, $[\mathbf{T}^*, \mathbf{T}] \geq 0$ if and only if $B^*A^{-1}B \leq [T_\varphi^*, T_\varphi]$. Since the matrix representation of $H_{\bar{z}^j} P_j H_{\bar{\varphi}}$ is given by

$$\begin{pmatrix} \bar{a}_j & \bar{a}_{j+1} & \dots & & & & \\ \vdots & \ddots & \ddots & & & & \\ \bar{a}_3 & & \ddots & \ddots & & & \\ \bar{a}_2 & \bar{a}_3 & & \ddots & \ddots & & \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \dots & \bar{a}_j & \bar{a}_{j+1} & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

we have

$$B^* = \left(\sum_{j=1}^n \alpha_j H_{\bar{z}^j} P_j H_{\bar{\varphi}} \right)^* R.$$

Since A is the compression of $[T_p^*, T_p]$ to $\bigvee\{1, z, \dots, z^{n-1}\}$, we can conclude that (T_p, T_φ) is hyponormal if and only if

$$\left(\sum_{j=1}^n \alpha_j H_{\bar{z}^j} P_j H_{\bar{\varphi}} \right)^* R C_p^{-1} R^* \left(\sum_{j=1}^n \alpha_j H_{\bar{z}^j} P_j H_{\bar{\varphi}} \right) \leq [T_\varphi^*, T_\varphi].$$

This completes the proof of (i).

The statement in (ii) is a special case of (i) with $p(z) = z^n$. Here $C_p = I_n$ and hence $RC_p^{-1}R^* = P_n$. Therefore by (i), (U^n, T_φ) is hyponormal if and only if

$$\big(P_n H_{\bar{\varphi}}\big)^* H_{\bar{z}^n}^* P_n H_{\bar{z}^n} \big(P_n H_{\bar{\varphi}}\big) \;\leq\; [T_\varphi^*, T_\varphi],$$

or equivalently,

$$\big(P_n H_{\bar{\varphi}}\big)^* \big(P_n H_{\bar{\varphi}}\big) \;\leq\; [T_\varphi^*, T_\varphi].$$

For (iii), suppose that $\varphi = f + g$ for some $f \in L^2 \ominus H^2$ and $g \in H^2$, and that $\psi := f + T_{\bar{z}^n} g \in L^\infty(\mathbb{T})$. Then it suffices to show that the operators $[T_\varphi^*, T_\varphi] - \big(P_n H_{\bar{\varphi}}\big)^* \big(P_n H_{\bar{\varphi}}\big)$ and $[T_\psi^*, T_\psi]$ have the same matrix representations. Suppose $\psi \in L^\infty(\mathbb{T})$ has Fourier coefficients $\hat{\psi}(m) = b_m$ for every $m \in \mathbb{Z}$. Then

$$b_m = \begin{cases} a_m & (m < 0) \\ a_{m+n} & (m \geq 0). \end{cases}$$

With respect to the canonical orthonormal basis of H^2, $[T_\psi^*, T_\psi]$ and $[T_\varphi^*, T_\varphi] - \big(P_n H_{\bar{\varphi}}\big)^* \big(P_n H_{\bar{\varphi}}\big)$ have matrix representation whose (μ, ν)-entries are, respectively,

$$\sum_{j=0}^\infty (\bar{b}_{j-\mu} b_{j-\nu} - b_{\mu-j} \bar{b}_{\nu-j}),$$

and

$$\sum_{j=0}^{\infty}(\bar{a}_{j-\mu}a_{j-\nu} - a_{\mu-j}\bar{a}_{\nu-j}) - \sum_{j=1}^{n} a_{\mu+j}\bar{a}_{\nu+j}.$$

Since for every sequence $\{c_k\}_{k\in\mathbb{Z}}$ of complex numbers,

$$\sum_{j=0}^{\mu+\nu}(\bar{c}_{j-\mu}c_{j-\nu} - c_{\mu-j}\bar{c}_{\nu-j}) = 0,$$

it follows that

$$\begin{aligned}
&\sum_{j=0}^{\infty}(\bar{a}_{j-\mu}a_{j-\nu} - a_{\mu-j}\bar{a}_{\nu-j}) - \sum_{j=1}^{n} a_{\mu+j}\bar{a}_{\nu+j}\\
&= \sum_{j=\mu+\nu+1}^{\infty}(\bar{a}_{j-\mu}a_{j-\nu} - a_{\mu-j}\bar{a}_{\nu-j}) - \sum_{j=1}^{n} a_{\mu+j}\bar{a}_{\nu+j}\\
&= \sum_{j=\mu+\nu+n+1}^{\infty}\bar{a}_{j-\mu}a_{j-\nu} - \sum_{j=\mu+\nu+1}^{\infty} a_{\mu-j}\bar{a}_{\nu-j}\\
&= \sum_{j=\mu+\nu+1}^{\infty}\bar{a}_{j-\mu+n}a_{j-\nu+n} - \sum_{j=\mu+\nu+1}^{\infty} b_{\mu-j}\bar{a}_{\nu-j}\\
&= \sum_{j=\mu+\nu+1}^{\infty}\bar{b}_{j-\mu}b_{j-\nu} - \sum_{j=\mu+\nu+1}^{\infty} b_{\mu-j}\bar{b}_{\nu-j}\\
&= \sum_{j=\mu+\nu+1}^{\infty}(\bar{b}_{j-\mu}b_{j-\nu} - b_{\mu-j}\bar{b}_{\nu-j})\\
&= \sum_{j=0}^{\infty}(\bar{b}_{j-\mu}b_{j-\nu} - b_{\mu-j}\bar{b}_{\nu-j}),
\end{aligned}$$

which says that the (μ,ν)-entries of $[T_\varphi^*, T_\varphi] - \big(P_n H_{\bar{\varphi}}\big)^*\big(P_n H_{\bar{\varphi}}\big)$ and $[T_\psi^*, T_\psi]$ coincide for all μ and ν. This proves (iii).

The backward implication of (1.6.5) follows from the fact that (M_{z^n}, M_φ) is a normal extension of (U^n, T_φ), where M_ψ denotes the multiplication operator on $L^2(\mathbb{T})$ by $\psi \in L^\infty(\mathbb{T})$. For the forward implication of (1.6.5), observe that the subnormality of (U^n, T_φ) implies the commutativity of U^n and T_φ. This forces φ to be analytic, and proves (iv). □

Consider the following two pairs:

$$\mathbf{T}_1 = (U, T_\varphi) \quad \text{and} \quad \mathbf{T}_2 = (U^2, T_\varphi).$$

For $\varphi = z^{-2}$, $T_\varphi + U$ is not hyponormal, but $T_\varphi + U^2$ is. This may suggest that, for general φ, $\mathbf{T}_1$ is less likely than $\mathbf{T}_2$ to be hyponormal. However, Proposition 1.6(iii) shows that the opposite is true: if $\varphi = z^{-1} + z^2$, then $\mathbf{T}_1$ is hyponormal whereas $\mathbf{T}_2$ is not.

In fact we have:

Corollary 1.7. *If (U^n, T_φ) is hyponormal and $0 \le m \le n$, then (U^m, T_φ) is hyponormal.*

Proof. Suppose $0 \le m \le n$. If $\psi_1 := f + T_{\bar{z}^n} g$ and $\psi_2 := f + T_{\bar{z}^m} g$ (cf. Proposition 1.6), Lemma 1.5 shows that the hyponormality of T_{ψ_1} implies that of T_{ψ_2}. Thus the result immediately follows from (1.6.4). $\square$

We next generalize the criterion (1.6.4). First we need two auxiliary results.

Lemma 1.8. *If p is an analytic polynomial of degree n, then*

$$[T_p^*, T_p]^{\frac{1}{2}} = H_{\bar{p}}^* W, \tag{1.8.1}$$

where W is a partial isometry whose initial and final space is $\bigvee\{1, z, \dots, z^{n-1}\}$.

Proof. In general, if $\varphi \in H^\infty(\mathbb{T})$, then $[T_\varphi^*, T_\varphi] = H_{\bar{\varphi}}^* H_{\bar{\varphi}} - H_\varphi^* H_\varphi = H_{\bar{\varphi}}^* H_{\bar{\varphi}}$. Suppose $H_{\bar{p}} = W|H_{\bar{p}}|$ is the polar decomposition, where W is a partial isometry whose initial and final space is $\bigvee\{1, z, \dots, z^{n-1}\}$. Then $|H_{\bar{p}}| = W^* H_{\bar{p}} = H_{\bar{p}}^* W$, so $[T_p^*, T_p]^{\frac{1}{2}} = \left(H_{\bar{p}}^* H_{\bar{p}}\right)^{\frac{1}{2}} = |H_{\bar{p}}| = H_{\bar{p}}^* W$, which gives (1.8.1). $\square$

Corollary 1.9. *If p is an analytic polynomial of the form $p(z) = \sum_{k=0}^n \alpha_k z^k$, with α_n nonzero, then*

$$[T_p^*, T_p]^{\frac{1}{2}} = \left(\sum_{k=1}^n \alpha_k S_k\right) H_{\bar{z}^n} W, \tag{1.9.1}$$

where W is the partial isometry in (1.8.1) and $S_k = Q_k \oplus 0_\infty$, for $Q_k = (a_{i-j})_{1 \le i,j \le n}$ the $n \times n$ upper triangular Toeplitz matrix given by

$$a_{i-j} := \begin{cases} 1 & \text{if } j - i = n - k \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If we write $[M]_{\{1,z,\dots,z^{n-1}\}}$ for the compression of M to $\bigvee\{1, z, \dots, z^{n-1}\}$, then we have $[\alpha_k S_k]_{\{1,z,\dots,z^{n-1}\}} = \alpha_k Q_k$, and hence

$$\left[\sum_{k=1}^n \alpha_k S_k\right]_{\{1,z,\dots,z^{n-1}\}} = \begin{pmatrix} \alpha_n & \alpha_{n-1} & \dots & \alpha_2 & \alpha_1 \\ & \alpha_n & \alpha_{n-1} & & \alpha_2 \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \alpha_{n-1} \\ & & & & \alpha_n \end{pmatrix}.$$

By (1.8.1),

$$\left(\sum_{k=1}^n \alpha_k S_k\right) H_{\bar{z}^n} W = \begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 & \dots & \bar{\alpha}_n & 0 & \dots \\ \bar{\alpha}_2 & & \cdot & 0 & & \\ \vdots & \cdot & \cdot & & & \\ \bar{\alpha}_n & 0 & & & & \\ 0 & & & & & \\ \vdots & & & & & \end{pmatrix}^* W = H_{\bar{p}}^* W = [T_p^*, T_p]^{\frac{1}{2}},$$

which completes the proof. $\square$

Our main result of this chapter is the following:

Theorem 1.10. *Let $\varphi \in L^\infty(\mathbb{T})$ be arbitrary and let p be an analytic polynomial of degree n. If $\varphi = f + g$ for some $f \in L^2 \ominus H^2$ and $g \in H^2$, and if $\psi := f + T_{\bar{z}^n} g \in L^\infty(\mathbb{T})$, then*

$$(T_p, T_\varphi) \text{ is hyponormal} \iff T_\psi \text{ is hyponormal.} \tag{1.10.1}$$

Proof. Write $p(z) = \sum_{k=0}^n \alpha_k z^k$ ($\alpha_n \neq 0$). Suppose T_ψ is hyponormal, hence (U^n, T_φ) is hyponormal, by (1.6.1). By Lemma 1.1,

$$[T_\varphi^*, U^n] = [U^{*n}, U^n]^{\frac{1}{2}}\ D\ [T_\varphi^*, T_\varphi]^{\frac{1}{2}}$$

with some contraction D. If $\varphi \in L^\infty(\mathbb{T})$ has Fourier coefficients $\hat{\varphi}(j) = a_j$ for every $j \in \mathbb{Z}$, then for every $k = 1, \dots, n$,

$$[T_\varphi^*, U^k] = \begin{pmatrix} \bar{a}_k & \bar{a}_{k+1} & \bar{a}_{k+2} & \cdots & \cdots \\ \bar{a}_{k-1} & \bar{a}_k & \bar{a}_{k+1} & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & & \end{pmatrix}.$$

Thus if S_k is given as in Corollary 1.9 then $S_k P_n = S_k$ for every $k = 1, \dots, n$, so

$$\begin{aligned} [T_\varphi^*, \alpha_k U^k] &= \alpha_k\, S_k\, [T_\varphi^*, U^n] \\ &= \alpha_k\, S_k\, P_n\, D\, [T_\varphi^*, T_\varphi]^{\frac{1}{2}} \\ &= \alpha_k\, S_k\, D\, [T_\varphi^*, T_\varphi]^{\frac{1}{2}} \qquad (1 \le k \le n). \end{aligned}$$

Let W be the partial isometry in (1.8.1). Since $H_{\bar{z}^n} W W^* H_{\bar{z}^n}$ is a projection onto $\bigvee\{1, z, \dots, z^{n-1}\}$, we have $\left(\sum_{k=1}^n \alpha_k\, S_k\right) H_{\bar{z}^n} W W^* H_{\bar{z}^n} = \sum_{k=1}^n \alpha_k\, S_k$. Thus it follows from Corollary 1.9 that

$$\begin{aligned} [T_\varphi^*, T_p] = \sum_{k=1}^n [T_\varphi^*, \alpha_k U^k] &= \left(\sum_{k=1}^n \alpha_k\, S_k\right)\ D\, [T_\varphi^*, T_\varphi]^{\frac{1}{2}} \\ &= \left(\sum_{k=1}^n \alpha_k\, S_k\right) H_{\bar{z}^n} W W^* H_{\bar{z}^n} D\, [T_\varphi^*, T_\varphi]^{\frac{1}{2}} \\ &= [T_p^*, T_p]^{\frac{1}{2}}\ W^*\, H_{\bar{z}^n} D\, [T_\varphi^*, T_\varphi]^{\frac{1}{2}} \\ &= [T_p^*, T_p]^{\frac{1}{2}}\ D'\ [T_\varphi^*, T_\varphi]^{\frac{1}{2}}, \end{aligned}$$

where $D' := W^*\, H_{\bar{z}^n}\, D$ is a contraction. Therefore, by Lemma 1.1, (T_p, T_φ) is hyponormal. This proves the backward implication of (1.10.1).

For the forward implication of (1.10.1), suppose (T_p, T_φ) is hyponormal. By Lemma 1.1, $[T_\varphi^*, T_p] = [T_p^*, T_p]^{\frac{1}{2}}\ E\ [T_\varphi^*, T_\varphi]^{\frac{1}{2}}$ for some contraction E. If W is the partial isometry in (1.8.1), then it follows that $[T_\varphi^*, T_p] = H_{\bar p}^*\,(W\,E)[T_\varphi^*, T_\varphi]^{\frac{1}{2}}$. Note that $D := (W\,E)$ is also a contraction. Since W is of the form $\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$, where the upper left corner is an $n \times n$ matrix, we can choose D of the form

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} & \dots & \dots \\ d_{21} & d_{22} & d_{23} & \dots & \dots \\ \vdots & \vdots & \vdots & & \\ d_{n1} & d_{n2} & d_{n3} & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & & \end{pmatrix}. \tag{1.10.2}$$

Then we have

$$H_{\bar p}^*\,D = \begin{pmatrix} \sum_{i=1}^n \alpha_i d_{i1} & \sum_{i=1}^n \alpha_i d_{i2} & \sum_{i=1}^n \alpha_i d_{i3} & \dots & \dots \\ \sum_{i=1}^{n-1} \alpha_{i+1} d_{i1} & \sum_{i=1}^{n-1} \alpha_{i+1} d_{i2} & \sum_{i=1}^{n-1} \alpha_{i+1} d_{i3} & \dots & \dots \\ \vdots & \vdots & \vdots & & \\ \alpha_{n-1} d_{11} + \alpha_n d_{21} & \alpha_{n-1} d_{12} + \alpha_n d_{22} & \alpha_{n-1} d_{13} + \alpha_n d_{23} & \dots & \dots \\ \alpha_n d_{11} & \alpha_n d_{12} & \alpha_n d_{13} & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & & \end{pmatrix}$$

and

$$[T_\varphi^*, T_p] = \begin{pmatrix} \sum_{i=1}^n \alpha_i \bar a_i & \sum_{i=2}^{n+1} \alpha_{i-1} \bar a_i & \sum_{i=3}^{n+2} \alpha_{i-2} \bar a_i & \dots & \dots \\ \sum_{i=1}^{n-1} \alpha_{i+1} \bar a_i & \sum_{i=2}^{n} \alpha_i \bar a_i & \sum_{i=3}^{n+1} \alpha_{i-1} \bar a_i & \dots & \dots \\ \vdots & \vdots & \vdots & & \\ \alpha_{n-1} \bar a_1 + \alpha_n \bar a_2 & \alpha_{n-1} \bar a_2 + \alpha_n \bar a_3 & \alpha_{n-1} \bar a_3 + \alpha_n \bar a_4 & \dots & \dots \\ \alpha_n \bar a_1 & \alpha_n \bar a_2 & \alpha_n \bar a_3 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & & \end{pmatrix}.$$

Let

$$V_k := I_k \bigoplus \begin{pmatrix} 1 & -\frac{\alpha_{n-1}}{\alpha_n} & -\frac{\alpha_{n-2}}{\alpha_n} & \dots & \dots & -\frac{\alpha_{k+1}}{\alpha_n} \\ & 1 & 0 & 0 & \dots & 0 \\ & & 1 & 0 & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \qquad (k = 0, 1, \dots, n-2).$$

Now recall that $[T_\varphi^*, T_p] = H_{\bar p}^*\ D\ [T_\varphi^*, T_\varphi]^{\frac{1}{2}}$; multiplying each side on the left by

$\frac{1}{\alpha_n}\prod_{k=0}^{n-2} V_k$ we obtain

(1.10.3)
$$\begin{pmatrix} d_{n1} & d_{n2} & d_{n3} & \dots \\ d_{n-1,1} & d_{n-1,2} & d_{n-1,3} & \dots \\ \vdots & \vdots & \vdots & \\ d_{11} & d_{12} & d_{13} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} [T_\varphi^*, T_\varphi]^{\frac{1}{2}} = \begin{pmatrix} \bar{a}_n & \bar{a}_{n+1} & \bar{a}_{n+2} & \dots \\ \bar{a}_{n-1} & \bar{a}_n & \bar{a}_{n+1} & \dots \\ \vdots & \vdots & \vdots & \\ \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Since the infinite matrix in the left-hand side of (1.10.3) represents $H_{\bar{z}^n} D$, we have

$$[U^{*n}, U^n]^{\frac{1}{2}}\, H_{\bar{z}^n} D\, [T_\varphi^*, T_\varphi]^{\frac{1}{2}} = [T_\varphi^*, U^n]. \tag{1.10.4}$$

Since $H_{\bar{z}^n}\, D$ is a contraction, (1.10.4) via Lemma 1.1 shows that (U^n, T_φ) is hyponormal and hence, by (1.6.4), T_ψ is hyponormal. This proves the forward implication of (1.10.1) and completes the proof of the theorem. □

If, in Theorem 1.10, φ is a trigonometric polynomial, we can get a simple necessary and sufficient condition for hyponormality of a Toeplitz pair (T_p, T_φ).

Corollary 1.11. *Suppose that φ is a trigonometric polynomial of co-analytic and analytic degrees m and N, and that p is an analytic polynomial of degree n.*

(i) *We have*

$$(T_p, T_\varphi) \text{ is hyponormal} \iff T_\varphi \text{ is hyponormal and } m \le \max\{N-n, 0\}. \tag{1.11.1}$$

(ii) *Moreover, if $N < n$, then*

$$(T_p, T_\varphi) \text{ is hyponormal} \iff \varphi \text{ is analytic} \iff (T_p, T_\varphi) \text{ is subnormal.} \tag{1.11.2}$$

(iii) *The hyponormality of (T_p, T_φ) is independent of the particular values of the Fourier coefficients $\hat{\varphi}(0), \hat{\varphi}(1), \dots, \hat{\varphi}(N-m)$ of φ.*

(iv) *In the cases where the pair $\mathbf{T} = (T_p, T_\varphi)$ is hyponormal, the rank of the self-commutator of $\mathbf{T}$ equals the rank of the self-commutator of T_φ:*

$$\operatorname{rank}[\mathbf{T}^*, \mathbf{T}] = \operatorname{rank}[T_\varphi^*, T_\varphi]. \tag{1.11.3}$$

Proof. Write $\varphi = f + g$ with

$$f(z) := \sum_{k=1}^{m} a_{-k} z^{-k} \quad \text{and} \quad g(z) := \sum_{k=0}^{N} a_k z^k.$$

Put $\psi := f + T_{\bar{z}^n} g$. If $n \le N - m$, then by Lemma 1.5, T_φ is hyponormal if and only if T_ψ is hyponormal. By Lemma 1.4(i), we now have

$$T_\psi \text{ is hyponormal} \iff T_\varphi \text{ is hyponormal and } m \le \max\{N-n, 0\}.$$

Therefore by Proposition 1.6(iii) and (1.10.1), we have

$$(T_p, T_\varphi) \text{ is hyponormal } \iff T_\varphi \text{ is hyponormal and } m \le \max\{N-n, 0\},$$

which proves (i).

(ii) Observe that if $N < n$ then ψ is co-analytic. Thus T_ψ is hyponormal if and only if $m = 0$, so φ is analytic. This together with (1.6.5) gives (1.11.2).

(iii) Recall that by Lemma 1.4(iii) the hyponormality of T_φ is independent of the particular values of the Fourier coefficients $a_0, a_1, \dots, a_{N-m}$ of φ; it follows that the hyponormality of T_ψ is also independent of these coefficients. Now apply (1.11.1).

The proof of (iv) will be given in Chapter 2 (Corollary 2.9). □

Corollary 1.12. *Suppose φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{k=-m}^{N} a_k z^k$ and p is an analytic polynomial of degree n. If $|a_{-m}| = |a_N| \neq 0$ and if $n \le N - m$, then*

$$(1.12.1) \qquad (T_p, T_\varphi) \textit{ is hyponormal } \iff \bar{a}_N \begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m} \begin{pmatrix} \bar{a}_{N-m+1} \\ \bar{a}_{N-m+2} \\ \vdots \\ \bar{a}_N \end{pmatrix}.$$

Proof. This follows from Lemma 1.4(iv) and (1.11.1). □

Example 1.13. *The operators $U+U^2$ and U^*+2U^2 are both hyponormal, whereas the pair $(U+U^2, U^*+2U^2)$ is not hyponormal.*

Proof. In general (cf. [**FL1**, Corollary 1.9]), $\lambda_1 U^{*m} + \lambda_2 U^N$ is hyponormal if and only if $m \le N$ and $|\lambda_1| \le |\lambda_2|$. Thus $U^* + 2U^2$ is hyponormal. Applying the criterion (1.11.1) with $n = 2, m = 1$, and $N = 2$, we see that $(U+U^2, U^*+2U^2)$ is not hyponormal. □

Example 1.14. *The pair $(U - U^2, U^* + U^k)$ is hyponormal if and only if $k \ge 3$.*

Proof. Observe that $U^* + U^k$ is hyponormal if and only if $k \ge 1$. Now applying (1.11.1) gives that $(U - U^2, U^* + U^k)$ is hyponormal if and only if $2 \le \max\{k-1, 0\}$ if and only if $k \ge 3$. □

Remark 1.15. In view of Theorem 1.10, one is tempted to guess that if $p(z) = \sum_{k=-l}^{n} \alpha_k z^k$, if $\varphi \in L^\infty(\mathbb{T})$ is such that $\varphi = \bar{f} + g$ for some $f, g \in H^2$, and if $\psi := \overline{T_{\bar{z}^l} f} + T_{\bar{z}^n} g \in L^\infty(\mathbb{T})$, then

$$(1.15.1) \qquad (T_p, T_\varphi) \text{ is hyponormal } \iff T_\psi \text{ is hyponormal.}$$

Note that if $l = 0$, then (1.15.1) reduces to (1.10.1). However, (1.15.1) cannot hold. For example, consider the following pairs:

$$\mathbf{T}_1 = (U^* + U,\ U^{*2} + U^3) \quad \text{and} \quad \mathbf{T}_2 = (1 + U,\ U^* + U^3).$$

Since $U^* + U$ is self-adjoint, and $U^* + U$ and $U^{*2} + U^3$ do not commute, it follows ([**CMX**, Example 1.14]) that $\mathbf{T}_1$ is not hyponormal. By contrast, $\mathbf{T}_2$ is hyponormal by Corollary 1.11. Note that if $\psi_{\mathbf{T}_1}$ (resp. $\psi_{\mathbf{T}_2}$) is defined as above, then $\psi_{\mathbf{T}_1}(z) \equiv \psi_{\mathbf{T}_2}(z) = \bar{z} + z^2$.

Recall that hyponormal $\Rightarrow$ weakly hyponormal, but that the converse is not true in general. These two notions coincide in some special cases (cf. [**CoS**], [**CMX**])–for example, every weakly hyponormal pair having a *normal* coordinate is hyponormal. On the other hand one might ask whether every weakly hyponormal Toeplitz pair is hyponormal. We now provide evidence that the answer may be affirmative. To see this, we need the following lemma, of independent interest.

Lemma 1.16. *Suppose $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ satisfies one of the following conditions:*

(a) *For some j $(0 \le j \le m-1)$, $|a_{-m+j}|$ is substantially larger than the absolute value of every element in the set $\{a_{N-j}, a_{N-j+1}, \dots, a_N\}$: more concretely,*

$$|a_{-m+j}| > \sum_{i=0}^{j} |a_{N-i}| \quad (0 \le j \le m-1); \tag{1.16.1}$$

(b) *For some j $(1 \le j \le m-1)$, $|a_{N-j}|$ is substantially larger than the absolute value of every element in the set $\{a_{-m+j}\} \cup \{a_{N-j+1}, a_{N-j+2}, \dots, a_N\}$: more concretely,*

$$|a_{N-j}| > \left|\frac{a_N}{a_{-m}}\right| \left(|a_{-m+j}| + \sum_{i=0}^{j-1} |a_{N-i}| \right) \quad (1 \le j \le m-1). \tag{1.16.2}$$

Then T_φ is not hyponormal.

Proof. Suppose $k \in H^\infty$ satisfies $\varphi - k\,\bar{\varphi} \in H^\infty$. Then k necessarily satisfies the property that

$$k \sum_{n=1}^{N} \bar{a}_n z^{-n} - \sum_{n=1}^{m} a_{-n} z^{-n} \in H^\infty.$$

Thus the Fourier coefficients $\hat{k}(0), \hat{k}(1), \dots, \hat{k}(N-1)$ of k are determined uniquely from the coefficients of φ by the recurrence relations

$$\begin{cases} \hat{k}(0) = \cdots = \hat{k}(N-m-1) = 0; \\ \hat{k}(N-m) = \frac{a_{-m}}{\bar{a}_N}; \\ \hat{k}(N-m+j) = \left(\bar{a}_N\right)^{-1} \left(a_{-m+j} - \sum_{i=1}^{j} \hat{k}(N-m+j-i)\, \bar{a}_{N-i} \right) \\ \qquad\qquad\qquad\qquad \text{for } j = 1, \dots, m-1. \end{cases} \tag{1.16.3}$$

Now suppose (1.16.1) holds. Without loss of generality, we can assume that $\hat{k}(N-m), \hat{k}(N-m+1), \dots, \hat{k}(N-m+j-1)$ all have absolute value at most 1 (otherwise $||k||_\infty \ge ||k||_2 \ge \sup_i |\hat{k}(i)| > 1$ and hence $\mathcal{E}(\varphi) = \emptyset$, which implies by Cowen's

Theorem (Theorem 1.3) that T_φ is not hyponormal). From the above recurrence relations we have

$$|\hat{k}(N-m+j)| \geq \frac{1}{|a_N|}\left(|a_{-m+j}| - \sum_{i=1}^{j} |\hat{k}(N-m+j-i)|\,|a_{N-i}|\right).$$

Since by hypothesis $|\hat{k}(N-m+i)| \leq 1$ for every $i = 0, 1, \dots, j-1$, it follows that

$$\begin{aligned} |\hat{k}(N-m+j)| &\geq \frac{1}{|a_N|}\left(|a_{-m+j}| - \sum_{i=1}^{j} |a_{N-i}|\right) \\ &> \frac{|a_N|}{|a_N|} = 1 \qquad\qquad \text{(by (1.16.1))}. \end{aligned}$$

Therefore we have $||k||_\infty > 1$ whenever $\varphi - k\,\bar{\varphi} \in H^\infty$. This implies that $\mathcal{E}(\varphi) = \emptyset$ and hence by Theorem 1.3, T_φ is not hyponormal. The argument for (1.16.2) is similar. □

We now have:

Theorem 1.17. *Let φ be a trigonometric polynomial of co-analytic and analytic degrees m and N, and let p be an analytic polynomial of degree n. Then the following four statements are equivalent:*

(i) (T_p, T_φ) *is hyponormal;*
(ii) (T_p, T_φ) *is weakly hyponormal;*
(iii) (U^n, T_φ) *is hyponormal;*
(iv) (U^n, T_φ) *is weakly hyponormal.*

Proof. The implication (i) $\Rightarrow$ (ii) is evident. For the implication (ii) $\Rightarrow$ (i), write $p(z) = \sum_{k=0}^{n} b_k z^k$ and $\varphi(z) = \sum_{k=-m}^{N} a_k z^k$. Suppose (T_p, T_φ) is weakly hyponormal. Then $T_{\alpha\, p+\beta\,\varphi}$ is hyponormal for every $\alpha, \beta \in \mathbb{C}$. There are three cases to consider.

Case 1 ($n > N$). Choose α, β so $\beta = 1$ and $|\alpha\, b_n| < |a_{-m}|$. Then, by Lemma 1.4(i), $T_{\alpha\, p+\beta\,\varphi}$ is not hyponormal, a contradiction.

Case 2 ($n = N$). Choose α, β so $\beta = 1$ and $|\alpha\, b_n + a_N| < |a_{-m}|$. Again by Lemma 1.4(i), $T_{\alpha\, p+\beta\,\varphi}$ is not hyponormal, a contradiction.

Case 3 ($N - m + 1 \leq n \leq N - 1$). Observe that $(\alpha\, p + \beta\,\varphi)^\wedge(j) = \beta\, a_j$ for every $j > n$ and for every $j < 0$. Choose α, β so $\beta = 1$ and $|\alpha|$ is sufficiently large to guarantee that

$$|\alpha\, b_n + a_n| > \left|\frac{a_N}{a_{-m}}\right| \left(|a_{N-m-n}| + \sum_{i=0}^{N-n-1} |a_{N-i}|\right).$$

Now applying Lemma 1.16 to $T_{\alpha\, p+\beta\,\varphi}$, we see that $T_{\alpha\, p+\beta\,\varphi}$ is not hyponormal, again a contradiction.

The above arguments show that if $T_{\alpha\, p+\beta\,\varphi}$ is hyponormal for every $\alpha, \beta \in \mathbb{C}$, then $n \leq N-m$, i.e., $m \leq N-n$. Therefore, by (1.11.1), we have (T_p, T_φ) is hyponormal.

The equivalence (i) $\Leftrightarrow$ (iii) comes from (1.6.4) and (1.10.1), and the equivalence (iii) $\Leftrightarrow$ (iv) is a special case of (i) $\Leftrightarrow$ (ii). □

Remark 1.18 (First "Joint" Version of Cowen's Theorem). If $\varphi \in L^\infty(\mathbb{T})$ is such that $\varphi = \bar{f} + g$ for some $f, g \in H^2$ and if p is an analytic polynomial of degree n, define

$$\mathcal{E}(p, \varphi) := \left\{ k \in H^\infty : ||k||_\infty \le 1 \text{ and } \bar{f} - k\, \overline{T_{\bar{z}^n} g} \in H^\infty \right\}.$$

Then Theorem 1.10 says that $\mathbf{T} = (T_p, T_\varphi)$ is hyponormal if and only if the set $\mathcal{E}(p, \varphi)$ is nonempty. Moreover there exists a finite Blaschke product b in $\mathcal{E}(p, \varphi)$ such that $\deg(b) = \operatorname{rank}[\mathbf{T}^*, \mathbf{T}]$ (this follows from [**NT**, Theorem 10] and Corollary 1.11(iv) above).

When (U^n, R) is a hyponormal pair, with R an operator on $H^2(\mathbb{T})$, R need not be a Toeplitz operator. In fact if (U^n, R) is a hyponormal pair then Lemma 1.1 implies that R is a block-Toeplitz operator T_Φ with a matrix valued symbol $\Phi \in L^\infty(\mathbb{T}) \otimes M_n(\mathbb{C})$. Furthermore, by contrast with Theorem 1.17, there exists a pair (U^n, R) which is weakly hyponormal but not hyponormal (cf. [**FM**]). On the other hand, if T_Φ is a block-Toeplitz operator of the form

$$\begin{pmatrix} A_0 & A_{-1} & A_{-2} & \dots & \dots & \dots \\ A_1 & A_0 & A_{-1} & A_{-2} & \dots & \dots \\ A_2 & A_1 & A_0 & A_{-1} & A_{-2} & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (A_i \in M_n(\mathbb{C}) \text{ for every } i \in \mathbb{Z})$$

for which (U^n, T_Φ) is a hyponormal pair then the diagonal of A_1 must be independent of the hyponormality of T_Φ because $\alpha\, U^n + \beta\, T_\Phi$ is hyponormal for every $\alpha, \beta \in \mathbb{C}$, and the perturbation of $\beta\, T_\Phi$ by $\alpha\, U^n$ affects only the diagonal of A_1; if $\Phi \in L^\infty(\mathbb{T})$ then (1.6.4) illustrates this fact. By analogy with (1.6.4) we have the following:

Conjecture 1.19. Let $\varphi \in L^\infty(\mathbb{T})$ be such that $\varphi = f + g$ for some $f \in L^2 \ominus H^2$ and $g \in H^2$, let $\psi := f + T_{\bar{z}^n} g \in L^\infty(\mathbb{T})$, and let $\Phi := \varphi \otimes A_n$ with $A_n \in M_n(\mathbb{C})$. Then (U^n, T_Φ) is hyponormal if and only if $T_{\psi \otimes A_n}$ is hyponormal.

CHAPTER 2

HYPONORMALITY OF TRIGONOMETRIC TOEPLITZ PAIRS

If both coordinates of a Toeplitz pair have non-analytic trigonometric polynomial symbols then hyponormality is rather rigid. Toeplitz operators with trigonometric polynomial symbols will be simply called *trigonometric Toeplitz operators* and Toeplitz pairs with both coordinates trigonometric Toeplitz operators will be called *trigonometric Toeplitz pairs*. We begin with:

Lemma 2.1. *Let $\varphi \in L^\infty(\mathbb{T})$ be arbitrary and let ψ be a non-analytic trigonometric polynomial. If (T_ψ, T_φ) is hyponormal then the co-analytic degree of φ is finite. More precisely,*

$$(\textit{co-analytic degree of } \varphi) \le (\textit{analytic degree of } \psi).$$

Proof. Suppose $\varphi \in L^\infty$ has Fourier coefficients $\hat{\varphi}(n) = a_n$ $(n \in \mathbb{Z})$ and write $\psi(z) = \sum_{k=-m}^{N} b_k z^k$ $(m \neq 0)$. Assume that (T_ψ, T_φ) is hyponormal. Then T_ψ is hyponormal and hence $m \le N$. With respect to the canonical orthonormal basis $\{e_n\}_{n=0}^\infty$ of $H^2(\mathbb{T})$, $[T_\psi^*, T_\psi]$ has a matrix representation $\left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where the upper left corner is an $N \times N$ matrix. It follows that $([T_\psi^*, T_\psi]e_k,\, e_k) = 0$, for every $k \ge N$. Thus by Lemma 1.1,

$$([T_\varphi^*,\, T_\psi]e_{m-1},\, e_k) = 0 \quad \text{for every } k \ge N. \tag{2.1.1}$$

Observe that for every $k \ge N$,

$$([T_\varphi^*, T_\psi]e_{m-1},\, e_k) = \big((I - P_N)[T_\varphi^*, T_\psi]e_{m-1},\, e_k\big).$$

Since

$$\begin{aligned}
(I - P_N)[T_\varphi^*, T_\psi] &= (I - P_N)(T_{\bar\varphi} T_\psi - T_\psi T_{\bar\varphi}) \\
&= (I - P_N)\Bigg[\left(\sum_{k=1}^{\infty} \bar{a}_k U^{*k} \sum_{k=1}^{N} b_k U^k - \sum_{k=1}^{n} b_k U^k \sum_{k=1}^{\infty} \bar{a}_k U^{*k}\right) \\
&\qquad + \left(\sum_{k=1}^{\infty} \bar{a}_{-k} U^k \sum_{k=1}^{m} b_{-k} U^{*k} - \sum_{k=1}^{m} b_{-k} U^{*k} \sum_{k=1}^{\infty} \bar{a}_{-k} U^k\right)\Bigg] \\
&= \sum_{k=N+1}^{\infty} \bar{a}_{-k} U^k \sum_{k=1}^{m} b_{-k} U^{*k} - \sum_{k=1}^{m} b_{-k} U^{*k} \sum_{k=N+1}^{\infty} \bar{a}_{-k} U^k \\
&= -\sum_{i=N+1}^{\infty} \bar{a}_{-i} \sum_{j=1}^{m} b_{-j} U^{i-j} P_{j-1},
\end{aligned}$$

it follows that for every $k \geq N$,

$$\begin{aligned}
\big([T_\varphi^*, T_\psi]e_{m-1},\, e_k\big) &= \left(-\sum_{i=N+1}^{\infty} \bar a_{-i} \sum_{j=1}^{m} b_{-j} U^{i-j} P_{j-1}(e_{m-1}),\ e_k\right) \\
&= \left(-b_{-m} \sum_{i=N+1}^{\infty} \bar a_{-i} U^{i-m} P_{m-1}(e_{m-1}),\ e_k\right) \qquad (2.1.2)\\
&= -b_{-m} \left(\sum_{i=N+1}^{\infty} \bar a_{-i} e_{i-1},\ e_k\right) \\
&= -b_{-m}\, \bar a_{-k-1}.
\end{aligned}$$

Since $b_{-m} \neq 0$, it follows from (2.1.1) that $a_{-k} = 0$ for every $k \geq N+1$. This completes the proof. □

The following theorem shows that hyponormal Toeplitz pairs with a normal coordinate are necessarily normal.

Theorem 2.2. *Let $\varphi \in L^\infty(\mathbb{T})$ be arbitrary and let $\psi \in L^\infty(\mathbb{T})$ be such that T_ψ is normal. Then*

$$(T_\psi, T_\varphi) \text{ is hyponormal} \iff \varphi = \alpha\,\psi + \beta \quad (\alpha, \beta \in \mathbb{C}). \qquad (2.2.1)$$

Thus every hyponormal Toeplitz pair having a normal coordinate is a normal pair.

Proof. Suppose T_ψ is normal. Then by Example 1.14 of [**CMX**], (T_ψ, T_φ) is hyponormal if and only if T_φ is hyponormal and $T_\psi T_\varphi = T_\varphi T_\psi$. Now recall that a necessary and sufficient condition that two Toeplitz operators commute is that either both be analytic, or both be co-analytic, or one be a linear function of the other (cf. [**BH**, Theorem 9]). The equivalence in (2.2.1) is now obvious, and the second assertion is an immediate consequence of (2.2.1). □

On the other hand we need not expect that if $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and ψ is a non-analytic trigonometric polynomial such that (T_ψ, T_φ) is hyponormal then φ must be a trigonometric polynomial. To see this, we reformulate Cowen's Theorem (Theorem 1.3).

Suppose $\varphi \in L^\infty(\mathbb{T})$ is of the form $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $k(z) = \sum_{n=0}^{\infty} c_n z^n$ is in $H^2(\mathbb{T})$. Then $\varphi - k\,\bar\varphi \in H^\infty$ has a solution $k \in H^\infty$ if and only if

$$\begin{pmatrix}
\bar a_1 & \bar a_2 & \bar a_3 & \dots & \bar a_n & \dots \\
\bar a_2 & \bar a_3 & \dots & \bar a_n & \dots & \\
\bar a_3 & \dots & \dots & \dots & & \\
\vdots & \bar a_n & \dots & & & \\
\bar a_n & \dots & & & & \\
\vdots & & & & &
\end{pmatrix}
\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}
=
\begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-3} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}, \qquad (2.2.2)$$

that is, $H_{\bar\varphi} k = H_\varphi e_0$, where $e_0 = (1, 0, 0, \dots)$. Thus, by Cowen's Theorem, T_φ is hyponormal if and only if there exists a solution $k \in H^\infty$ of the equation (2.2.2) such that $||k||_\infty \leq 1$.

Example 2.3. *Suppose $\varphi, \psi \in L^\infty(\mathbb{T})$ are functions of the form*

$$\varphi(z) = \frac{1}{6}\, z^{-1} + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} z^n \quad \textit{and} \quad \psi(z) = b\, z^{-1} + a\, z \quad (a, b \in \mathbb{C}).$$

Then (T_ψ, T_φ) is hyponormal if and only if $a + 6\, b = 0$.

Remark. In view of Lemma 2.1, if $\psi(z) = b\, z^{-1} + a\, z$, then the hyponormality of (T_ψ, T_φ) forces φ to be of the form $\varphi(z) = \sum_{n=-1}^{\infty} a_n z^n$.

Proof. Since the hyponormality of the pair is independent of a scalar multiple of every factor, we may assume that $a = 1$. Thus it suffices to show that (T_ψ, T_φ) is hyponormal if and only if $b = -\frac{1}{6}$. Observe that $k(z) = \sum_{n=0}^{\infty} c_n z^n$ satisfies $\varphi - k\, \bar{\varphi} \in H^\infty$ if and only if

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots \\ \frac{1}{2} & \frac{1}{4} & \frac{!}{8} & \cdots & \\ \frac{1}{4} & \frac{1}{8} & \cdots & & \\ \frac{1}{8} & \cdots & & & \\ \vdots & & & & \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}.$$

A straightforward calculation shows that

$$k(z) = -\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} z^n$$

satisfies (2.2.2). Also, it is easy to see that $k(z) = \frac{1}{3} \frac{z - \frac{1}{2}}{1 - \frac{1}{2} z}$, so $||k||_\infty = \frac{1}{3}$. Therefore T_φ is hyponormal. Now, T_ψ is hyponormal if and only if $|b| \le 1$. On the other hand a calculation shows that

$$[T_\psi^*, T_\psi] = (\, 1 - |b|^2\,) \oplus 0_\infty;$$

$$[T_\psi^*, T_\varphi] = \begin{pmatrix} -\frac{\bar{b}}{6} & 0 & \cdots \\ \frac{1}{2} & 0 & \cdots \\ \frac{1}{4} & 0 & \cdots \\ \frac{1}{8} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix};$$

$$[T_\varphi^*, T_\varphi] = \begin{pmatrix} \frac{11}{36} & \frac{1}{6} & \frac{1}{12} & \frac{1}{24} & \frac{1}{48} & \frac{1}{96} & \cdots \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{12} & \frac{1}{24} & \frac{1}{48} & \cdots \\ \vdots & \vdots & \frac{1}{12} & \frac{1}{24} & \frac{1}{48} & \frac{1}{96} & \cdots \\ \vdots & \vdots & \vdots & \frac{1}{48} & \frac{1}{96} & \frac{1}{192} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \frac{1}{192} & \frac{1}{384} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $|b| = 1$, then T_ψ is normal, so by Theorem 2.2, (T_ψ, T_φ) is not hyponormal. Assume $|b| \neq 1$. Note that the Moore-Penrose inverse $[T_\psi^*, T_\psi]^\#$ of $[T_\psi^*, T_\psi]$ is given by

$$[T_\psi^*, T_\psi]^\# = \left(\tfrac{1}{1-|b|^2}\right) \oplus 0_\infty.$$

Thus we have

$$
\begin{aligned}
&[T_\psi^*, T_\varphi]\,[T_\psi^*, T_\psi]^\#\,[T_\psi^*, T_\varphi]^* \\
&= \begin{pmatrix} -\frac{\bar b}{6} & 0 & \cdots \\ \frac{1}{2} & 0 & \cdots \\ \frac{1}{4} & 0 & \cdots \\ \frac{1}{8} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}
\begin{pmatrix} \frac{1}{1-|b|^2} & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}
\begin{pmatrix} -\frac{b}{6} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= \frac{1}{1-|b|^2}
\begin{pmatrix} \frac{|b|^2}{36} & -\frac{\bar b}{12} & -\frac{\bar b}{24} & -\frac{\bar b}{48} & \cdots & \cdots \\ -\frac{b}{12} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ \vdots & \vdots & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots \\ \vdots & \vdots & \vdots & \frac{1}{64} & \frac{1}{128} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{aligned}
$$

By Lemma 1.2, (T_ψ, T_φ) is hyponormal if and only if

$$[T_\varphi^*, T_\varphi] - [T_\psi^*, T_\varphi]\,[T_\psi^*, T_\psi]^\#\,[T_\psi^*, T_\varphi]^* \geq 0.$$

Therefore if (T_ψ, T_φ) is hyponormal then restricting to the 2×2 matrix in the upper left corner gives

$$\det \begin{pmatrix} \frac{11}{36} - \frac{|b|^2}{36(1-|b|^2)} & \frac{1}{6} + \frac{\bar b}{12(1-|b|^2)} \\ \frac{1}{6} + \frac{b}{12(1-|b|^2)} & \frac{1}{3} - \frac{1}{4(1-|b|^2)} \end{pmatrix} \geq 0.$$

Now a simple calculation shows that this is equivalent to

$$(|b|^2 - 1)\,(36\,|b|^2 + 12\,\mathrm{Re}\,b + 1) \geq 0.$$

Since $|b| < 1$, we must have $b = -\frac{1}{6}$. Conversely if $b = -\frac{1}{6}$ then

(2.3.1)

$$
\begin{aligned}
&[T_\varphi^*, T_\varphi] - [T_\psi^*, T_\varphi]\,[T_\psi^*, T_\psi]^\#\,[T_\psi^*, T_\varphi]^* \\
&\quad= \begin{pmatrix}
\frac{11}{36} - \frac{1}{35\cdot 36} & \frac{1}{6} - \frac{1}{35\cdot 2} & \frac{1}{12} - \frac{1}{35\cdot 4} & \frac{1}{24} - \frac{1}{35\cdot 8} & \cdots & \cdots \\
\frac{1}{6} - \frac{1}{35\cdot 2} & \frac{1}{3} - \frac{9}{35} & \frac{1}{6} - \frac{9}{35\cdot 2} & \frac{1}{12} - \frac{9}{35\cdot 4} & \frac{1}{24} - \frac{9}{35\cdot 8} & \cdots \\
\vdots & \vdots & \frac{1}{12} - \frac{9}{35\cdot 4} & \frac{1}{24} - \frac{9}{35\cdot 8} & \frac{1}{48} - \frac{9}{35\cdot 16} & \cdots \\
\vdots & \vdots & \vdots & \frac{1}{48} - \frac{9}{35\cdot 16} & \frac{1}{96} - \frac{9}{35\cdot 32} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\end{aligned}
$$

A simple observation shows that the rank of the matrix in (2.3.1) is 1: more concretely, if $\mathbf{v}$ denotes the first row vector of the matrix in (2.3.1) then this matrix can be represented by $(\mathbf{v}, \frac{1}{2}\,\mathbf{v}, \frac{1}{2^2}\,\mathbf{v}, \frac{1}{2^3}\,\mathbf{v}, \dots)^T$. Now recall ([**Smu**], [**CF4**, Proposition 2.2]) that if A is a positive semi-definite matrix and if $\tilde{A} := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is an *extension* of A satisfying $\operatorname{rank} \tilde{A} = \operatorname{rank} A$ (such extensions are called *flat extensions*), then $\tilde{A}$ is necessarily positive semi-definite. This immediately shows that every hermitian matrix of rank 1 whose $(1,1)$-entry is positive must be positive semi-definite. Therefore we can conclude that the matrix in (2.3.1) is positive semi-definite and therefore (T_ψ, T_φ) is hyponormal. This completes the proof. $\square$

We now give a condition that *non-analytic* trigonometric polynomials φ and ψ must necessarily satisfy in order for (T_ψ, T_φ) to be a weakly hyponormal pair. It is somewhat surprising when we compare it with Theorem 1.10.

Theorem 2.4 (A Necessary Condition for Weak Hyponormality). *Suppose φ and ψ are non-analytic trigonometric polynomials. If (T_ψ, T_φ) is weakly hyponormal then φ and ψ have the same co-analytic and analytic degrees.*

Proof. Suppose $\varphi(z) = \sum_{k=-m}^{N} a_k z^k$ $(m \neq 0)$ and $\psi(z) = \sum_{k=-l}^{n} b_k z^k$ $(l \neq 0)$. Assume without loss of generality that $n \le N$. Suppose (T_ψ, T_φ) is weakly hyponormal. Then $T_{\alpha\,\psi + \beta\,\varphi}$ is hyponormal for every $\alpha, \beta \in \mathbb{C}$. We wish to show that $n = N$. Assume to the contrary that $n < N$. Then we claim that $l < m$: indeed, if this were not the case then choosing $\alpha, \beta \in \mathbb{C}$ such that

$$\begin{cases} \alpha = 1 \text{ and } |\beta\, a_N| < |b_{-l}| & \text{if } l > m, \\ \beta = 1 \text{ and } |a_N| < |a_{-m} + \alpha\, b_{-l}| & \text{if } l = m, \end{cases} \tag{2.4.1}$$

$T_{\alpha\,\psi + \beta\,\varphi}$ would not be hyponormal (by Lemma 1.4(i)), a contradiction. Now choose $\beta = 1$ and write $\zeta_\alpha := \alpha\psi + \varphi$. We have

$$\hat{\zeta}_\alpha(j) = \begin{cases} a_j & (-m \le j \le -l-1) \\ \alpha\, b_j + a_j & (-l \le j \le n) \\ a_j & (n+1 \le j \le N). \end{cases}$$

Three cases arise.

Case 1 $(N-n < m-l)$. Since $N-m-n < -l$, we have $\hat{\zeta}_\alpha(N-m-n) = a_{N-m-n}$. Now apply Lemma 1.16(b) with ζ_α and n in place of φ and $N-j$, respectively. Then (1.16.2) can be written as

$$\left|\hat{\zeta}_\alpha(n)\right| > \left|\frac{\hat{\zeta}_\alpha(N)}{\hat{\zeta}_\alpha(-m)}\right| \left(|\hat{\zeta}_\alpha(N-m-n)| + \sum_{i=0}^{N-n-1} |\hat{\zeta}_\alpha(N-i)| \right),$$

or equivalently,

$$|\alpha\, b_n + a_n| > \left|\frac{a_N}{a_{-m}}\right| \left(|a_{N-m-n}| + \sum_{i=0}^{N-n-1} |a_{N-i}| \right). \tag{2.4.2}$$

Thus we can choose α sufficiently large such that (2.4.2) holds. Therefore by Lemma 1.16, T_{ζ_α} is not hyponormal, a contradiction.

Case 2 $(N-n>m-l)$. Since $N-m+l>n$, we have $\hat{\zeta}_\alpha(N-i)=a_{N-i}$ for every $i=0,1,\dots,m-l$. Now apply Lemma 1.16(a) with ζ_α and $-l$ in place of φ and $-m+j$, respectively. Then (1.16.1) can be written as

$$\left|\hat{\zeta}(-l)\right| > \sum_{i=0}^{m-l}\left|\hat{\zeta}_\alpha(N-i)\right|, \quad \text{that is,} \quad |\alpha\, b_{-l}+a_{-l}| > \sum_{i=0}^{m-l}|a_{N-i}|. \tag{2.4.3}$$

Thus we can choose α sufficiently large such that (2.4.3) holds. Therefore by Lemma 1.16, T_{ζ_α} is not hyponormal, a contradiction.

Case 3 $(N-n=m-l)$. Since $N-m+l=n$, we have $\hat{\zeta}_\alpha(N-i)=a_{N-i}$ for every $i=0,1,\dots,m-l-1$. Since T_{ζ_α} is hyponormal, we can imitate the proof of Lemma 1.16 to show that there exists a function $k\in\mathcal{E}(\zeta_\alpha)$ whose Fourier coefficients $\hat{k}(N-m),\dots,\hat{k}(N-1)$ are uniquely determined by

$$\hat{k}(N-m)=\frac{\hat{\zeta}_\alpha(-m)}{\overline{\hat{\zeta}_\alpha}(N)}=\frac{a_{-m}}{\bar{a}_N}$$

and for $j=1,\dots,m-1$,

$$\hat{k}(N-m+j)=\left(\bar{a}_N\right)^{-1}\left(\hat{\zeta}_\alpha(-m+j)-\sum_{i=1}^{j}\hat{k}(N-m+j-i)\,\overline{\hat{\zeta}_\alpha}(N-i)\right).$$

This implies that $\hat{k}(N-m+i)$ is independent of the value of α and the Fourier coefficients of ψ for every $i=0,1,\dots,l-1$. Since

$$\begin{aligned}
&\hat{k}(N-l)\\
&=\left(\bar{a}_N\right)^{-1}\left(\hat{\zeta}_\alpha(-l)-\sum_{i=1}^{m-l}\hat{k}(N-l-i)\overline{\hat{\zeta}_\alpha}(N-i)\right)\\
&=\left(\bar{a}_N\right)^{-1}\Bigg((a_{-l}+\alpha\, b_{-l})-\hat{k}(N-m)(\bar{a}_{N-m+l}+\bar{\alpha}\,\bar{b}_{N-m+l})\\
&\qquad\qquad\qquad -\sum_{i=1}^{m-l-1}\hat{k}(N-l-i)\overline{\hat{\zeta}_\alpha}(N-i)\Bigg)\\
&=\left(\bar{a}_N\right)^{-1}\left(\alpha\, b_{-l}-\bar{\alpha}\,\hat{k}(N-m)\bar{b}_n+h\right) \qquad\qquad (\text{because } N-m+l=n),
\end{aligned}$$

where

$$h:=a_{-l}-\hat{k}(N-m)\,\bar{a}_n-\sum_{i=1}^{m-l-1}\hat{k}(N-l-i)\overline{\hat{\zeta}_\alpha}(N-i)$$

is independent of the value of α, let $\alpha:=re^{i\theta}$ for $\theta\in[0,2\pi)$. We have

$$\alpha\, b_{-l}-\bar{\alpha}\,\hat{k}(N-m)\,\bar{b}_n=re^{i\theta}(b_{-l}-e^{-2i\theta}\hat{k}(N-m)\,\bar{b}_n).$$

Choose $\theta \in [0, 2\pi)$ such that $b_{-l} - e^{-2i\theta}\hat{k}(N-m)\,\bar{b}_n \neq 0$. Then choose r sufficiently large such that

$$r\left|b_{-l} - e^{-2i\theta}\hat{k}(N-m)\,\bar{b}_n\right| > |a_N| + |h|.$$

It follows that $|\hat{k}(N-l)| > 1$ and hence $||k||_\infty > 1$, a contradiction.

The above arguments show that $n = N$. Next we will show that $l = m$. If this is not the case, we can assume without loss of generality that $l < m$. Choose $\beta = 1$. Then

$$\alpha\,\psi + \varphi = \sum_{k=l+1}^{m} a_{-k} z^{-k} + \sum_{k=-l}^{N} (a_k + \alpha\, b_k)\, z^k.$$

Therefore if we choose $\alpha \in \mathbb{C}$ such that $|a_N + \alpha\, b_N| < |a_{-m}|$, then by Lemma 1.4(i), $T_{\alpha\,\psi+\beta\,\varphi}$ is not hyponormal, a contradiction. Therefore we must have $l = m$. This completes the proof. $\square$

To prove the main theorem of this chapter we need the following observation.

Lemma 2.5. *Suppose $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ and $\psi(z) = \varphi(z) + \sum_{n=0}^{N-m}(b_n - a_n)$. If $\varphi_0 := f + T_{\bar{z}^{N-m}} g$, where $f = \sum_{n=1}^{m} a_{-n} z^{-n}$ and $g = \sum_{n=0}^{N} a_n z^n$, then*

$$[T_\psi^*, T_\varphi] = [T_{\varphi_0}^*, T_{\varphi_0}] + \sum_{k=1}^{N-m} \left(A_k^* B_k \oplus 0_\infty\right), \tag{2.5.1}$$

where

$$A_k = (\bar{a}_k, \bar{a}_{k+1}, \ldots, \bar{a}_N) \in \mathbb{C}^{N-k+1}$$

and

$$B_k = (\bar{b}_k, \ldots, \bar{b}_{N-m}, \bar{a}_{N-m+1}, \ldots, \bar{a}_N) \in \mathbb{C}^{N-k+1}.$$

Thus in particular we have

$$[T_\varphi^*, T_\varphi] - [T_{\varphi_0}^*, T_{\varphi_0}] = \sum_{k=1}^{N-m} \left(A_k^* A_k \oplus 0_\infty\right), \tag{2.5.2}$$

which is a positive semi-definite matrix expressible as a sum of positive rank 1 terms.

Proof. Let $h := \sum_{n=0}^{N-m} b_n z^n + \sum_{n=N-m+1}^{N} a_n z^n$, so $\psi = f + h$, and for $\zeta \in H^2$, let $\zeta_k := T_{\bar{z}^k}\zeta$. Then $h_0 = h$ and $g_0 = g$. Since $\bar{f}$ and g are analytic it follows that

$$\begin{aligned}
[T_\psi^*, T_\varphi] &= [T_{f+h}^*, T_{f+g}] \\
&= [T_f^*, T_f] + [T_h^*, T_g] \\
&= [T_f^*, T_f] + \sum_{k=1}^{N-m}\Big([T_{h_{k-1}}^*, T_{g_{k-1}}] - [T_{h_k}^*, T_{g_k}]\Big) + [T_{h_{N-m}}^*, T_{g_{N-m}}].
\end{aligned}$$

Note that $g_{N-m} = h_{N-m} = T_{\bar{z}^{N-m}} g$, so that

$$\begin{aligned}[][T_\psi^*, T_\varphi] &= [T^*_{f+g_{N-m}}, T_{f+g_{N-m}}] + \sum_{k=1}^{N-m} \Big([T^*_{h_{k-1}}, T_{g_{k-1}}] - [T^*_{h_k}, T_{g_k}] \Big) \\ &= [T^*_{\varphi_0}, T_{\varphi_0}] + \sum_{k=1}^{N-m} \big(C_k \oplus 0_\infty \big),\end{aligned}$$

for some $(N-k+1) \times (N-k+1)$ matrix C_k. We now claim that $C_k = A_k^* B_k$ for every $k = 1, \dots, N-m$. Indeed, a straightforward calculation shows that

$$\begin{aligned} &C_k \oplus 0_\infty \\ &= \left[\left(\sum_{j=1}^{N-m-k+1} \bar{b}_{k+j-1} U^{*j} + \sum_{j=N-m-k+2}^{N-k+1} \bar{a}_{k+j-1} U^{*j} \right), \sum_{j=1}^{N-k+1} a_{k+j-1} U^j \right] \\ &\quad - \left[\left(\sum_{j=1}^{N-m-k} \bar{b}_{k+j} U^{*j} + \sum_{j=N-m-k+1}^{N-k} \bar{a}_{k+j} U^{*j} \right), \sum_{j=1}^{N-k} a_{k+j} U^j \right] \\ &= \sum_{j=0}^{N-k} a_{k+j} \left(\sum_{j=1}^{N-m-k+1} \bar{b}_{k+j-1} U^{*j} + \sum_{j=N-m-k+2}^{N-k+1} \bar{a}_{k+j-1} U^{*j} \right) \big(U^{j+1} - U^j \big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(where } U^0 := 0) \\ &= \begin{pmatrix} a_k \bar{b}_k & \dots & a_k \bar{b}_{N-m} & \dots & a_k \bar{a}_N \\ a_{k+1} \bar{b}_k & \dots & a_{k+1} \bar{b}_{N-m} & \dots & a_{k+1} \bar{a}_N \\ \vdots & \dots & \vdots & \dots & \vdots \\ a_N \bar{b}_k & \dots & a_N \bar{b}_{N-m} & \dots & a_N \bar{a}_N \end{pmatrix} \bigoplus 0_\infty \\ &= \begin{pmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_N \end{pmatrix} \left(\bar{b}_k, \quad \dots, \quad \bar{b}_{N-m}, \quad \dots, \quad \bar{a}_N \right) \bigoplus 0_\infty \\ &= A_k^* B_k \oplus 0_\infty. \end{aligned}$$

This completes the proof. $\square$

Theorem 2.6 (A Necessary and Sufficient Condition for Hyponormality). *Suppose φ and ψ are non-analytic trigonometric polynomials such that T_φ and T_ψ are hyponormal.*

(1) *The following statements are equivalent.*
  - (i) *(T_ψ, T_φ) is hyponormal.*
  - (ii) *(T_ψ, T_φ) is weakly hyponormal.*
  - (iii) *φ and ψ have the same co-analytic and analytic degrees m and N, and furthermore there exists a constant $c \in \mathbb{C}$ such that $\varphi - c\,\psi = \sum_{j=0}^{N-m} d_j z^j$ for some $d_0, \dots, d_{N-m} \in \mathbb{C}$.*

(2) *The hyponormality of* (T_ψ, T_φ) *is independent of the particular values of the Fourier coefficients* $\hat\varphi(0), \hat\varphi(1), \dots, \hat\varphi(N-m)$ *and* $\hat\psi(0), \hat\psi(1), \dots, \hat\psi(N-m)$.

(3) *In the cases where the pair* $\mathbf{T} = (T_\psi, T_\varphi)$ *is hyponormal, the rank of the self-commutator of* $\mathbf{T}$ *equals the rank of the self-commutator of* T_φ *(or* T_ψ*): more precisely, if* $\varphi = f + g$ *for some* $f \in L^2 \ominus H^2$ *and* $g \in H^2$ *and if* $\varphi_0 := f + T_{\bar z^{N-m}} g$*, then*

$$\text{rank}\,[\mathbf{T}^*, \mathbf{T}] = \text{rank}\,[T_\varphi^*, T_\varphi] = N - m + \text{rank}\,[T_{\varphi_0}^*, T_{\varphi_0}]. \tag{2.6.1}$$

Proof. We first prove (1).

(i)⇒(ii): This is trivial.

(ii)⇒(iii): The first assertion follows from Theorem 2.4. For the second assertion write $\varphi(z) = \sum_{k=-m}^N a_k z^k$ and $\psi(z) = \sum_{k=-m}^N b_k z^k$. Suppose (T_ψ, T_φ) is weakly hyponormal. Then $T_{\alpha\,\psi+\beta\,\varphi}$ is hyponormal for every $\alpha, \beta \in \mathbb{C}$. We may assume without loss of generality that $a_{-m} = b_{-m} = 1$. Choose $\beta = 1$ and write $\zeta_\alpha := \alpha\,\psi + \varphi$. Since T_{ζ_α} is hyponormal, we can imitate the proof of Lemma 1.16 to show that there exists a function $k \in \mathcal{E}(\zeta_\alpha)$ such that the Fourier coefficients $\hat k(0), \hat k(1), \dots, \hat k(N-m)$ of k are uniquely determined by

$$\begin{cases} \hat k(0) = \hat k(1) = \cdots = \hat k(N-m+1) = 0; \\ \hat k(N-m) = \frac{\hat\zeta_\alpha(-m)}{\overline{\hat\zeta_\alpha(N)}} = \frac{\alpha\, b_{-m} + a_{-m}}{\bar\alpha\, \bar b_N + \bar a_N} = \frac{\alpha+1}{\bar\alpha\, \bar b_N + \bar a_N}; \\ \hat k(N-m+j) \\ \quad = \frac{1}{\bar\alpha\, \bar b_N + \bar a_N} \left(\alpha\, b_{-m+j} + a_{-m+j} - \sum_{i=1}^j \hat k(N-m+j-i)\, (\bar\alpha\, \bar b_{N-i} + \bar a_{N-i}) \right) \\ \qquad\qquad\qquad\qquad \text{for } j = 1, 2, \dots, m-1. \end{cases}$$

If we let $\alpha := \frac{\epsilon - a_N}{b_N}$ $(\epsilon > 0)$, then

$$\hat k(N-m) = \frac{\epsilon + (b_N - a_N)}{\epsilon\, b_N}.$$

Thus if $a_N \neq b_N$ then we can choose ϵ sufficiently small such that $|\hat k(N-m)| > 1$ and hence $||k||_\infty > 1$, a contradiction. Therefore we must have $a_N = b_N$. Also note that

$$\hat k(N-m) = \left(\frac{\alpha+1}{\bar\alpha+1} \right) \frac{1}{\bar a_N}.$$

We next show that $a_{N-1} = b_{N-1}$ and $a_{-m+1} = b_{-m+1}$. Observe that

$$\hat k(N-m+1) = \frac{1}{(\bar\alpha+1)\, \bar a_N} \left(\alpha\, b_{-m+1} + a_{-m+1} - \left(\frac{\alpha+1}{\bar\alpha+1} \right) \frac{1}{\bar a_N} \left(\bar\alpha\, \bar b_{N-1} + \bar a_{N-1} \right) \right).$$

Assume that $r := (a_{-m+1} - b_{-m+1}) - \left(\bar a_{N-1} - \bar b_{N-1} \right) / \bar a_N \neq 0$. Let $\alpha := -1 + \epsilon$ $(\epsilon > 0)$. Then we have

$$\lim_{\substack{\epsilon \to 0 \\ (\alpha = -1+\epsilon)}} \left(\alpha\, b_{-m+1} + a_{-m+1} - \left(\frac{\alpha+1}{\bar\alpha+1} \right) \frac{1}{\bar a_N} \left(\bar\alpha\, \bar b_{N-1} + \bar a_{N-1} \right) \right) = r.$$

Since $(\bar\alpha+1)\,\bar a_N=\epsilon\,\bar a_N$, we can choose ϵ sufficiently small such that $|\hat k(N-m+1)|>1$, a contradiction. Thus we have

$$a_{-m+1}-b_{-m+1}=\left(\bar a_{N-1}-\bar b_{N-1}\right)/\bar a_N=:s.$$

We now claim that $s=0$. To see this, let $\alpha:=-1+\epsilon\, i$ $(\epsilon>0)$. Thus $\frac{\alpha+1}{\bar\alpha+1}=-1$. Then

$$\begin{aligned}&\lim_{\substack{\epsilon\to 0\\(\alpha=-1+\epsilon\, i)}}\left(\alpha\, b_{-m+1}+a_{-m+1}-\left(\frac{\alpha+1}{\bar\alpha+1}\right)\frac{1}{\bar a_N}\left(\bar\alpha\,\bar b_{N-1}+\bar a_{N-1}\right)\right)\\&=2(a_{-m+1}-b_{-m+1})=2s.\end{aligned}$$

Since $(\bar\alpha+1)\bar a_N=-\epsilon\, i\,\bar a_N$, it follows that if $s\neq 0$ then we can choose ϵ sufficiently small such that $|\hat k(N-m+1)|>1$, a contradiction. Therefore we have

$$a_{N-1}=b_{N-1}\quad\text{and}\quad a_{-m+1}=b_{-m+1}.$$

Next observe that

$$\begin{aligned}\hat k(N-m+2)=\frac{1}{(\bar\alpha+1)\,\bar a_N}\Big(\alpha\, b_{-m+2}+a_{-m+2}-\hat k(N-m)\left(\bar\alpha\,\bar b_{N-2}+\bar a_{N-2}\right)\\-\hat k(N-m+1)\big(\bar\alpha\,\bar b_{N-1}+\bar a_{N-1}\big)\Big).\end{aligned}$$

Since $\lim_{\alpha\to-1}(\bar\alpha\,\bar b_{N-1}+\bar a_{N-1})=0$, once again it follows that

$$\lim_{\substack{\alpha\to-1\\(\alpha\in\mathbb{C})}}\left(\alpha\, b_{-m+2}+a_{-m+2}-\left(\frac{\alpha+1}{\bar\alpha+1}\right)\frac{1}{\bar a_N}\left(\bar\alpha\,\bar b_{N-2}+\bar a_{N-2}\right)\right)=0.$$

A similar argument with $-m+2$ and $N-2$ in place of, respectively, $-m+1$ and $N-1$ gives $a_{N-2}=b_{N-2}$ and $a_{-m+2}=b_{-m+2}$. Continuing this process with $\hat k(N-m+3),\dots,\hat k(N-1)$ gives $a_k=b_k$ for $k=-m,\dots,-1$ and for $k=N-m+1,\dots,N$. This proves the implication (ii)⇒(iii).

(iii)⇒(i): Suppose $\varphi-c\,\psi=\sum_{j=0}^{N-m}d_jz^j$. We can then write

$$\varphi(z)=\sum_{n=-m}^{N}a_nz^n\quad\text{and}\quad c\,\psi=\varphi+\sum_{n=0}^{N-m}(b_n-a_n)\quad\text{for some } b_0,\dots,b_{N-m}\in\mathbb{C}.$$

Since the hyponormality of the pair is independent of a scalar multiple of each coordinate, we may assume that $c=1$. Write

$$f:=\sum_{n=1}^{m}a_{-n}z^{-n}\quad\text{and}\quad g:=\sum_{n=0}^{N}a_nz^n.$$

If $\varphi_0 := f + T_{\bar{z}^{N-m}} g$, then by Lemma 1.5 T_{φ_0} is hyponormal, so $[T^*_{\varphi_0}, T_{\varphi_0}] \geq 0$. Observe that if $\mathbf{T} = (T_\psi, T_\varphi)$, then applying Lemma 2.5 gives

(2.6.2)

$$\begin{aligned}[\mathbf{T}^*, \mathbf{T}] &= \begin{pmatrix} [T^*_\psi, T_\psi] & [T^*_\varphi, T_\psi] \\ [T^*_\psi, T_\varphi] & [T^*_\varphi, T_\varphi] \end{pmatrix} \\ &= \begin{pmatrix} [T^*_{\varphi_0}, T_{\varphi_0}] & [T^*_{\varphi_0}, T_{\varphi_0}] \\ [T^*_{\varphi_0}, T_{\varphi_0}] & [T^*_{\varphi_0}, T_{\varphi_0}] \end{pmatrix} + \begin{pmatrix} \sum_{k=1}^{N-m} \left(B^*_k B_k \oplus 0_\infty\right) & \sum_{k=1}^{N-m} \left(B^*_k A_k \oplus 0_\infty\right) \\ \sum_{k=1}^{N-m} \left(A^*_k B_k \oplus 0_\infty\right) & \sum_{k=1}^{N-m} \left(A^*_k A_k \oplus 0_\infty\right) \end{pmatrix},\end{aligned}$$

where $A_k := (\bar{a}_k, \bar{a}_{k+1}, \ldots, \bar{a}_N)$ and $B_k := (\bar{b}_k, \ldots, \bar{b}_{N-m}, \bar{a}_{N-m+1}, \ldots, \bar{a}_N)$. Note that the second matrix in (2.6.2) is unitarily equivalent to the following matrix:

$$\sum_{k=1}^{N-m} \left[\begin{pmatrix} B^*_k B_k & B^*_k A_k \\ A^*_k B_k & A^*_k A_k \end{pmatrix} \oplus 0_\infty \right].$$

Since for every $k = 1, \ldots, N-m$,

$$\begin{pmatrix} B^*_k B_k & B^*_k A_k \\ A^*_k B_k & A^*_k A_k \end{pmatrix} = \begin{pmatrix} B_k & A_k \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} B_k & A_k \\ 0 & 0 \end{pmatrix} \tag{2.6.3}$$

and since the first term in the right-hand side of (2.6.2) is also positive, we can conclude that (T_ψ, T_φ) is hyponormal.

Assertion (2) follows from (1) and the fact that those coefficients are independent of the hyponormality of T_ψ and T_φ, respectively.

To establish assertion (3), write

$$[T^*_{\varphi_0}, T_{\varphi_0}] := C \oplus 0_\infty,$$

where C is an $m \times m$ matrix. Then

$$[T^*_\varphi, T_\varphi] = \left[\left(C \oplus 0_{N-m}\right) + \sum_{k=1}^{N-m} \left(A^*_k A_k \oplus 0_{k-1}\right) \right] \bigoplus 0_\infty.$$

By a similar argument we have, by (2.6.2),

(2.6.4)

$$[\mathbf{T}^*, \mathbf{T}] = \left[\left(\begin{array}{cc:cc} C & 0 & C & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline C & 0 & C & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) + \sum_{k=1}^{N-m} \begin{pmatrix} B^*_k B_k \oplus 0_{k-1} & B^*_k A_k \oplus 0_{k-1} \\ A^*_k B_k \oplus 0_{k-1} & A^*_k A_k \oplus 0_{k-1} \end{pmatrix} \right] \bigoplus 0_\infty,$$

where every block of the first matrix in the square bracket is an $N \times N$ matrix. Since, for every $k = 1, \ldots, N-m$, $B^*_k B_k, B^*_k A_k, A^*_k B_k$ and $A^*_k A_k$ are $(N-k+1) \times (N-k+1)$ matrices of rank 1 (see Lemma 2.5), a straightforward calculation via elementary row operations applied *simultaneously* to every $(N-k+1)$ row $(k=1,\dots,N-m)$ shows that

$$\text{rank}\left[\sum_{k=1}^{N-m}\begin{pmatrix} B_k^*B_k\oplus 0_{k-1} & B_k^*A_k\oplus 0_{k-1}\\ A_k^*B_k\oplus 0_{k-1} & A_k^*A_k\oplus 0_{k-1}\end{pmatrix}\right]=\text{rank}\begin{pmatrix} 0 & 0 & \vdots & 0 & 0\\ D & E & \vdots & F & E\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \vdots & 0 & 0\\ D & E & \vdots & F & E\end{pmatrix},$$

where D and F are $(N-m)\times m$ Toeplitz matrices, and E is a $(N-m)\times(N-m)$ lower triangular Toeplitz matrix whose diagonal entry is $\bar a_N$; more concretely,

$$D=\begin{pmatrix} \bar b_{N-m} & \bar a_{N-m+1} & \dots & \bar a_{N-1}\\ \bar b_{N-m-1} & \ddots & \ddots & \vdots\\ \vdots & \ddots & & \\ \bar b_1 & \dots & & \end{pmatrix},$$

$$F=\begin{pmatrix} \bar a_{N-m} & \bar a_{N-m+1} & \dots & \bar a_{N-1}\\ \bar a_{N-m-1} & \ddots & \ddots & \vdots\\ \vdots & \ddots & & \\ \bar a_1 & \dots & & \end{pmatrix}$$

and

$$E=\begin{pmatrix} \bar a_N & & & \\ \bar a_{N-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ & \dots & \bar a_{N-1} & \bar a_N\end{pmatrix}.$$

Note that the above mentioned elementary row operations do not affect the matrix C. Evidently, $\text{rank}\,E=N-m$. Therefore we have

$$\begin{aligned}\text{rank}\,[\mathbf{T}^*,\mathbf{T}]&=\text{rank}\left[\begin{pmatrix} C & 0 & C & 0\\ 0 & 0 & 0 & 0\\ C & 0 & C & 0\\ 0 & 0 & 0 & 0\end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & 0\\ D & E & F & E\\ 0 & 0 & 0 & 0\\ D & E & F & E\end{pmatrix}\right]\\ &=\text{rank}\begin{pmatrix} C & 0 & C & 0\\ D & E & F & E\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}=\text{rank}\begin{pmatrix} C & 0\\ D & E\end{pmatrix}\\ &=\text{rank}\,C+\text{rank}\,E\\ &=\text{rank}\,[T_{\varphi_0}^*,T_{\varphi_0}]+(N-m).\end{aligned}$$

Moreover,

$$\text{rank}\,[T_\varphi^*,T_\varphi]=\text{rank}\begin{pmatrix} C & 0\\ F & E\end{pmatrix}=\text{rank}\begin{pmatrix} C & 0\\ D & E\end{pmatrix}=\text{rank}\,[T_\psi^*,T_\psi].$$

This completes the proof. $\square$

Corollary 2.7. *Let $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$ and $\psi(z) = \sum_{n=-N}^{N} b_n z^n$, where a_{-N} and b_{-N} are nonzero. Then (T_ψ, T_φ) is hyponormal if and only if φ is a linear function of ψ.*

Proof. This immediately follows from Theorem 2.6. $\square$

Remark 2.8. From Lemma 1.4(ii), we know that if $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is such that T_φ is hyponormal then $N - m \le \text{rank}\,[T_\varphi^*, T_\varphi] \le N$. The second equality of (2.6.1), however, gives a more concrete formula for the self-commutator of T_φ: i.e., if $\varphi = \bar f + g$ for some $f, g \in H^2$, then

$$\text{rank}\,[T_\varphi^*, T_\varphi] = N - m + \text{rank}\,\left[T^*_{\bar f + T_{\bar z^{N-m}} g},\ T_{\bar f + T_{\bar z^{N-m}} g}\right]. \tag{2.8.1}$$

Furthermore, from (2.8.1), we can recapture the second assertion of Lemma 1.4(iii) because $\bar f + T_{\bar z^{N-m}} g$ is independent of the values of $a_0, a_1, \dots, a_{N-m}$. In particular if $|a_{-m}| = |a_N|$ and T_φ is hyponormal then by Lemma 1.4(v), $T_{\bar f + T_{\bar z^{N-m}} g}$ is normal, so $\text{rank}\,[T_\varphi^*, T_\varphi] = N - m$. Therefore we again obtain the second assertion of Lemma 1.4(iv).

We are now ready for the proof of Corollary 1.11(iv) in Chapter 1.

Corollary 2.9. *Let φ be an arbitrary trigonometric polynomial and let p be an analytic polynomial of degree less than or equal to the analytic degree of φ. If the pair $\mathbf{T} = (T_p, T_\varphi)$ is hyponormal then* $\text{rank}\,[\mathbf{T}^*, \mathbf{T}] = \text{rank}\,[T_\varphi^*, T_\varphi]$.

Proof. Suppose $\varphi(z) = \sum_{k=-m}^{N} a_k z^k$ and $p(z) = \sum_{k=0}^{n} b_k z^k$ $(n \le N)$ are such that (T_ψ, T_φ) is hyponormal. By Corollary 1.11(i), we have $n \le N - m$. Let $\varphi_0 := f + T_{\bar z^{N-m}} g$,

$$A_k := (\bar a_k, \bar a_{k+1}, \dots, \bar a_N) \in \mathbb{C}^{N-k+1} \quad \text{and} \quad B_k := (\bar b_k, \bar b_{k+1}, \dots, \bar b_n) \in \mathbb{C}^{n-k+1}.$$

As in the proofs of Lemma 2.5 and Theorem 2.6, we then have

$$\begin{aligned}
[T_\varphi^*, T_\varphi] &= [T_{\varphi_0}^*, T_{\varphi_0}] + \sum_{k=1}^{N-m} \big(A_k^* A_k \oplus 0_\infty\big);\\
[T_\varphi^*, T_p] &= \sum_{k=1}^{n} \big(B_k^* A_k \oplus 0_\infty\big);\\
[T_p^*, T_p] &= \sum_{k=1}^{n} \big(B_k^* B_k \oplus 0_\infty\big).
\end{aligned}$$

Since $n \le N - m$, we can write

$$\begin{aligned}
[\mathbf{T}^*, \mathbf{T}] = &\begin{pmatrix} [T_{\varphi_0}^*, T_{\varphi_0}] + \sum_{k=n+1}^{N-m} \big(A_k^* A_k \oplus 0_\infty\big) & 0 \\ 0 & 0 \end{pmatrix}\\
&+ \begin{pmatrix} \sum_{k=1}^{n} (A_k^* A_k \oplus 0_\infty) & \sum_{k=1}^{n} (A_k^* B_k \oplus 0_\infty) \\ \sum_{k=1}^{n} (B_k^* A_k \oplus 0_\infty) & \sum_{k=1}^{n} (B_k^* B_k \oplus 0_\infty) \end{pmatrix}.
\end{aligned}$$

Arguing as in the proof of Theorem 2.6, we obtain

$$\begin{aligned}\operatorname{rank}[\mathbf{T}^*,\mathbf{T}] &= \operatorname{rank}\left([T^*_{\varphi_0},T_{\varphi_0}] + \sum_{k=1}^{N-m}\left(A_k^*A_k\oplus 0_\infty\right)\right)\\ &= \operatorname{rank}[T^*_{\varphi_0},T_{\varphi_0}] + N-m\\ &= \operatorname{rank}[T^*_{\varphi},T_{\varphi}].\end{aligned}$$

□

Remark 2.10 (Second "Joint" Version of Cowen's Theorem). Theorem 2.6 shows that if φ and ψ are non-analytic trigonometric polynomials and (T_ψ, T_φ) is hyponormal, then the set

$$\mathcal{E}(\varphi,\psi) := \left\{k\in H^\infty : ||k||_\infty\le 1,\ \begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}k & 0\\ 0 & ke^{i\theta}\end{pmatrix}\begin{pmatrix}\bar\varphi\\ \bar\psi\end{pmatrix}\in H^\infty\oplus H^\infty \right.$$
$$\left.\text{for some } \theta\in[0,2\pi)\right\}$$

is nonempty.

However the converse of Remark 2.10 is not true in general. For example, consider the following trigonometric polynomials:

$$\varphi(z) = z^{-2}+2z^2 \quad\text{and}\quad \psi(z) = z^{-2}+z^{-1}+2z+2z^2.$$

Then T_φ and T_ψ are hyponormal, because

$$[T^*_\varphi,T_\varphi] = \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}\oplus 0_\infty \quad\text{and}\quad [T^*_\psi,T_\psi] = \begin{pmatrix}6 & 3\\ 3 & 3\end{pmatrix}\oplus 0_\infty.$$

Choose $k(z) := \frac{1}{2}$. Then

$$\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\begin{pmatrix}\bar\varphi\\ \bar\psi\end{pmatrix} = \begin{pmatrix}\frac{3}{2}z^2\\ \frac{3}{2}z+\frac{3}{2}z^2\end{pmatrix} \ \in\ H^\infty\oplus H^\infty,$$

which implies that $\mathcal{E}(\varphi,\psi)\ne\emptyset$. By Corollary 2.7, however, (T_ψ,T_φ) is not hyponormal. In fact, the self-commutator of $\mathbf{T}=(T_\psi,T_\varphi)$ is not positive:

$$[\mathbf{T}^*,\mathbf{T}] = \begin{pmatrix}3 & 0 & 3 & -1\\ 0 & 3 & 4 & 3\\ 3 & 4 & 6 & 3\\ -1 & 3 & 3 & 3\end{pmatrix}\ \oplus\ 0_\infty.$$

We now summarize our results on hyponormality of trigonometric Toeplitz pairs.

Hyponormality of Trigonometric Toeplitz Pairs. Let φ and ψ be trigonometric polynomials of analytic degrees N and n ($n \le N$) and co-analytic degrees m and l, respectively. Then the hyponormality of the Toeplitz pair $\mathbf{T} = (T_\psi, T_\varphi)$ can be described as follows:

(i) If $l = m = 0$, then $\mathbf{T}$ is necessarily subnormal (Corollary 1.11(ii));
(ii) If $l = 0$ and $m \neq 0$, then $\mathbf{T}$ is hyponormal if and only if T_φ is hyponormal and $n \le N - m$ (Corollary 1.11(i));
(iii) If $l \neq 0$ and $m = 0$, then $\mathbf{T}$ is not hyponormal (Corollary 1.11(i));
(iv) If $l \neq 0$ and $m \neq 0$, then $\mathbf{T}$ is hyponormal if and only if T_φ is hyponormal and $\varphi - c\,\psi = \sum_{j=0}^{N-m} d_j z^j$ for some $c, d_0, \dots, d_{N-m} \in \mathbb{C}$ (Theorem 2.6).

In the cases where $\mathbf{T}$ is hyponormal, the rank of the self-commutator of $\mathbf{T}$ equals the rank of the self-commutator of T_φ. Moreover weak hyponormality and hyponormality for $\mathbf{T}$ are equivalent properties.

We wish to note that the above characterization of hyponormality for trigonometric Toeplitz pairs can be extended to trigonometric Toeplitz n-tuples.

Corollary 2.11 (Hyponormality of Trigonometric Toeplitz n-Tuples). *Let $\mathbf{T} = (T_{\varphi_1}, \dots, T_{\varphi_n})$ be an n-tuple of trigonometric Toeplitz operators. Then the following three statements are equivalent.*

(i) *The tuple $\mathbf{T}$ is hyponormal.*
(ii) *Every subpair of $\mathbf{T}$ is hyponormal.*
(iii) *The symbols φ_i satisfy the following properties:*
  (a) *All non-analytic trigonometric polynomials φ_i are of the form $\varphi_i(z) = \sum_{k=-m}^{N} a_k z^k$ (where a_{-m} and a_N are nonzero), every T_{φ_i} is hyponormal, and for every pair $\{\varphi_i, \varphi_j\}$ ($i \neq j$) we have $\varphi_i - c\,\varphi_j = \sum_{k=0}^{N-m} d_k z^k$ for some $c, d_0, \dots, d_{N-m} \in \mathbb{C}$.*
  (b) $\max\{\deg(\varphi_i) : \varphi_i$ *is an analytic polynomial*$\} \le N - m$.

Proof. The implication (i)⇒(ii) is evident, and the implication (ii)⇒(iii) follows from the above characterization. For the implication (iii)⇒(i), we shall use the notion of flatness for hermitian matrices. To see this assume first that the φ_i's are non-analytic trigonometric polynomials. Then with the same notations as in the proof of Theorem 2.6, we can write

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_{\varphi_1}^*, T_{\varphi_1}] & [T_{\varphi_2}^*, T_{\varphi_1}] & \dots & [T_{\varphi_n}^*, T_{\varphi_1}] \\ [T_{\varphi_1}^*, T_{\varphi_2}] & [T_{\varphi_2}^*, T_{\varphi_2}] & \dots & [T_{\varphi_n}^*, T_{\varphi_2}] \\ \vdots & \vdots & \ddots & \vdots \\ [T_{\varphi_1}^*, T_{\varphi_n}] & [T_{\varphi_2}^*, T_{\varphi_n}] & \dots & [T_{\varphi_n}^*, T_{\varphi_n}] \end{pmatrix}$$

$$= \left[\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \bigotimes \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} + \sum_{k=1}^{N-m} \begin{pmatrix} A_{1k}^* A_{1k} \oplus 0_{k-1} & \dots & A_{1k}^* A_{nk} \oplus 0_{k-1} \\ A_{2k}^* A_{1k} \oplus 0_{k-1} & \dots & A_{2k}^* A_{nk} \oplus 0_{k-1} \\ \vdots & \ddots & \vdots \\ A_{nk}^* A_{1k} \oplus 0_{k-1} & \dots & A_{nk}^* A_{nk} \oplus 0_{k-1} \end{pmatrix} \right] \bigoplus 0_\infty$$

(cf. (2.6.4)). (Here for every $1 \le j \le n$, $A_{jk} := \big(\bar{\hat{\varphi}}_j(k), \bar{\hat{\varphi}}_j(k+1), \dots, \bar{\hat{\varphi}}_j(N)\big)$, where $\hat{\varphi}_j(k)$ denotes the k-th Fourier coefficient of φ_j.) Arguing as in the proof of Theorem 2.6, we obtain that $\operatorname{rank}[\mathbf{T}^*, \mathbf{T}] = N - m$. Therefore $[\mathbf{T}^*, \mathbf{T}]$ is a flat (i.e., rank-preserving) extension of $[T^*_{\varphi_1}, T_{\varphi_1}]$, and hence $[\mathbf{T}^*, \mathbf{T}] \ge 0$. If φ_i is an analytic polynomial for some $i = 1, \dots, n$, then apply the argument in the proof of Corollary 2.9. □

One might conjecture that the conditions (i) and (ii) in Corollary 2.11 are equivalent for every tuple of operators. However, this is not the case, as the following example, which uses weighted shifts, shows. The structure of k-hyponormal weighted shifts has been studied in [**Cu1**], [**Cu2**], [**CF1**], [**CF2**], [**CF3**], [**Fa2**], and [**St**]; we will use the techniques in those papers to construct our example. Let W_α be the unilateral weighted shift defined by $W_\alpha e_n := \alpha_n e_{n+1}$ $(n \ge 0)$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ_2 and $\{\alpha_n\}_{n=0}^\infty$ is a bounded sequence of positive numbers called the "weight sequence" (or "weights"). Our example uses the weighted shift W_x with weights $\alpha_0 = \sqrt{x}$, $\alpha_1 = \sqrt{\frac{2}{3}}$, $\alpha_2 = \sqrt{\frac{3}{4}}$, $\alpha_3 = \sqrt{\frac{4}{5}}, \dots$. This shift is a close relative of the Bergman shift B_+ (which corresponds to the case $x = \frac{1}{2}$). We now have:

Example 2.12. *For $x > 0$, let W_x be the unilateral weighted shift whose weights are given by $\alpha_0 := \sqrt{x}$, $\alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \ge 1)$. Then for $\frac{8}{15} < x \le \frac{9}{16}$, the triple (W_x, W_x^2, W_x^3) is not hyponormal while all subpairs are hyponormal. More precisely,*

- (i) *W_x is subnormal $\Longleftrightarrow 0 < x \le \frac{1}{2}$;*
- (ii) *W_x is 3-hyponormal $\Longleftrightarrow 0 < x \le \frac{8}{15}$;*
- (iii) *W_x is 2-hyponormal $\Longleftrightarrow 0 < x \le \frac{9}{16}$;*
- (iv) *(W_x, W_x^3) is hyponormal $\Longleftrightarrow 0 < x \le \frac{32}{55}$;*
- (v) *(W_x^2, W_x^3) is hyponormal $\Longleftrightarrow 0 < x \le \frac{2}{3}$.*

Proof. Assertions (i), (ii), and (iii) are obtained in [**Cu1**]; we shall establish here (iv) and (v), using the technique in [**Cu1**].

(iv) In general, suppose T is a unilateral weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. Applying Lemma 1.1 we see that (T, T^3) is hyponormal if and only if for some contraction D,

$$[T^{*3}, T] = [T^*, T]^{\frac{1}{2}}\, D\, [T^{*3}, T^3]^{\frac{1}{2}}.$$

Observe that both $[T^*, T]$ and $[T^{*3}, T^3]$ are diagonal, and that $[T^{*3}, T]$ is a backward weighted shift of multiplicity 2. Thus D must be a backward weighted shift of multiplicity 2, so it suffices to check the $(k, k+2)$-entries of the matrix D. Thus (T, T^3) is hyponormal if and only if

$$\big|\big([T^{*3}, T]\, e_{k+2},\, e_k\big)\big|^2 \le \big([T^*, T]\, e_k,\, e_k\big)\big([T^{*3}, T^3]\, e_{k+2},\, e_{k+2}\big), \tag{2.12.1}$$

or equivalently,

$$\alpha_k^2 \alpha_{k+1}^2 \left(\alpha_{k+2}^2 - \alpha_{k-1}^2\right)^2 \le \left(\alpha_k^2 - \alpha_{k-1}^2\right)\left(\alpha_{k+2}^2 \alpha_{k+3}^2 \alpha_{k+4}^2 - \alpha_{k+1}^2 \alpha_k^2 \alpha_{k-1}^2\right) \tag{2.12.2}$$

for all $k \geq 0$.

For $k \geq 2$ the inequality in (2.12.2) is always true for W_x with no restriction on x because W_x is subnormal when $\alpha_0 = \sqrt{\frac{1}{2}}$. (Note that the inequality in (2.12.2) is independent of the value of α_0 whenever $k \geq 2$).

For $k = 0$ we have

$$\begin{aligned} &\alpha_0^2\alpha_1^2\big(\alpha_2^2 - \alpha_{-1}^2\big)^2 \leq \big(\alpha_0^2 - \alpha_{-1}^2\big)\big(\alpha_2^2\alpha_3^2\alpha_4^2 - \alpha_1^2\alpha_0^2\alpha_{-1}^2\big) \\ \iff\ & x \cdot \frac{2}{3}\left(\frac{3}{4}\right)^2 \leq x \cdot \frac{1}{2} \qquad (\alpha_{-1} = 0), \end{aligned}$$

which implies that (2.12.2) is true for every $x > 0$.

Finally, for $k = 1$ we have

$$\begin{aligned} &\alpha_1^2\alpha_2^2\big(\alpha_3^2 - \alpha_0\big)^2 \leq \big(\alpha_1^2 - \alpha_0^2\big)\big(\alpha_3^2\alpha_4^2\alpha_5^2 - \alpha_2^2\alpha_1^2\alpha_0^2\big) \\ \iff\ & \frac{1}{2}\cdot\frac{3}{4}\left(\frac{4}{5} - x\right)^2 \leq \left(\frac{2}{3} - x\right)\left(\frac{4}{7} - \frac{x}{2}\right) \\ \iff\ & x \leq \frac{32}{55}, \end{aligned}$$

which proves (iv).

(v) Observe first that if T is a unilateral weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$, then $[T^{*3}, T^2]$ is a backward weighted shift of multiplicity 1. Thus by the argument used in the proof of (iv) above, we have (T^2, T^3) is hyponormal if and only if

$$\big|\big([T^{*3}, T^2]\, e_{k+1},\, e_k\big)\big|^2 \leq \big([T^{*2}, T^2]\, e_k,\, e_k\big)\big([T^{*3}, T^3]\, e_{k+1},\, e_{k+1}\big), \tag{2.12.3}$$

or equivalently,

$$\begin{aligned} &\big(\alpha_k\alpha_{k+1}^2\alpha_{k+2}^2 - \alpha_k\alpha_{k-1}^2\alpha_{k-2}^2\big)^2 \\ &\quad\leq \big(\alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2\big)\big(\alpha_{k+1}^2\alpha_{k+2}^2\alpha_{k+3}^2 - \alpha_k^2\alpha_{k-1}^2\alpha_{k-2}^2\big) \end{aligned} \tag{2.12.4}$$

for all $k \geq 0$.

For $k \geq 3$ the inequality in (2.12.4) is always true for W_x because W_x is subnormal when $\alpha_0 = \sqrt{\frac{1}{2}}$.

For $k = 0$ we have

$$\begin{aligned} &\alpha_0^2\big(\alpha_1^2\alpha_2^2 - \alpha_{-1}^2\alpha_{-2}^2\big)^2 \leq \big(\alpha_0^2\alpha_1^2 - \alpha_{-1}^2\alpha_{-2}^2\big)\big(\alpha_1^2\alpha_2^2\alpha_3^2 - \alpha_0^2\alpha_{-1}^2\alpha_{-2}^2\big) \\ \iff\ & x\left(\frac{1}{2}\right)^2 \leq \left(x \cdot \frac{2}{3}\right)\left(\frac{2}{5}\right) \qquad (\alpha_{-1} = \alpha_{-2} = 0), \end{aligned}$$

which implies that (2.12.4) is true for every $x > 0$.

For $k = 1$ we have

$$\begin{aligned} &\alpha_1^2\big(\alpha_2^2\alpha_3^2 - \alpha_0^2\alpha_{-1}^2\big)^2 \leq \big(\alpha_1^2\alpha_2^2 - \alpha_0^2\alpha_{-1}^2\big)\big(\alpha_2^2\alpha_3^2\alpha_4^2 - \alpha_1^2\alpha_0^2\alpha_{-1}^2\big) \\ \iff\ & \frac{2}{3}\left(\frac{3}{5}\right)^2 \leq \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \qquad (\alpha_{-1} = 0), \end{aligned}$$

which again implies that (2.12.4) is true for every $x > 0$.

Finally, for $k = 2$ we have

$$\begin{aligned}
&\alpha_2^2\big(\alpha_3^2\alpha_4^2 - \alpha_1^2\alpha_0^2\big)^2 \le \big(\alpha_2^2\alpha_3^2 - \alpha_1^2\alpha_0^2\big)\big(\alpha_3^2\alpha_4^2\alpha_5^2 - \alpha_2^2\alpha_1^2\alpha_0^2\big)\\
\iff\ & \frac{3}{4}\left(\frac{2}{3} - \frac{2}{3}x\right)^2 \le \left(\frac{3}{5} - \frac{2}{3}x\right)\left(\frac{4}{7} - \frac{1}{2}x\right)\\
\iff\ & x \le \frac{2}{3},
\end{aligned}$$

which proves (v). □

J. Stampfli [**St**, Theorem 6] found a propagation phenomenon for subnormal weighted shifts with two (consecutive) equal weights: if T is a subnormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$ and $\alpha_n = \alpha_{n+1}$ for some $n \ge 0$ then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$. In [**Cu1**], as a strategy to construct non-subnormal *polynomially* hyponormal weighted shifts (such weighted shifts have not yet been found concretely even though it is known ([**CP1**], [**CP2**]) that they exist), Stampfli's propagation was extended as follows. Let T be a unilateral weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$; the following statements hold.

(i) If T is 2-hyponormal and $\alpha_n = \alpha_{n+1}$ for some n, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., T is subnormal.

(ii) If T is *quadratically* hyponormal (i.e., (T, T^2) is weakly hyponormal) and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some n, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.

In fact, propagation also occurs in the hyponormality of (T, T^n) for every $n \ge 2$. To see this we first prove a characterization of the hyponormality of (T, T^n), where T is a unilateral weighted shift.

Proposition 2.13 (cf. [Cu1, Corollary 5]). *Let T be a hyponormal weighted shift with weight sequence $\{\alpha_i\}_{i=0}^\infty$ and let $n \ge 2$. The following statements are equivalent.*

(i) *(T, T^n) is hyponormal;*

(ii) *For all $k \ge 1$,*

$$\left(\prod_{j=k}^{k+n-2}\alpha_j^2\right)\left(\alpha_{k+n-1}^2 - \alpha_{k-1}^2\right)^2 \le \left(\alpha_k^2 - \alpha_{k-1}^2\right)\left(\prod_{j=k+n-1}^{k+2n-2}\alpha_j^2 - \prod_{j=k-1}^{k+n-2}\alpha_j^2\right);$$

(iii) *For all $k \ge 1$,*

$$\begin{aligned}
&\alpha_{k-1}^2\left(\prod_{j=k+1}^{k+n-2}\alpha_j^2\right)\left(\alpha_{k+n-1}^2 - \alpha_k^2\right)^2\\
&\qquad\qquad \le \alpha_{k+n-1}^2\left(\alpha_k^2 - \alpha_{k-1}^2\right)\left(\prod_{j=k+n}^{k+2n-2}\alpha_j^2 - \prod_{j=k+1}^{k+n-1}\alpha_j^2\right).
\end{aligned}$$

Proof. By the argument used in the proof of (2.12.1), (T, T^n) is hyponormal for some $n \geq 2$ if and only if

$$|([T^{*n}, T]e_{k+n-1}, e_k)|^2 \leq ([T^*, T]e_k, e_k)\,([T^{*n}, T^n]e_{k+n-1}, e_{k+n-1}).$$

Now a straightforward calculation gives the equivalence (i) $\Leftrightarrow$ (ii). Also, (iii) is just (ii) suitably rewritten. $\square$

Condition (iii) in Proposition 2.13 exhibits a two-way propagation phenomenon: Let $k \geq 2$. If $\alpha_{k-1} = \alpha_k$ then $\alpha_k = \alpha_{k+n-1}$, so $\alpha_k = \alpha_{k+1}$ (Outer Propagation), and if $\alpha_{k+1} = \alpha_{k+2}$ then by Outer Propagation, $\prod_{j=k+1}^{k+n-1} \alpha_j^2 = \prod_{j=k+n}^{k+2n-2} \alpha_j^2$, so $\alpha_k = \alpha_{k+n-1}$ and hence $\alpha_k = \alpha_{k+1}$ (Inner Propagation). We record this fact.

Corollary 2.14 (Propagation; cf. [Cu1, Corollary 6]). *Suppose T is a hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. If (T, T^n) is hyponormal for some $n \geq 2$ and $\alpha_k = \alpha_{k+1}$ for some k, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., T is subnormal.*

In [**Cu1**], it was shown that if T is as in Example 2.12 then T is quadratically hyponormal if and only if $0 < x \leq \frac{2}{3}$. Thus, in light of Example 2.12, Corollary 2.14 and the remarks following Example 2.12, it is natural to ask a question on the relationship between the quadratic hyponormality of T and the hyponormality of (T, T^n) $(n \geq 2)$: *If T is a unilateral weighted shift, does it follow that*

(i) (T, T^n) *hyponormal* $\Longrightarrow (T, T^{n+1})$ *hyponormal for every* $n \geq 2$ *?*
(ii) (T, T^n) *hyponormal for some* $n \geq 2 \Longrightarrow T$ *quadratically hyponormal ?*

We have not been able to answer these questions; however the following example provides evidence towards affirmative answers.

Example 2.15. *Let W_x be as in Example 2.12 and let*

$$H_{m,n} := \sup\left\{x : (W_x^m, W_x^n) \text{ is hyponormal}\right\} \quad (1 \leq m < n).$$

Then we have

$$H_{m,n} = \frac{(m+1)^2(n+1)^2}{2(m+n+1)(m+n+1+2mn)}. \tag{2.15.1}$$

Moreover we have

(i) $H_{m,n} < H_{m,n+1}$ *for every* m, n*;*
(ii) $\lim_n H_{m,n} = \frac{(m+1)^2}{2(2m+1)}$ *for fixed* m*;*
(iii) (W_x^m, W_x^n) *is hyponormal* $\Longleftrightarrow 0 < x \leq H_{m,n}$*;*
(iv) *If W_x is hyponormal then*

$$(W_x^m, W_x^n) \text{ is hyponormal} \Longleftrightarrow 0 < x \leq \min\{H_{m,n}, \frac{2}{3}\}.$$

Therefore the following implications hold for W_x*: for every* $n \geq 2$,

$$\begin{aligned}(W_x, W_x^n) \text{ is hyponormal} &\Longrightarrow (W_x, W_x^{n+1}) \text{ is hyponormal} \\ &\Longrightarrow W_x \text{ is quadratically hyponormal.}\end{aligned}$$

Proof. By the argument used in the proof of (2.12.3), (W_x^m, W_x^n) is hyponormal for $1 \leq m < n$ if and only if

$$|([W_x^{*n}, W_x^m]\, e_{k+n-m},\, e_k)|^2 \leq ([W_x^{*m}, W_x^m] e_k,\, e_k)\, ([W_x^{*n}, W_x^n]\, e_{k+n-m},\, e_{k+n-m}),$$

or equivalently,

$$\begin{aligned}&\prod_{j=k}^{k+n-m-1} \alpha_j^2 \left(\prod_{j=k+n-m}^{k+n-1} \alpha_j^2 - \prod_{j=k-m}^{k-1} \alpha_j^2 \right)^2 \\ &\qquad \leq \left(\prod_{j=k}^{k+m-1} \alpha_j^2 - \prod_{j=k-m}^{k-1} \alpha_j^2 \right) \left(\prod_{j=k+n-m}^{k+2n-m-1} \alpha_j^2 - \prod_{j=k-m}^{k+n-m-1} \alpha_j^2 \right)\end{aligned} \tag{2.15.2}$$

for all $k \geq 0$. Note that for $k \geq m+1$, the inequality in (2.15.2) is always true for W_x because it is independent of the value of α_0. Thus we need to check only the cases when $k = 0, 1, \ldots, m$.

For $k = 0$ we have

$$\begin{aligned}&(\alpha_0 \cdots \alpha_{n-m-1})^2 \left(\alpha_{n-m}^2 \cdots \alpha_{n-1}^2\right)^2 \leq \left(\alpha_0^2 \cdots \alpha_{m-1}^2\right) \left(\alpha_{n-m}^2 \cdots \alpha_{2n-m-1}^2\right) \\ &\Longleftrightarrow x \cdot \frac{2}{n-m+1} \left(\frac{n-m+1}{n+1} \right)^2 \leq x \cdot \frac{2}{m+1} \cdot \frac{n-m+1}{2n-m+1} \\ &\Longleftrightarrow 0 \leq (n-m)^2,\end{aligned}$$

which implies that (2.15.2) is true for every $x > 0$.

For $1 \leq k \leq m-1$ the same argument used in the case $k = 0$ gives

$$\begin{aligned}&(\alpha_k \cdots \alpha_{k+n-m-1})^2 \left(\alpha_{k+n-m}^2 \cdots \alpha_{k+n-1}^2\right)^2 \\ &\qquad\qquad \leq \left(\alpha_k^2 \cdots \alpha_{k+m-1}^2\right) \left(\alpha_{k+n-m}^2 \cdots \alpha_{k+2n-m-1}^2\right) \\ &\Longleftrightarrow \frac{k}{k+n-m+1} \left(\frac{k+n-m+1}{k+n+1} \right)^2 \leq \frac{k}{k+m+1} \cdot \frac{k+n-m+1}{k+2n-m+1} \\ &\Longleftrightarrow 0 \leq (n-m)^2,\end{aligned}$$

which again implies that (2.15.2) is true for every $x > 0$.

For $k = m$ we have

$$\begin{aligned}&(\alpha_m \cdots \alpha_{n-1})^2 \left(\alpha_n^2 \cdots \alpha_{m+n-1}^2 - \alpha_0^2 \cdots \alpha_{m-1}^2\right)^2 \\ &\qquad\qquad \leq \left(\alpha_m^2 \cdots \alpha_{2m-1}^2 - \alpha_0^2 \cdots \alpha_{m-1}^2\right) \left(\alpha_n^2 \cdots \alpha_{2n-1}^2 - \alpha_0^2 \cdots \alpha_{n-1}^2\right) \\ &\Longleftrightarrow \frac{m+1}{n+1} \left(\frac{n+1}{m+n+1} - \frac{2}{m+1}\, x \right)^2 \\ &\qquad\qquad \leq \left(\frac{m+1}{2m+1} - \frac{2}{m+1}\, x \right) \left(\frac{n+1}{2n+1} - \frac{2}{n+1}\, x \right) \\ &\Longleftrightarrow x \leq \frac{(m+1)^2(n+1)^2}{2(m+n+1)(m+n+1+2mn)},\end{aligned}$$

which proves (2.15.1). The remaining assertions immediately follow from elementary calculations. □

Example 2.15 shows that if $0 < x \le H_{1,2} = \frac{9}{16}$ then (W_x^m, W_x^n) is hyponormal for every m, n. In other words, 2-hyponormality of W_x implies hyponormality of (W_x^m, W_x^n) for every m, n.

We conclude this chapter with a remark about non-subnormal polynomially hyponormal weighted shifts. In Example 2.12 we showed that even though all subpairs of $\mathbf{T} := (T, T^2, T^3)$ are hyponormal, $\mathbf{T}$ need not be hyponormal. We have not been able to decide, however, whether the following question has an affirmative answer: If all subpairs of $\mathbf{T} = (T, T^2, \dots, T^k)$ are hyponormal, does it follow that $\mathbf{T}$ is weakly hyponormal?

If this were true, then in Example 2.12 we would have

$$0 < x \le \frac{9}{16} \implies W_x \text{ is } \textit{cubically} \text{ hyponormal} \tag{2.15.3}$$
$$\text{(i.e., } (W_x, W_x^2, W_x^3) \text{ is weakly hyponormal)}$$

and furthermore, via Example 2.15,

$$0 < x \le \frac{9}{16} \implies W_x \text{ is polynomially hyponormal.} \tag{2.15.4}$$

(T is said to be *polynomially hyponormal* if $(T, T^2, \dots, T^k)$ is weakly hyponormal for every k.) We don't know if either (2.15.3) or (2.15.4) holds. If (2.15.4) were true, Examples 2.12 and 2.15 would show that $W_{\frac{9}{16}}$ provides an example of a unilateral shift which is not subnormal (even not 3-hyponormal) yet polynomially hyponormal.

CHAPTER 3

THE GAP BETWEEN 2–HYPONORMALITY AND SUBNORMALITY

The Bram-Halmos characterization of subnormality indicates that 2 - hyponormality is generally far from subnormality. There are special classes of operators, however, for which these two notions are equivalent. A trivial example is given by the class of operators whose square is compact. We present here a nontrivial example. Let $W_{\hat{\alpha}} := W_{(\alpha_0,\dots,\alpha_m)^\wedge}$ be the *recursively generated subnormal completion* of α, i.e., $W_{\hat{\alpha}}$ is the subnormal weighted shift with an initial segment of positive weights $\alpha\colon \alpha_0,\dots,\alpha_m$, followed by recursively generated weights (cf. [**CF2**], [**CF3**]). Also, let $W_{x,(\alpha_0,\dots,\alpha_m)^\wedge}$ denote the weighted shift whose weight sequence consists of the initial weight x followed by the weight sequence of $W_{(\alpha_0,\dots,\alpha_m)^\wedge}$.

Example 3.1. *Let W_α be the weighted shift with weights $\alpha\colon \sqrt{x}, (\sqrt{a},\sqrt{b},\sqrt{c})^\wedge$ $(0<a<b<c)$. Then W_α is 2-hyponormal if and only if W_α is subnormal.*

Proof. The weights α_i of W_α are given by (cf. [**CF3**])

$$\alpha_0^2 = x;\ \alpha_1^2 = a;\ \alpha_2^2 = b;\ \alpha_3^2 = c;\ \alpha_{k+1}^2 = \psi_1 + \frac{\psi_0}{\alpha_k^2}\ (k\ge 3),\ \text{where}$$
$$\psi_1 = \frac{b(c-a)}{b-a} \quad \text{and} \quad \psi_0 = -\frac{ab(c-b)}{b-a}.$$

The *moments* of W_α are defined by $\gamma_0 := 1$, $\gamma_n := \alpha_0^2\cdots\alpha_{n-1}^2$ $(n\ge 1)$. Thus $\gamma_0 = 1$, $\gamma_1 = x$, $\gamma_2 = ax$, $\gamma_3 = abx$, $\gamma_4 = abcx, \dots$ etc. We use the criterion of k-hyponormality of weighted shift in [**Cu1**, Theorem 4]. From [**CF3**, Theorem 4.3], it is known that W_α is 2-hyponormal if and only if $0 < x \le \left(\frac{ab(c-b)}{(b-a)^2+b(c-b)}\right)^{\frac{1}{2}}$. Define the $(k+1)\times(l+1)$ "Hankel" matrix $A(n;k;l)$ $(k\le l)$ by

$$A(n;k;l) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+l} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+1+l} \\ \vdots & & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+l} & \cdots & \gamma_{n+k+l} \end{pmatrix}.$$

Then we have

$$W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k;k)\ge 0 \ \text{ for all } n\ge 0.$$

Since W_α is the one step extension of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, it suffices to check the case $n=0$; let $\hat{A}(n;k;l)$ denote the Hankel matrix corresponding to the subnormal completion $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$. Since $0<a<b$, it follows from the recursive relation of the subnormal completion (cf. [**CF3**], [**CF4**]) that

$$\begin{aligned}\operatorname{rank}\hat{A}(0;k;l) &= \operatorname{rank}\hat{A}(0;2;2)\\ &= \operatorname{rank}\begin{pmatrix} 1 & a & ab\\ a & ab & abc\\ ab & abc & \frac{bc^2-2abc+ab^2}{c(b-a)}\end{pmatrix} = 2 \quad \text{for all } k\geq 2.\end{aligned}$$

Note that if $\tilde{B}$ denotes the $(k-1)\times k$ matrix obtained by eliminating the first row of a $k\times k$ matrix B, then

$$\tilde{A}(0;k;k) = x\,\hat{A}(0;k-1;k) \quad \text{for all } k\geq 3.$$

Therefore for every $k\geq 3$, $A(0;k;k)$ is a flat extension of $A(0;2;2)$. This implies that if W_α is 2-hyponormal (and hence $A(0;2;2)\geq 0$) then it is k-hyponormal for every $k\geq 3$, and therefore it is subnormal. □

In [**Hal1**], P.R. Halmos raised the following question: *Is every subnormal Toeplitz operator either normal or analytic?* Cowen and Long [**CoL**] answered this question negatively, by resorting to a Toeplitz operator $T_{\psi+\alpha\bar{\psi}}$, with ψ a continuous symbol which is a conformal map of the unit disk onto the interior of an ellipse. It is also known that the answer to Halmos's question is yes for (i) Toeplitz operators with bounded type symbols (e.g., trigonometric polynomials) [**NT**], and (ii) for quasinormal Toeplitz operators [**AIW**]. We would like to pose the following question:

(3.1.1) *Is every* 2*-hyponormal Toeplitz operator subnormal?*

An affirmative answer to (3.1.1) would show that there exists no gap between 2-hyponormality and subnormality for Toeplitz operators. A negative answer would give rise to a challenging problem:

Characterize non-subnormal k-hyponormal Toeplitz operators.

We have noted above that Nakazi and Takahashi gave an affirmative answer to Halmos's question for trigonometric Toeplitz operators; our next result shows that this is also the case for question (3.1.1).

Theorem 3.2. *Every hyponormal trigonometric Toeplitz operator whose square is hyponormal must be either normal or analytic. Thus in particular every* 2*-hyponormal trigonometric Toeplitz operator is subnormal.*

Remark. A related case of Theorem 3.2 was considered in [**Hal2**, Problem 209], where it was shown that there exists a hyponormal operator whose square is not hyponormal, e.g., U^*+2U, which is a trigonometric Toeplitz operator.

Proof of Theorem 3.2. Suppose T_φ is a hyponormal Toeplitz operator with trigonometric polynomial symbol φ of the form $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$, with a_N nonzero.

By Lemma 1.4(i), $m \le N$ and $|a_{-m}| \le |a_N|$. Without loss of generality we can assume $a_0 = 0$. Write $f := \sum_{n=1}^{m} \bar{a}_{-n}\, z^n$ and $g := \sum_{n=1}^{N} a_n z^n$, so $\varphi = \bar{f} + g$. Since f and g are analytic, it follows that

$$\begin{aligned}
T_\varphi^2 &= \left(T_{\bar{f}^2} + T_{\bar{f}g} + T_{g^2}\right) + T_g\, T_{\bar{f}} \\
&= \left(T_{\bar{f}^2} + T_{\bar{f}g} + T_{g^2}\right) + \left(\sum_{j=1}^{N} a_j U^j\right) T_{\bar{f}} \\
&= \left(T_{\bar{f}^2} + T_{\bar{f}g} + T_{g^2}\right) + \sum_{j=1}^{N} a_j U^j U^{*j} T_{\bar{f}} U^j \\
&= \left(T_{\bar{f}^2} + T_{\bar{f}g} + T_{g^2}\right) + \sum_{j=1}^{N} a_j (I - P_j) T_{\bar{f}} U^j \\
&= \left(T_{\bar{f}^2} + T_{\bar{f}g} + T_{g^2}\right) + T_{\bar{f}g} - \sum_{j=1}^{N} a_j P_j T_{\bar{f}} U^j \\
&= T_{\varphi^2} - \sum_{j=1}^{N} a_j P_j T_{\bar{f}} U^j.
\end{aligned}$$

Write $F := \sum_{j=1}^{N} a_j P_j T_{\bar{f}} U^j$, and hence $T_\varphi^2 = T_{\varphi^2} - F$. Since

$$R(F) \subseteq \bigvee \{e_0, \dots, e_{N-1}\},$$

we have

$$(F(x), e_k) = 0 \qquad \text{for } k \ge N \text{ and for every } x \in H^2. \tag{3.2.1}$$

Similarly since $F^* = \sum_{j=1}^{N} \bar{a}_j U^{*j} T_f P_j = \sum_{j=1}^{N} \bar{a}_j U^{*j} \left(\sum_{k=1}^{m} \bar{a}_{-k} U^k\right) P_j$, we have

$$(F^*(x), e_k) = 0 \qquad \text{for } k \ge m \text{ and for every } x \in H^2. \tag{3.2.2}$$

Also since φ^2 is a trigonometric polynomial of co-analytic and analytic degrees $2m$ and $2N$, respectively, it follows that $[T_{\varphi^2}^*, T_{\varphi^2}]$ is the sum of a $2N \times 2N$ matrix and an infinite zero matrix. Without loss of generality we can assume $m \ne 0$. Then

$$[T_{\varphi^2}^*, T_{\varphi^2}]\, e_{2N+m-1} = 0. \tag{3.2.3}$$

On the other hand, since for every $1 \le j \le N$,

$$\left(T_{\bar{f}} U^j\right)\, e_{2N+m-1} \in \bigvee \{e_{2N-1}, e_{2N}, \dots, e_{2N+m-2+j}\},$$

it follows that

$$\left(P_j T_{\bar{f}} U^j\right)\, e_{2N+m-1} = 0 \ \ (1 \le j \le N), \ \ \text{and hence} \ \ F(e_{2N+m-1}) = 0. \tag{3.2.4}$$

Also,

$$F^*(e_{2N+m-1}) = \sum_{j=1}^{N} \bar{a}_j U^{*j} \left(\sum_{k=1}^{m} \bar{a}_k U^k \right) P_j e_{2N+m-1} = 0. \tag{3.2.5}$$

Therefore from (3.2.1)–(3.2.5), we have

(3.2.6)
$$\left([T_\varphi^{2*}, T_\varphi^2] e_{2N+m-1},\, e_{2N+m-1}\right) = \left([T_{\varphi^2}^* - F^*,\, T_{\varphi^2} - F] e_{2N+m-1},\, e_{2N+m-1}\right) = 0.$$

Similarly, we have

(3.2.7)
$$\begin{aligned}
&\left([T_\varphi^{*2}, T_\varphi^2] e_{2N+m-1},\, e_{N-1}\right)\\
&= \left(([T_{\varphi^2}^*, T_{\varphi^2}] - [F^*, T_{\varphi^2}] - [T_{\varphi^2}^*, F] + [F^*, F]) e_{2N+m-1},\, e_{N-1}\right)\\
&= \left((F T_{\varphi^2}^* - F^* T_{\varphi^2}) e_{2N+m-1}, e_{N-1}\right).
\end{aligned}$$

Since

$$T_{\varphi^2}(e_{2N+m-1}) \in \bigvee \{e_{2N-m-1}, e_{2N-m}, \dots, e_{4N+m-1}\},$$

it follows that $\left(P_j T_{\varphi^2}\right) e_{2N+m-1} = 0$ for every $1 \le j \le 2N - m - 1$. Therefore

$$\begin{aligned}
&\left(F^* T_{\varphi^2}\right) e_{2N+m-1}\\
&= \left(\bar{a}_1 U^* T_f P_1 + \cdots + \bar{a}_N U^{*N} T_f P_N\right) T_{\varphi^2} e_{2N+m-1}\\
&= \begin{cases} 0 & (m < N) \\ \bar{a}_N U^{*N} T_f P_N T_{\varphi^2}(e_{2N+m-1}) = \bar{a}_N a_{-m}^2 U^{*N} T_f(e_{N-1}) & (m = N), \end{cases}
\end{aligned}$$

so

$$\left((F^* T_{\varphi^2}) e_{2N+m-1},\, e_{N-1}\right) = \begin{cases} 0 & (m < N) \\ \bar{a}_N \bar{a}_{-m} a_{-m}^2 & (m = N). \end{cases} \tag{3.2.8}$$

Also since

$$\left(T_{\bar{f}} U^N T_{\varphi^2}^*\right) e_{2N+m-1} \in \bigvee \{e_{N-1}, e_{N-2}, \dots, e_{3N+3m-2}\}$$

and since

$$(P_j(x),\, e_{N-1}) = 0 \qquad \text{for every } 1 \le j \le N-1,$$

it follows that

$$\begin{aligned}
\left((F T_{\varphi^2}^*) e_{2N+m-1},\, e_{N-1}\right) &= a_N \,\left((P_N T_{\bar{f}} U^N T_{\varphi^2}^*) e_{2N+m-1},\, e_{N-1}\right)\\
&= a_N \bar{a}_N^2 a_{-m} \,\left((P_N U^{*m} U^N U^{*2N}) e_{2N+m-1},\, e_{N-1}\right)\\
&= a_N \bar{a}_N^2 a_{-m} (e_{N-1}, e_{N-1}) = a_N \bar{a}_N^2 a_{-m}.
\end{aligned} \tag{3.2.9}$$

Therefore from (3.2.7), (3.2.8), and (3.2.9) we have

$$\left([T_\varphi^{2*}, T_\varphi^2] e_{2N+m-1},\, e_{N-1}\right) = \begin{cases} a_N \bar{a}_N^2 a_{-m} & (m < N) \\ a_N \bar{a}_N^2 a_{-m} - \bar{a}_N \bar{a}_{-m} a_{-m}^2 & (m = N). \end{cases} \tag{3.2.10}$$

Now recall that if $Q \geq 0$ and $(Qe_j, e_j) = 0$ for some $j \geq 0$, then $(Qe_j, e_k) = 0$ for every $k \geq 0$. Thus if T_φ^2 is hyponormal then by (3.2.6) and (3.2.10) we have

$$\begin{cases} a_N \bar{a}_N^2 a_{-m} = 0 & (m < N) \\ |a_N|^2 a_{-m} = |a_{-m}|^2 a_{-m} & (m = N). \end{cases}$$

Therefore we can conclude that either $|a_{-m}| = |a_N|$ or $a_{-m} = 0$. If $|a_{-m}| = |a_N|$ then by Lemma 1.4(v), T_φ is normal. If $a_{-m} = 0$ then induction shows that $a_{-1} = a_{-2} = \cdots = a_{-m} = 0$, so φ is analytic. This completes the proof of the first assertion. The second assertion follows from the first. □

The subnormal Toeplitz operator T_φ which Cowen and Long constructed in [**CoL**] is a unilateral weighted shift with weights $\left\{(1-\alpha^{2n+2})^{\frac{1}{2}}||T_\varphi||\right\}_{n=0}^{\infty}$ for some $0 < \alpha < 1$. Thus one might expect that the idea in [**CoL**] can be used to find a negative answer to (3.1.1), namely by constructing a symbol $\varphi \in L^\infty$ such that T_φ is unitarily equivalent to a weighted shift which is k-hyponormal but not subnormal. This idea doesn't work; the reason is this. In [**Ab**], M. Abrahamse asked "Is the Bergman shift unitarily equivalent to a Toeplitz operator ?". Sun Shunhua [**Sun**] showed that if a Toeplitz operator T_φ is unitarily equivalent to a hyponormal weighted shift W_α with *strictly increasing* weight sequence α, then α must be of the form

$$\alpha = \left\{(1-\beta^{2n+2})^{\frac{1}{2}}||T_\varphi||\right\}_{n=0}^{\infty} \quad \text{for some } \beta \ (0 < \beta < 1), \tag{3.2.11}$$

thus answering Abrahamse's question in the negative. Cowen and Long [**CoL**] showed that a unilateral weighted shift with weight sequence of the form (3.2.11) must be subnormal (also see [**Fa3**]). On the other hand, if W_α is a 2-hyponormal unilateral weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$ and $\alpha_n = \alpha_{n+1}$ for some n, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal (see the remark following Example 2.12). Combining the results in [**Sun**], [**CoL**], and [**Cu1**] gives:

Corollary 3.3. *Every 2-hyponormal Toeplitz operator which is unitarily equivalent to a unilateral weighted shift is subnormal.*

Recall that every Toeplitz operator need not be unitarily equivalent to a unilateral weighted shift because the spectrum of every weighted shift has circular symmetry. Corollary 3.3 thus reduces the question (3.1.1) to the class of Toeplitz operators which contains no weighted shifts in their unitary orbits.

CHAPTER 4

APPLICATIONS

In this chapter we consider the notion of *flatness* for Toeplitz pairs and Toeplitz extensions of positive moment matrices, and we give an application to hyponormality of single Toeplitz operators.

§**4.1. Flatness of Toeplitz pairs.** If $A = A^* \in \mathcal{L}(\mathcal{H}_1)$, then an operator matrix (whose entries have possibly infinite-matrix representations)

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$$

is called an *extension* of A. If A is of finite rank, we refer to a rank-preserving extension $\tilde{A}$ of A as a *flat extension* of A. It is known ([**CF2**]) that if A is of finite rank and $A \geq 0$, then $\tilde{A}$ is a flat extension of A if and only if $\tilde{A}$ is of the form

$$\tilde{A} = \begin{pmatrix} A & AV \\ V^*A & V^*AV \end{pmatrix}$$

for an operator $V : \mathcal{H}_2 \to \mathcal{H}_1$. Moreover $\tilde{A}$ is positive whenever A is positive.

We shall introduce the notion of flatness for a pair of operators.

Definition 4.1. Let $\mathbf{T} = (T_1, T_2)$ be a pair of operators on $\mathcal{H}$. Then we shall say that $\mathbf{T}$ is *a flat pair* if $[\mathbf{T}^*, \mathbf{T}]$ is flat relative to $[T_1^*, T_1]$ or $[T_2^*, T_2]$.

Remark 4.2. The following facts are evident from the definition.

(i) Flatness of (T_1, T_2) is not affected by permuting the operators T_i.
(ii) If $(T_1,\, T_2)$ is flat, then so is $(\lambda_1 T_1,\, \lambda_2 T_2)$ for every $\lambda_1,\, \lambda_2 \in \mathbb{C}$.
(iii) If $(T_1,\, T_2)$ is flat, then so is $(T_1 - \lambda_1 I,\, T_2 - \lambda_2 I)$ for every $\lambda_1, \lambda_2 \in \mathbb{C}$.
(iv) If $S \in \mathcal{L}(\mathcal{H})$ is hyponormal with finite-rank self-commutator then $(\mu_1 S - \mu_2 I,\ \lambda_1 S - \lambda_2 I)$ is flat for every $\lambda_1, \lambda_2 \in \mathbb{C}$.
(v) If T_1 or T_2 is hyponormal and if (T_1, T_2) is flat, then (T_1, T_2) is hyponormal.

In the sequel, for convenience, we will assume that rank $[T_2^*, T_2] \leq$ rank $[T_1^*, T_1] < \infty$ whenever we discuss the flatness of a pair (T_1, T_2).

The following gives a criterion for the flatness of a positive operator matrix whose upper left-hand corner is of finite rank.

Proposition 4.3. *Let $A \geq 0$ be of finite rank and let $\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$. Then $\tilde{A}$ is flat if and only if $C = B^* A^\# B$, where $A^\#$ is the Moore-Penrose inverse of A.*

Proof. Write, as in the proof of Lemma 1.2,

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : R(A) \oplus N(A) \longrightarrow R(A) \oplus N(A),$$

where A_0 is invertible. Then $A^\# = \begin{pmatrix} A_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. But since $\tilde{A} \geq 0$, it follows from [**Smu**] that there exists $V : H_2 \to R(A)$ such that $B = AV$. Since $R(V) \subseteq R(A)$, V is uniquely determined by $V = A^\# B$, so $\tilde{A}$ is flat if and only if $C = V^* A V = B^* A^\# A A^\# B = B^* A^\# B$. □

Corollary 4.4. *If $\mathbf{T} = (T_1, T_2)$ is a hyponormal pair and if $[T_1^*, T_1]$ is of finite rank, then $\mathbf{T}$ is flat if and only if $[T_2^*, T_2] = [T_2^*, T_1]^* \, [T_1^*, T_1]^\# \, [T_2^*, T_1]$.*

Proof. This follows from Proposition 4.1. □

The following theorem is an immediate consequence of the results in Chapter 2.

Theorem 4.5.

(i) *Every hyponormal Toeplitz pair having a normal coordinate is flat.*
(ii) *Every hyponormal trigonometric Toeplitz pair is flat.*
(iii) *Every "jointly quasinormal" pair (T_1, T_2) (i.e., $\{T_1, T_2, T_1^* T_1, T_2^* T_2\}$ is a commutative family) satisfying the inclusion $R[T_2^*, T_2] \subseteq R[T_1^*, T_1]$ and* $\operatorname{rank}[T_1^*, T_1] < \infty$ *must be flat.*

Proof. (i) follows from Theorem 2.2.

(ii) follows from Theorem 2.6(3) and Corollary 2.9.

For (iii), we first assume that $\mathbf{T} = (T_1, T_2)$ is a subnormal pair of operators on $\mathcal{H}$. Suppose N_i is a commuting normal extension of T_i for $i = 1, 2$. Thus we can write

$$N_i = \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix} \quad (i = 1, 2). \tag{4.5.1}$$

Since by Fuglede's Theorem, $N_j^* N_i = N_i N_j^*$, we have $[T_j^*, T_i] = A_i A_j^*$ (cf. [**At**, Proposition 2]), so

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix} = \begin{pmatrix} A_1 A_1^* & A_1 A_2^* \\ A_2 A_1^* & A_2 A_2^* \end{pmatrix}.$$

Since $R[T_2^*, T_2] \subseteq R[T_1^*, T_1]$, it follows that $R(A_2) \subseteq R(A_1)$ and $\operatorname{rank}(A_2 A_2^*) \leq \operatorname{rank}(A_1 A_1^*) (< \infty)$. Since $A_1 A_1^*$ is of finite rank, $A_1 A_1^*$ has a Moore-Penrose inverse $(A_1 A_1^*)^\#$, and hence so have both A_1 and A_1^*: moreover $(A_1 A_1^*)^\# = (A_1^\#)^* A_1^\#$. Since $(A_1^\# A_1)^* = A_1^\# A_1$ (see Lemma 1.2), it follows that

$$\begin{aligned} (A_2 A_1^*) \, (A_1 A_1^*)^\# \, (A_1 A_2^*) &= A_2 \, (A_1^* A_1^{\#*}) \, (A_1^\# A_1) \, A_2^* \\ &= A_2 \, (A_1^\# A_1 A_1^\# A_1) \, A_2^* \\ &= A_2 \, (A_1^\# A_1) \, A_2^*. \end{aligned}$$

Since $A_1^{\#}A_1$ is the projection onto $R(A_1^*)$, it follows that if $R(A_2^*) \subseteq R(A_1^*)$, then $A_2\,(A_1^{\#}A_1)\,A_2^* = A_2A_2^*$, which implies that $[T_2^*, T_2] = [T_2^*, T_1]^*\,[T_1^*, T_1]^{\#}\,[T_2^*, T_1]$. Therefore by Corollary 4.4, $\mathbf{T} = (T_1, T_2)$ is flat. It remains to show that $R(A_2^*) \subseteq R(A_1^*)$. It is here that we use the assumption that (T_1, T_2) is jointly quasinormal. It is known ([**Lu**]) that if (T_1, T_2) is jointly quasinormal then it is subnormal. It is instructive to give a concrete commuting normal extension. For this recall ([**Hal2**, Problem 195]) that if $T_i \in \mathcal{L}(\mathcal{H})$ is a quasinormal operator with polar decomposition $T_i = U_i\,|T_i|$ $(i = 1, 2)$, then

$$N_i = \begin{pmatrix} T_i & (1 - U_iU_i^*)|T_i| \\ 0 & T_i^* \end{pmatrix}$$

is a normal extension of T_i. Then joint quasinormality implies $N_1N_2 = N_2N_1$ (cf. [**Lu**], [**Yo**]), and therefore (N_1, N_2) is a desired commuting normal extension of (T_1, T_2). Thus in (4.5.1) we may choose

$$A_i := (1 - U_iU_i^*)|T_i| \quad \text{and} \quad B_i := T_i^*.$$

Furthermore since quasinormality implies that $|T_i|$ commutes with U_i and U_i^*, it follows that A_i is self-adjoint, and therefore $R(A_2^*) = R(A_2) \subseteq R(A_1) = R(A_1^*)$. This completes the proof. $\square$

One might think that every Toeplitz pair is flat. However, this is not the case; for example, the Toeplitz pair (T_ψ, T_φ) given in Example 2.3 is not flat because by (2.3.1), $[T_\varphi^*, T_\varphi] - [T_\varphi^*, T_\psi^*]^*\,[T_\psi^*, T_\psi]^{\#}\,[T_\varphi^*, T_\psi]$ is of rank one (observe that in this case φ is not a trigonometric polynomial).

§**4.2. Toeplitz extensions of positive moment matrices.** Given a doubly indexed finite sequence of complex numbers $\gamma\colon \gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \dots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the *truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i\, z^j\, d\mu \quad (0 \le i + j \le 2n);$$

μ is called a *representing measure* for γ, which is called a *truncated moment sequence*; the *quadratic moment problem* is the case of the truncated moment problem when $n = 1$ (cf. [**CF4**]). Given $m, n \ge 0$, let $M[m, n]$ be the $(m + 1) \times (n + 1)$ Toeplitz-like matrix whose first row has entries given by $\gamma_{m,n}, \gamma_{m+1,n-1}, \dots, \gamma_{m+n,0}$ and whose first column has entries given by $\gamma_{m,n}, \gamma_{m-1,n+1}, \dots, \gamma_{0,n+m}$. For example, $M[1, 1] = \begin{pmatrix} \gamma_{1,1} & \bar{\gamma}_{0,2} \\ \gamma_{0,2} & \gamma_{1,1} \end{pmatrix}$. Then the *moment matrix* $M(n) \equiv M(n)(\gamma)$ is defined as follows:

$$M(n) := \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,n] \\ M[1,0] & M[1,1] & \dots & M[1,n] \\ \vdots & \vdots & & \vdots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix};$$

for example,

$$M(1) = \begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \bar{\gamma}_{0,1} \\ \bar{\gamma}_{0,1} & \gamma_{1,1} & \bar{\gamma}_{0,2} \\ \gamma_{0,1} & \gamma_{0,2} & \gamma_{1,1} \end{pmatrix}.$$

We shall say that the moment matrix $M(n)$ is *induced* by a trigonometric Toeplitz tuple $\mathbf{T} = (T_{\varphi_1}, \ldots, T_{\varphi_n})$ if $M(n) \oplus 0_\infty = [\mathbf{T}^*, \mathbf{T}]$. In general a moment matrix (even a positive one) need not be induced by a trigonometric Toeplitz tuple. For example if $M(1) := \left(\begin{smallmatrix} r & 1 & 1 \\ 1 & s & 1 \\ 1 & 1 & s \end{smallmatrix}\right)$, then there is a trigonometric polynomial φ such that $M(1) \oplus 0_\infty = [T_\varphi^*, T_\varphi]$ if and only if $r = s = 1$. To see this, observe first that if $M(1)$ is positive and $M(1) \oplus 0_\infty = [T_\varphi^*, T_\varphi]$, then φ must be of the form $\varphi(z) = \sum_{n=-3}^{3} a_n z^n$. (*Proof.* Since $M(1)$ is positive, T_φ is hyponormal. Thus if $\varphi(z) = \sum_{n=-k}^{k} a_n z^n$ $(k > 3)$, then either $[T_\varphi^*, T_\varphi]$ contains a $k \times k$ nonzero matrix if $|a_{-k}| \neq |a_k|$, or $[T_\varphi^*, T_\varphi] = 0$ if $|a_{-k}| = |a_k|$, using Lemma 1.4(v). In either case we get a contradiction.) A straightforward calculation now gives the following equations:

$$\begin{cases} |a_3|^2 - |a_{-3}|^2 = s \\ |a_2|^2 - |a_{-2}|^2 = 0 \\ |a_1|^2 - |a_{-1}|^2 = r - s \\ \bar{a}_3 a_2 - a_{-3}\bar{a}_{-2} = 1 \\ \bar{a}_2 a_1 - a_{-2}\bar{a}_{-1} = 0 \\ \bar{a}_3 a_1 - a_{-3}\bar{a}_{-1} = 1, \end{cases}$$

which admits a solution only if $r = s = 1$. Since $\det M(1) = (s-1)(r + rs - 2)$, it follows that $M(1) \geq 0$ precisely when $s \geq 1$ and $r \geq \frac{2}{1+s}$, and $\operatorname{rank} M(1) = 1$, 2, or 3 depending upon $r = s = 1$, (($s = 1$ and $r > 1$) or ($s > 1$ and $r = \frac{2}{1+s}$)), or $s > 1$ and $r > \frac{2}{1+s}$. Thus, the above system of equations admits a solution precisely when $\operatorname{rank} M(1) = 1$. In general, if $M(1)$ is a rank-one positive moment matrix, then it *can* be induced by a trigonometric Toeplitz operator. To see this, let $M(1)$ be such a moment matrix. Then $M(1)$ can be written in the form

$$M(1) = a \begin{pmatrix} 1 & \alpha & \bar{\alpha} \\ \bar{\alpha} & |\alpha|^2 & \bar{\alpha}^2 \\ \alpha & \alpha^2 & |\alpha|^2 \end{pmatrix} \quad (a, \alpha \in \mathbb{C}).$$

Now if we choose φ of the form

$$\varphi(z) = \sqrt{a}\left(\bar{\alpha} z^{-2} + \alpha z^{-1} + z + \bar{\alpha} z^2 + \alpha z^3\right), \tag{4.5.2}$$

then a straightforward calculation shows that $[T_\varphi^*, T_\varphi] = M(1) \oplus 0_\infty$. This says that every rank-one moment matrix $M(1)$ is induced by a trigonometric Toeplitz operator. Now one might ask whether a positive moment matrix $M(n)$ admits a positive extension $M(n+1)$ induced by a trigonometric Toeplitz pair, i.e., whether for some trigonometric Toeplitz pair (T_ψ, T_φ) we have

$$M(n) = [T_\psi^*, T_\psi]_0 \geq 0 \quad \text{and} \quad M(n+1) = \begin{pmatrix} [T_\psi^*, T_\psi]_0 & [T_\varphi^*, T_\psi]_0 \\ [T_\psi^*, T_\varphi]_0 & [T_\varphi^*, T_\varphi]_0 \end{pmatrix} \geq 0\,.$$

Here $[T_\zeta^*, T_\eta]_0$ denotes the truncated matrix such that $[T_\zeta^*, T_\eta] = [T_\zeta^*, T_\eta]_0 \oplus 0_\infty$, This will be referred to as the (*positive*) *Toeplitz extension problem* for positive moment matrices. In view of Theorem 4.5(ii), every Toeplitz extension is a flat extension. The *quadratic Toeplitz extension problem* is the case of the Toeplitz extension problem when $n = 1$. In this section we give a solution of the quadratic Toeplitz extension problem. To do this we need:

Lemma 4.6. *If A is a finite hermitian Toeplitz matrix whose diagonal entry is positive then there exists a trigonometric polynomial φ such that $[T_\varphi^*, T_\varphi]_0 = A$: more concretely, if A is given by*

$$A \equiv \begin{pmatrix} a_n & a_{n-1} & \dots & a_1 \\ \bar{a}_{n-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1} \\ \bar{a}_1 & \dots & \bar{a}_{n-1} & a_n \end{pmatrix} \qquad (a_n > 0) \tag{4.6.1}$$

and if φ is a trigonometric polynomial of the form

$$\varphi(z) := \frac{e^{i\theta}}{\sqrt{a_n}} \left(a_n z^n + \sum_{k=1}^{n-1} \left(\bar{a}_k z^{-k} + a_k z^k \right) \right) \quad \textit{for some } \theta \in [0, 2\pi), \tag{4.6.2}$$

then $[T_\varphi^*, T_\varphi]_0 = A$. Thus, in particular, we have T_φ is hyponormal if and only if $A \geq 0$; moreover, if $|a_{n-1}| = a_n$, then $A \geq 0$ if and only if $a_k = a_1 e^{i(k-1)\theta}$ $(k = 1, \dots, n;\ \theta \in [0, 2\pi))$.

Proof. Recall that if $\varphi \in L^\infty(\mathbb{T})$ has Fourier coefficients $\hat{\varphi}(n) = b_n$ for every $n \in \mathbb{Z}$, then with respect to the canonical orthonormal basis $[T_\varphi^*, T_\varphi]$ admits a matrix representation whose (μ, ν)-entry is

$$\sum_{j=0}^{\infty} (\bar{b}_{j-\mu} b_{j-\nu} - b_{\mu-j} \bar{b}_{\nu-j}).$$

Thus if φ is given by (4.6.2), and since $b_{-k} = \bar{b}_k$ whenever $k \neq n$, it follows that for $\mu \leq \nu \leq n$ the (μ, ν)-entry of $[T_\varphi^*, T_\varphi]_0$ is $\bar{b}_n b_{n+\mu-\nu}$. Since

$$b_n = \sqrt{a_n} e^{i\theta} \quad \text{and} \quad b_{n+\mu-\nu} = \frac{a_{n+\mu-\nu}}{\sqrt{a_n}} e^{i\theta},$$

we see that the (μ, ν)-entry of $[T_\varphi^*, T_\varphi]_0$ is $a_{n+\mu-\nu}$, which proves the first assertion. The second assertion is evident. For the last assertion, observe from Lemma 1.4(iv) that if $|a_{n-1}| = a_n$, then

$$T_\varphi \text{ is hyponormal} \iff a_n \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{pmatrix} = a_{n-1} \begin{pmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix},$$

which together with the second assertion proves the last assertion. □

We now have:

Theorem 4.7 (Quadratic Toeplitz Extension). *If $M(1)$ is a positive moment matrix, then the following statements are equivalent.*

(i) $M(1)$ *admits a Toeplitz extension* $M(2)$.
(ii) $\operatorname{rank} M(1) = 1$.
(iii) *There exists a unique representing measure; this measure is* 1*-atomic.*

Proof. (i)⇒(ii): The moment matrices $M(1)$ and $M(2)$ are of the form

$$M(1) = \begin{pmatrix} a & d & \bar{d} \\ \bar{d} & b & e \\ d & \bar{e} & b \end{pmatrix} \tag{4.7.1}$$

and

$$M(2) = \begin{pmatrix} a & d & \bar{d} & \vdots & \bar{e} & b & e \\ \bar{d} & b & e & \vdots & f & \bar{f} & g \\ d & \bar{e} & b & \vdots & \bar{g} & f & \bar{f} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ e & \bar{f} & g & \vdots & c & h & k \\ b & f & \bar{f} & \vdots & \bar{h} & c & h \\ \bar{e} & \bar{g} & f & \vdots & \bar{k} & \bar{h} & c \end{pmatrix}. \tag{4.7.2}$$

Assume that $M(1)$ admits a (positive) Toeplitz extension $M(2)$. If $b = 0$, then the positivity of $M(1)$ requires that $d = e = 0$, so evidently, $M(1) = \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ has rank 1. Thus we assume that $b \neq 0$. If ψ is a trigonometric polynomial such that $M(1) = [T_\psi^*, T_\psi]_0 \geq 0$, then by an argument used earlier in this section, ψ is of the form $\psi(z) = \sum_{n=-m}^{3} a_n z^n$ $(m \leq 3)$. Suppose a trigonometric polynomial φ satisfies

$$M(2) = \begin{pmatrix} [T_\psi^*, T_\psi]_0 & [T_\varphi^*, T_\psi]_0 \\ [T_\psi^*, T_\varphi]_0 & [T_\varphi^*, T_\varphi]_0 \end{pmatrix} \geq 0.$$

Then φ is also of the form $\varphi(z) = \sum_{n=-l}^{3} b_n z^n$ $(l \leq 3)$. Observe that

$$[T_\psi^*, T_\psi]_0 = \begin{pmatrix} \sum_{i=1}^{3}(|a_i|^2 - |a_{-i}|^2) & \sum_{i=1}^{2}(\bar{a}_{i+1}a_i - a_{-(i+1)}\bar{a}_{-i}) & \bar{a}_3 a_1 - a_{-3}\bar{a}_{-1} \\ * & \sum_{i=2}^{3}(|a_i|^2 - |a_{-i}|^2) & \bar{a}_3 a_2 - a_{-3}\bar{a}_{-2} \\ * & * & |a_3|^2 - |a_{-3}|^2 \end{pmatrix}.$$

There are three cases to consider.

Case 1 $(m = 0)$. $M(1) = [T_\psi^*, T_\psi]_0$ implies that $\psi(z) = \sqrt{b}\, z^3$, so $a = b \neq 0$. Thus by Corollary 1.11, φ must be of the form $\varphi(z) = \sum_{n=0}^{3} b_n z^n$. Since $[T_\varphi^*, T_\varphi]_0 = \left(\begin{smallmatrix} c & h & k \\ \bar{h} & c & h \\ \bar{k} & \bar{h} & c \end{smallmatrix}\right)$, we see that φ is also of the form $\varphi(z) = \sqrt{c}\, z^3$. Then $[T_\varphi^*, T_\psi]_0 = \sqrt{bc}\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \bar{e} & b & e \\ f & \bar{f} & g \\ \bar{g} & f & \bar{f} \end{smallmatrix}\right)$, which is impossible because $b \neq 0$.

Case 2 ($m = 1$). By Theorem 2.6, we have $l = m = 1$. $M(1) = [T_\psi^*, T_\psi]_0$ gives

$$\begin{cases} |a_3|^2 = b \\ |a_2|^2 = 0 \\ \bar{a}_3 a_2 = e \\ \bar{a}_3 a_1 = \bar{d} \\ \bar{a}_3 a_2 + \bar{a}_2 a_1 = d, \end{cases}$$

which implies $a_1 = a_2 = 0$, so that $\psi(z) = a_{-1} z^{-1} + a_3 z^3$. Similarly, equating $[T_\varphi^*, T_\varphi]_0 = \left(\begin{smallmatrix} c & h & k \\ \bar{h} & c & h \\ \bar{k} & \bar{h} & c \end{smallmatrix}\right)$ gives $b_2 = 0$ and therefore $\varphi(z) = b_{-1} z^{-1} + b_1 z + b_3 z^3$. Then

$$[T_\varphi^*, T_\psi]_0 = \begin{pmatrix} \bar{a}_3 b_3 - \bar{a}_{-1} b_{-1} & 0 & \bar{a}_3 b_1 \\ 0 & \bar{a}_3 b_3 & 0 \\ a_3 \bar{b}_1 & 0 & \bar{a}_3 b_3 \end{pmatrix},$$

so the equation $[T_\varphi^*, T_\psi]_0 = \left(\begin{smallmatrix} \bar{e} & b & e \\ f & \bar{f} & g \\ \bar{g} & f & \bar{f} \end{smallmatrix}\right)$ has no solution because $b \neq 0$. Thus this case is also impossible.

Case 3 ($m \geq 2$). By Theorem 2.6 we have $\varphi - \alpha\,\psi = d_0 + d_1\, z$ for some $0 \neq \alpha \in \mathbb{C}$ ($d_1 = 0$ if $m = 3$), so $[T_\psi^*, T_\psi]_0$ and $(1/\bar{\alpha})\,[T_\varphi^*, T_\psi]_0$ have equal corresponding second and third columns because they are independent of the particular value of b_1. Therefore

$$\begin{pmatrix} d & \bar{d} \\ b & e \\ \bar{e} & b \end{pmatrix} = (1/\bar{\alpha}) \begin{pmatrix} b & e \\ \bar{f} & g \\ f & \bar{f} \end{pmatrix}, \tag{4.7.3}$$

which gives the following equations:

$$\begin{cases} d = \frac{b}{\bar{\alpha}} = \frac{\bar{e}}{\alpha} \\ e = \frac{\bar{f}}{\alpha} = \frac{g}{\bar{\alpha}} \\ b = \frac{\bar{f}}{\bar{\alpha}}, \end{cases}$$

so

$$\frac{f}{\bar{\alpha}} = \bar{e} = \frac{\alpha}{\bar{\alpha}} b = \frac{\alpha}{\bar{\alpha}} \cdot \frac{\bar{f}}{\bar{\alpha}} \quad \text{and hence} \quad f = \frac{\alpha}{\bar{\alpha}} \bar{f}.$$

We thus have

$$[T_\varphi^*, T_\psi]_0 = \begin{pmatrix} \bar{e} & b & e \\ f & \bar{f} & g \\ \bar{g} & f & \bar{f} \end{pmatrix} = \begin{pmatrix} \frac{1}{\bar{\alpha}} f & \frac{1}{\bar{\alpha}} \bar{f} & \frac{1}{\bar{\alpha}} g \\ f & \bar{f} & g \\ \frac{\alpha}{\bar{\alpha}} f & \frac{\alpha}{\bar{\alpha}} \bar{f} & \frac{\alpha}{\bar{\alpha}} g \end{pmatrix},$$

which is of rank 1. Thus $\operatorname{rank} M(1) = \operatorname{rank}\,[T_\psi^*, T_\psi]_0 = \operatorname{rank}\,[T_\varphi^*, T_\psi]_0 = 1$ (see the proof of Theorem 2.6). This proves the implication (i)$\Rightarrow$(ii).

(ii)⇒(i): Suppose $M(1)$ is a moment matrix of rank 1, that is

$$M(1) = a \begin{pmatrix} 1 & \alpha & \bar{\alpha} \\ \bar{\alpha} & |\alpha|^2 & \bar{\alpha}^2 \\ \alpha & \alpha^2 & |\alpha|^2 \end{pmatrix} \quad (a, \alpha \in \mathbb{C});$$

$M(2)$ must in turn be of the form

$$M(2) = \begin{pmatrix} a & a\alpha & a\bar{\alpha} & \vdots & a\alpha^2 & a|\alpha|^2 & a\bar{\alpha}^2 \\ a\bar{\alpha} & a|\alpha|^2 & a\bar{\alpha}^2 & \vdots & f & \bar{f} & g \\ a\alpha & a\alpha^2 & a|\alpha|^2 & \vdots & \bar{g} & f & \bar{f} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a\bar{\alpha}^2 & \bar{f} & g & \vdots & c & h & k \\ a|\alpha|^2 & f & \bar{f} & \vdots & \bar{h} & c & h \\ a\alpha^2 & \bar{g} & f & \vdots & \bar{k} & \bar{h} & c \end{pmatrix}. \tag{4.7.4}$$

By (4.5.2), if we define

$$\psi(z) := \sqrt{a}\left(\bar{\alpha}z^{-2} + \alpha z^{-1} + z + \bar{\alpha}z^2 + \alpha z^3\right),$$

then $[T_\psi^*, T_\psi]_0 = M(1)$. On the other hand, since $[T_\varphi^*, T_\varphi]_0$ must be a hermitian Toeplitz matrix, it follows from Lemma 4.6 that we can choose φ of the form

$$\varphi(z) := \frac{e^{2i(\arg\alpha)}}{\sqrt{c}}\left(\bar{h}\, z^{-2} + \bar{k}\, z^{-1} + k\, z + h\, z^2 + c\, z^3\right) \quad (c \neq 0).$$

Then we have $[T_\varphi^*, T_\varphi] = \left(\begin{smallmatrix} c & h & k \\ \bar{h} & c & h \\ \bar{k} & \bar{h} & c \end{smallmatrix}\right)$. Now we will show that

$$[T_\psi^*, T_\varphi]_0 = \begin{pmatrix} a\bar{\alpha}^2 & \bar{f} & g \\ a|\alpha|^2 & f & \bar{f} \\ a\alpha^2 & \bar{g} & f \end{pmatrix} \tag{4.7.5}$$

for some $f, g \in \mathbb{C}$. A straightforward calculation shows that

$$[T_\psi^*, T_\varphi]_0 = \frac{\sqrt{a}}{\sqrt{c}} e^{2i\,(\arg\alpha)} \begin{pmatrix} \bar{\alpha}c + \alpha h + k - \bar{\alpha}\bar{k} - \alpha\bar{h} & \bar{\alpha}h + \alpha k - \bar{\alpha}\bar{h} & \bar{\alpha}k \\ \alpha c + h - \alpha\bar{k} & \bar{\alpha}c + \alpha h - \alpha\bar{h} & \bar{\alpha}h \\ c & \alpha c & \bar{\alpha}c \end{pmatrix}. \tag{4.7.6}$$

Therefore solving the equations (4.7.5) and (4.7.6) simultaneously gives the following solution:

$$\begin{pmatrix} c \\ h \\ k \\ f \\ g \end{pmatrix} = a \begin{pmatrix} |\alpha|^4 \\ |\alpha|^2\bar{\alpha}^2 \\ \bar{\alpha}^4 \\ |\alpha|^2\alpha \\ \bar{\alpha}^3 \end{pmatrix}.$$

This shows that $M(2)$ is a Toeplitz extension. We now argue that this extension is positive. To do this, observe that

$$\begin{aligned}\varphi(z) &= \frac{e^{2i\,(\arg\alpha)}}{\sqrt{c}}\left(\bar{h}z^{-2} + \bar{k}z^{-1} + kz + hz^2 + cz^3\right)\\ &= \sqrt{a}\left(|\alpha|^2 z^{-2} + \alpha^2 z^{-1} + \bar{\alpha}^2 z + |\alpha|^2 z^2 + \alpha^2 z^3\right).\end{aligned}$$

Then $\varphi - \alpha\,\psi = \sqrt{a}\left(\bar{\alpha}^2 - \alpha\right)z$, so it follows from Theorem 2.6 and Theorem 4.5 that the pair (T_ψ, T_φ) is hyponormal and flat. Therefore $M(2)$ is the desired Toeplitz extension of $M(1)$.

(ii)⇔(iii): This immediately follows from the extension-uniqueness criteria for the quadratic moment problem (see [**CF4**, Theorem 6.1])—note that the unique representing measure μ is defined by $\mu := \gamma_{00}\,\delta_\omega$ ($\omega := \frac{\gamma_{01}}{\gamma_{00}}$), and that if $\operatorname{rank} M(1) > 1$, then there exist infinitely many representing measures. $\square$

We conclude with a result about Toeplitz representations of positive moment matrices of rank 1. In the preceding, we have shown that if $\operatorname{rank} M(1) = 1$, then $M(1)$ is induced by a trigonometric Toeplitz operator. If $M(2)$ is a moment matrix of rank 1, then $M(2)$ is also induced by a trigonometric Toeplitz operator. Indeed if $\operatorname{rank} M(2) = 1$, then $M(2)$ can be written as

$$M(2) = a\begin{pmatrix} 1 & \alpha & \bar{\alpha} & \alpha^2 & |\alpha|^2 & \bar{\alpha}^2\\ \bar{\alpha} & |\alpha|^2 & \bar{\alpha}^2 & |\alpha|^2\alpha & |\alpha|^2\bar{\alpha} & \bar{\alpha}^3\\ \alpha & \alpha^2 & |\alpha|^2 & \alpha^3 & |\alpha|^2\alpha & |\alpha|^2\bar{\alpha}\\ \bar{\alpha}^2 & |\alpha|^2\bar{\alpha} & \bar{\alpha}^3 & |\alpha|^4 & |\alpha|^2\bar{\alpha}^2 & \bar{\alpha}^4\\ |\alpha|^2 & |\alpha|^2\alpha & |\alpha|^2\bar{\alpha} & |\alpha|^2\alpha^2 & |\alpha|^4 & |\alpha|^2\bar{\alpha}^2\\ \alpha^2 & \alpha^3 & |\alpha|^2\alpha & \alpha^4 & |\alpha|^2\alpha^2 & |\alpha|^4 \end{pmatrix} \quad (a, \alpha \in \mathbb{C}).$$

Thus if we choose an analytic polynomial f of the form

$$f(z) = \sqrt{a}\,\left(z + \bar{\alpha}\,z^2 + \alpha\,z^3 + \bar{\alpha}^2\,z^4 + |\alpha|^2\,z^5 + \alpha^2\,z^6\right),$$

then a straightforward calculation shows that $[T^*_{f+z\bar{f}}, T_{f+z\bar{f}}]_0 = M(2)$. More generally we can show that if $M(n)$ is a moment matrix of rank 1, then $M(n)$ can be induced by a trigonometric Toeplitz operator. To do this, recall ([**CF4**]) that the following lexicographic order can be used for the rows and columns of $M(n)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, \dots, Z^n, \bar{Z}Z^{n-1}, \dots, \bar{Z}^{n-1}Z, \bar{Z}^n;$$

e.g., the first column is labeled 1, the second column is labeled Z, the third $\bar{Z}$, the fourth Z^2, etc.

We now have:

Theorem 4.8. *Every moment matrix $M(n)$ of rank 1 is induced by a trigonometric Toeplitz operator. More precisely, if $\operatorname{rank} M(n) = 1$ and if we define*

$$f(z) := \sqrt{a}\sum_{k=1}^{(n+1)(n+2)/2} \bar{a}_k z^k,$$

where a_k is the complex number obtained by substituting $\frac{\gamma_{01}}{\gamma_{00}}$ for Z in the k-th slot of the above lexicographic order (e.g., $a_1 = 1$, $a_2 = \frac{\gamma_{01}}{\gamma_{00}}$, $a_3 = \frac{\bar\gamma_{01}}{\gamma_{00}}$, $a_4 = \left(\frac{\gamma_{01}}{\gamma_{00}}\right)^2, \ldots$), then

$$M(n) = [T^*_{f+z\bar f}, T_{f+z\bar f}]_0. \tag{4.8.1}$$

Proof. Observe that if $\varphi \in L^\infty$ then $[T^*_\varphi, T_\varphi]$ has matrix representation whose (μ,ν)-entries are

$$\sum_{j=0}^{\infty}\Big(\bar{\hat\varphi}(j-\mu)\,\hat\varphi(j-\nu) - \hat\varphi(\mu-j)\bar{\hat\varphi}(\nu-j)\Big).$$

Put $\varphi := f + z\bar f$. Then evidently, $\hat\varphi(-j+1) = \bar{\hat\varphi}(j)$ for every $j = 2,3,\ldots,(n+1)(n+2)/2$. Thus we have

$$(\mu,\nu)\text{-entry of } [T^*_\varphi, T_\varphi] = \hat\varphi(\mu)\,\bar{\hat\varphi}(\nu) = \bar a_\mu\, a_\nu. \tag{4.8.2}$$

Let Z_j denote the j-th slot of the above lexicographic order, and let α_j denote the complex number obtained by substituting $\frac{\gamma_{01}}{\gamma_{00}}$ for Z in Z_j for every j. Then since $\operatorname{rank} M(n) = 1$, the (μ,ν)-entry of $M(n)$ must be given by $\bar\alpha_\mu\,\alpha_\nu$. Therefore, by (4.8.2) and the definition of a_k, we get the equality (4.8.1). □

§**4.3. Hyponormality of single Toeplitz operators.** We now apply the preceding results to examine hyponormality of Toeplitz operators with trigonometric polynomial symbols of a prescribed form. We will first consider the hyponormality of a Toeplitz operator T_φ with symbol φ of the form

$$\varphi(z) = a_{-N}z^{-N} + a_{-m}z^{-m} + a_m z^m + a_N z^N.$$

P. Fan [**Fa1**] has shown that if $\varphi(z) = a_{-2}z^{-2} + a_{-1}z^{-1} + a_1 z + a_2 z^2$, then

$$T_\varphi \text{ is hyponormal} \iff \left|\det\begin{pmatrix} a_{-1} & a_{-2} \\ \bar a_1 & \bar a_2\end{pmatrix}\right| \le |a_2|^2 - |a_{-2}|^2. \tag{4.8.3}$$

We begin with:

Lemma 4.9. *Suppose $\varphi(z)$ is a trigonometric polynomial such that $\varphi = \bar z^n\,\bar f + z^n\,g$, where f and g are analytic polynomials. If $\psi := \bar f + g$, then*

$$T_\varphi \textit{ is hyponormal} \implies T_\psi \textit{ is hyponormal.} \tag{4.9.1}$$

Proof. Suppose T_φ is hyponormal. By Theorem 1.3, there exists a function k in the closed unit ball of $H^\infty(\mathbb{T})$ such that $\varphi - k\,\bar\varphi \in H^\infty(\mathbb{T})$, so $\left(\bar z^n\,\bar f + z^n\,g\right) - k\left(\bar z^n\,\bar g + z^n\,f\right) \in H^\infty(\mathbb{T})$. Then $\bar z^n\,\bar f - k\,\bar z^n\,\bar g \in H^\infty(\mathbb{T})$ and hence $\bar f - k\bar g \in H^\infty(\mathbb{T})$. Therefore we have $\psi - k\bar\psi \in H^\infty(\mathbb{T})$, which says that T_ψ is hyponormal. □

The converse of (4.9.1) is not true in general. For example if

$$\varphi(z) = z^{-2} + z^{-1} + 4z + 2z^2 \quad \text{and} \quad \psi(z) = z^{-3} + z^{-2} + 4z^2 + 2z^3,$$

then T_φ is hyponormal, while T_ψ is not: indeed we have

$$[T_\varphi^*, T_\varphi] = \begin{pmatrix} 18 & 7 \\ 7 & 18 \end{pmatrix} \oplus 0_\infty \quad \text{and} \quad [T_\psi^*, T_\psi] = \begin{pmatrix} 18 & 7 & 0 \\ 7 & 18 & 7 \\ 0 & 7 & 3 \end{pmatrix} \oplus 0_\infty,$$

which implies that $[T_\varphi^*, T_\varphi]$ is positive semi-definite, while $[T_\psi^*, T_\psi]$ is not.

In spite of the above example we have:

Theorem 4.10. *If $\varphi(z) = a_{-N}z^{-N} + a_{-m}z^{-m} + a_m z^m + a_N z^N$, where $m \le \frac{N}{2}$, then*

$$\text{(4.10.1)} \qquad T_\varphi \text{ is hyponormal } \iff \left| \det \begin{pmatrix} a_{-m} & a_{-N} \\ \bar{a}_m & \bar{a}_N \end{pmatrix} \right| \le |a_N|^2 - |a_{-N}|^2.$$

Proof. Suppose T_φ is hyponormal. If $\psi(z) := a_{-N}z^{-N+m-1} + a_{-m}z^{-1} + a_m z + a_N z^{N-m+1}$, then by Lemma 4.9, T_ψ is hyponormal. If $k \in \mathcal{E}(\varphi)$ then the Fourier coefficients $\hat{k}(0), \hat{k}(1), \dots, \hat{k}(N-m)$ are determined uniquely by the formulas

$$\begin{aligned} \hat{k}(0) &= \frac{a_{-N}}{\bar{a}_N}; \\ \hat{k}(1) = \cdots = \hat{k}(N-m-1) &= 0; \\ \hat{k}(N-m) &= \frac{\bar{a}_N a_{-m} - a_{-N}\bar{a}_m}{\bar{a}_N^2}. \end{aligned}$$

Then we can find a function $h \in H^\infty$ such that

$$\hat{h}(0) = \hat{k}(0), \quad \hat{h}(1) = \hat{k}(N-m), \quad \text{and} \quad ||h||_\infty \le 1$$

(apply the classical interpolation theorem of I. Schur in [**Sch**] or [**Zh**]). Thus if

$$\tilde{\psi}(z) := a_{-N}z^{-2} + a_{-m}z^{-1} + a_m z + a_N z^2,$$

then $h \in \mathcal{E}(\tilde{\psi})$, and hence $T_{\tilde{\psi}}$ is hyponormal. Therefore by (4.8.3), we see that with no restriction on m and N,

$$\left| \det \begin{pmatrix} a_{-m} & a_{-N} \\ \bar{a}_m & \bar{a}_N \end{pmatrix} \right| \le |a_N|^2 - |a_{-N}|^2,$$

which proves the forward implication of (4.10.1).

For the backward implication suppose $m \le \frac{N}{2}$ and the inequality in (4.10.1) holds. If $|a_N| = |a_{-N}|$ then this is a special case of Lemma 1.4(iv). Thus we assume $|a_N| \neq |a_{-N}|$. By Proposition 1.6 we see that if $T_1 := U^m$ and $T_2 := U^{N-m}$ then $\mathbf{T} := (c_1 T_1, c_2 T_2)$ is hyponormal for every $c_1, c_2 \in \mathbb{C}$. Take

$$c_1 = \frac{\bar{a}_N a_m - a_{-N}\bar{a}_{-m}}{\sqrt{|a_N|^2 - |a_{-N}|^2}} \quad \text{and} \quad c_2 = \sqrt{|a_N|^2 - |a_{-N}|^2}.$$

Then we have

$$\begin{aligned}
&[\mathbf{T}^*, \mathbf{T}]\\
&= \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}\\
&= \begin{pmatrix} \frac{\left|\bar{a}_N a_m - a_{-N}\bar{a}_{-m}\right|^2}{|a_N|^2 - |a_{-N}|^2} I_m & \left(\bar{a}_N a_m - a_{-N}\bar{a}_{-m}\right) P_m U^{*(N-2m)} \\ \left(a_N \bar{a}_m - \bar{a}_{-N} a_{-m}\right) U^{N-2m} P_m & \left(|a_N|^2 - |a_{-N}|^2\right) I_{N-m} \end{pmatrix} \oplus 0_\infty \geq 0,
\end{aligned}$$

where the skew-diagonal entries should be understood as truncated matrices and the second equality is up to unitary equivalence. Now a straightforward calculation shows that

$$\begin{aligned}
&[T_\varphi^*, T_\varphi]\\
&= \begin{pmatrix} \left[(|a_N|^2 - |a_{-N}|^2) + (|a_m|^2 - |a_{-m}|^2)\right] I_m & \left(\bar{a}_N a_m - a_{-N}\bar{a}_{-m}\right) P_m U^{*(N-2m)} \\ \left(a_N \bar{a}_m - \bar{a}_{-N} a_{-m}\right) U^{(N-2m)} P_m & \left(|a_N|^2 - |a_{-N}|^2\right) I_{N-m} \end{pmatrix} \oplus 0_\infty,
\end{aligned}$$

and hence

$$\begin{aligned}
&[T_\varphi^*, T_\varphi] - [\mathbf{T}^*, \mathbf{T}]\\
&= \left[\left(|a_N|^2 - |a_{-N}|^2 + |a_m|^2 - |a_{-m}|^2\right) - \frac{\left|\bar{a}_N a_m - a_{-N}\bar{a}_{-m}\right|^2}{|a_N|^2 - |a_{-N}|^2}\right] I_m \bigoplus 0_\infty.
\end{aligned}$$

Since

$$\left|\bar{a}_N a_m - a_{-N}\bar{a}_{-m}\right|^2 = \left|\bar{a}_N a_{-m} - a_{-N}\bar{a}_m\right|^2 + (|a_N|^2 - |a_{-N}|^2)(|a_m|^2 - |a_{-m}|^2),$$

it follows that

$$\begin{aligned}
&|a_N|^2 - |a_{-N}|^2 + |a_m|^2 - |a_{-m}|^2 \;\geq\; \frac{\left|\bar{a}_N a_m - a_{-N}\bar{a}_{-m}\right|^2}{|a_N|^2 - |a_{-N}|^2}\\
\Longleftrightarrow\; &\left|\det \begin{pmatrix} a_{-m} & a_{-N} \\ \bar{a}_m & \bar{a}_N \end{pmatrix}\right| \leq |a_N|^2 - |a_{-N}|^2.
\end{aligned}$$

Therefore if the inequality in (4.10.1) holds then $[T_\varphi^*, T_\varphi] \geq 0$, and therefore T_φ is hyponormal. □

We next consider the following hyponormal extension problem of a Toeplitz operator: *Suppose T_ψ is a hyponormal Toeplitz operator with a trigonometric polynomial symbol ψ of the form $\psi(z) = \sum_{k=-(N-1)}^{N-1} a_k z^k$. If $\varphi := a_{-N} z^{-N} + \psi + a_N z^N$, when is T_φ hyponormal?*

Theorem 4.11. *Let $\psi(z) \equiv \sum_{k=-(N-1)}^{N-1} a_k z^k$ be such that T_ψ is hyponormal and let $\varphi(z) := a_{-N} z^{-N} + \psi(z) + a_N z^N$ $(|a_{-N}| \le |a_N|)$.*

(i) *Let $\tilde{\varphi}(z) := \sum_{k=-(N-1)}^{(N-1)} b_k z^k$, where*

$$b_k := \begin{cases} \det \begin{pmatrix} a_{k+1} & a_{-N} \\ \bar{a}_{-(k+1)} & \bar{a}_N \end{pmatrix} & \text{for } k = 0, 1, \dots, N-1 \\ \bar{b}_{-(k+1)} & \text{for } k = -1, -2, \dots, -(N-1). \end{cases} \tag{4.11.1}$$

If $T_{\tilde{\varphi}}$ is hyponormal then T_φ is hyponormal.

(ii) *Assume T_ψ is normal; then T_φ is hyponormal if and only if $T_{\tilde{\varphi}}$ is hyponormal.*

Proof. If $A = (c_{ij})$ is an $N \times N$ hermitian Toeplitz matrix whose first row satisfies

$$\begin{pmatrix} c_{11} \\ c_{12} \\ \vdots \\ \vdots \\ c_{1N} \end{pmatrix}^T = \bar{a}_N \begin{pmatrix} a_N \\ a_{N-1} \\ \vdots \\ \vdots \\ a_1 \end{pmatrix}^T - a_{-N} \begin{pmatrix} \bar{a}_{-N} \\ \bar{a}_{-(N-1)} \\ \vdots \\ \vdots \\ \bar{a}_{-1} \end{pmatrix}^T,$$

then a straightforward calculation shows that

$$\begin{aligned} [T_\varphi^*, T_\varphi] - [T_\psi^*, T_\psi] &= (|a_N|^2 - |a_{-N}|^2)[U^{*N}, U^N] + [\bar{a}_{-N} U^N + \bar{a}_N U^{*N}, T_\psi] \\ &\quad + [T_\psi^*, a_{-N} U^{N*} + a_N U^N] = A \oplus 0_\infty. \end{aligned} \tag{4.11.2}$$

Since T_ψ is hyponormal it follows that T_φ is hyponormal whenever $A \ge 0$. On the other hand since $A = (c_{ij})$ is a hermitian Toeplitz matrix and $c_{11} \ge 0$, Lemma 4.6 shows that if

$$\zeta(z) := \frac{1}{\sqrt{c_{11}}} \Big(c_{11} z^n + \sum_{k=1}^{N-1} \big(\bar{c}_{1,(N-k+1)} z^{-k} + c_{1,(N-k+1)} z^k \big) \Big),$$

then $[T_\zeta^*, T_\zeta]_0 = A$. Thus a straightforward calculation together with Lemma 1.5 shows that if $\tilde{\varphi}$ is given by (4.11.1) then $T_{\tilde{\varphi}}$ is hyponormal if and only if T_ζ is hyponormal. Therefore if $T_{\tilde{\varphi}}$ is hyponormal then T_φ is hyponormal. This proves (i). The statement in (ii) follows from the observation that if T_ψ is normal then by (4.11.2), T_φ is hyponormal if and only if $T_{\tilde{\varphi}}$ is hyponormal. □

We conclude this chapter with:

Example 4.12. *Let $\psi(z) \equiv \sum_{k=-2}^{2} a_k z^k$ be such that T_ψ is normal and let $\varphi(z) := \sum_{k=-3}^{3} a_k z^k$. Then T_φ is hyponormal if and only if $|a_{-3}| \le |a_3|$ and*

$$\begin{aligned} &\big(|a_3|^2 - |a_{-3}|^2\big)^2 - |\bar{a}_3 a_2 - a_{-3} \bar{a}_{-2}|^2 \\ &\qquad \ge \Big| (|a_3|^2 - |a_{-3}|^2)(\bar{a}_3 a_1 - a_{-3} \bar{a}_{-1}) - |\bar{a}_3 a_2 - a_{-3} \bar{a}_{-2}|^2 \Big|. \end{aligned} \tag{4.12.1}$$

In particular, necessary for T_φ to be hyponormal is that

$$|a_3|^2 - |a_{-3}|^2 \geq \left|\det \begin{pmatrix} a_k & a_{-3} \\ \bar{a}_{-k} & \bar{a}_3 \end{pmatrix}\right| \quad \textit{for } k = 1, 2. \tag{4.12.2}$$

Proof. By Lemma 1.4(i), "$|a_{-3}| \leq |a_3|$" is necessary for hyponormality of T_φ. Thus if $|a_{-3}| \leq |a_3|$ and if we define

$$\begin{aligned}\tilde{\varphi}(z) := (\bar{a}_3 a_2 - a_{-3}\bar{a}_{-2})\, z^{-2} + (\bar{a}_3 a_1 - a_{-3}\bar{a}_{-1})\, z^{-1} \\ + (\bar{a}_3 a_2 - a_{-3}\bar{a}_{-2})\, z + (|a_3|^2 - |a_{-3}|^2)\, z^2,\end{aligned}$$

then by Theorem 4.11 and Lemma 1.5, T_φ is hyponormal if and only if $T_{\tilde{\varphi}}$ is hyponormal. Since by (4.8.3), $T_{\tilde{\varphi}}$ is hyponormal if and only if

$$\begin{aligned}&\left(|a_3|^2 - |a_{-3}|^2\right)^2 - |\bar{a}_3 a_2 - a_{-3}\bar{a}_{-2}|^2 \\ &\qquad \geq \left|(|a_3|^2 - |a_{-3}|^2)(a_3\bar{a}_1 - \bar{a}_{-3}a_{-1}) - |\bar{a}_3 a_2 - a_{-3}\bar{a}_{-2}|^2\right|,\end{aligned}$$

the first assertion immediately follows. The necessary condition (4.12.2) for $k = 2$ is evident from (4.12.1), and for $k = 1$ it follows from the observation

$$\begin{aligned}&\left(|a_3|^2 - |a_{-3}|^2\right)^2 - |\bar{a}_3 a_2 - a_{-3}\bar{a}_{-2}|^2 \\ &\qquad \geq \left(|a_3|^2 - |a_{-3}|^2\right)\left|(\bar{a}_3 a_1 - a_{-3}\bar{a}_{-1})\right| - \left|\bar{a}_3 a_2 - a_{-3}\bar{a}_{-2}\right|^2.\end{aligned}$$

$\square$

CHAPTER 5

CONCLUDING REMARKS AND OPEN PROBLEMS

1. Hyponormality for trigonometric Toeplitz tuples. In Corollary 2.11 we showed that a trigonometric Toeplitz n-tuple $\mathbf{T}$ is hyponormal if and only if every subpair of $\mathbf{T}$ is hyponormal. Also in Example 2.12 we gave an example which shows that the equivalence need not be true for n-tuples of operators in general. However it is not clear whether for every Toeplitz tuple, the conditions (i) and (ii) in Corollary 2.11 are equivalent; we conjecture that they are.

Conjecture 5.1. For every Toeplitz tuple $\mathbf{T}$, $\mathbf{T}$ is hyponormal if and only if every subpair of $\mathbf{T}$ is hyponormal.

2. Rigidity of hyponormal Toeplitz pairs. In Example 2.3, we constructed a hyponormal Toeplitz pair (T_ψ, T_φ), where ψ is a non-analytic trigonometric polynomial and $\varphi \in L^\infty(\mathbb{T})$ is not a trigonometric polynomial. It is possible, however, for the hyponormality of the Toeplitz pair (T_ψ, T_φ) with trigonometric Toeplitz operator coordinate T_ψ to force the pair to be a *trigonometric* Toeplitz pair. For example, this is the case for Toeplitz pairs with a normal coordinate T_ψ (cf. Theorem 2.2). More generally we have:

Problem 5.2. Let ψ be a non-analytic trigonometric polynomial and let $\varphi \in L^\infty(\mathbb{T})$ be arbitrary. When does the hyponormality of (T_ψ, T_φ) force φ to be a trigonometric polynomial?

3. Existence of non-subnormal k-hyponormal Toeplitz operators. In Chapter 3, we discussed the existence of non-subnormal 2-hyponormal Toeplitz operators. In Theorem 3.2, we showed that every 2-hyponormal trigonometric Toeplitz operator is subnormal. The following open problem is of particular interest in single operator theory.

Problem 5.3. Is every 2-hyponormal Toeplitz operator subnormal? If the answer is no, characterize k-hyponormal Toeplitz operators.

4. Hyponormality of (U^n, R). In the paragraph following Remark 1.18 we discussed the hyponormality of the pair (U^n, R), where $R \in \mathcal{L}(H^2(\mathbb{T}))$; we saw that the hyponormality of (U^n, R) forces R to be a block-Toeplitz operator. Certainly, as in [**FM**], there exists a weakly hyponormal pair (U^n, R) with non-Toeplitz operator R for every $n \in \mathbb{Z}$: for example if R is an n-power of a unilateral weighted shift with strictly increasing positive weight sequence then (U^n, R) is a weakly hyponormal pair. Since R is not a block-Toeplitz operator, (U^n, R) cannot be hyponormal.

Problem 5.4. For $n > 1$, find a non-Toeplitz block-Toeplitz operator R for which the pair (U^n, R) is hyponormal.

5. Subnormality and flatness. In [**CMX**] and [**Cu2**], it was conjectured that if $\mathbf{T} = (T_1, T_2)$ is a hyponormal pair of commuting subnormal operators then $\mathbf{T}$ must be a subnormal pair. This conjecture is true for some pairs of operators. For example this is the case for jointly quasinormal pairs as we remarked in the proof of Theorem 4.5(iii). Also this conjecture is true for Toeplitz pairs because two hyponormal Toeplitz operators commute if and only if either both are analytic or one is a linear function of the other (cf. [**BH**, Theorem 9]). In fact, joint subnormality of a commuting Toeplitz pair is equivalent to the subnormality of one coordinate operator. In particular, by Theorem 2.2, every non-analytic trigonometric polynomial cannot induce a non-normal, subnormal Toeplitz pair, because the trigonometric Toeplitz operator T_ψ is hyponormal if and only if T_ψ is either normal or T_ψ is analytic (cf. [**IW**]). However the above-mentioned conjecture remains open. We pose a weaker problem related to this conjecture.

Problem 5.5. If $\mathbf{T}$ is a subnormal pair whose self-commutator is of finite rank, is $\mathbf{T}$ flat? More generally, if $\mathbf{T}$ is a hyponormal pair of commuting subnormal (or quasinormal) operators whose self-commutator is of finite rank, is $\mathbf{T}$ flat?

6. Toeplitz extension problem of positive moment matrices. In Theorem 4.7 we obtained a solution of the quadratic Toeplitz extension problem for positive moment matrices. When $n \geq 2$, it is not even clear when $M(n)$ is induced by a trigonometric Toeplitz operator. For example, in Theorem 4.8 we showed that if $M(n)$ is of rank 1 then $M(n)$ can be induced by a trigonometric Toeplitz operator. But if $n \geq 2$ and $\operatorname{rank} M(n) \geq 2$, the Toeplitz extension problem seems intractable at present.

Problem 5.6. Solve the Toeplitz extension problem for positive moment matrices $M(n)$ with $n \geq 2$.

From a different viewpoint, one might suggest a Toeplitz extension problem as follows. If $M(n)$ is a positive moment matrix induced by a trigonometric Toeplitz n-tuple $\mathbf{T}$, does there exist a positive extension $M(n+1)$ induced by a trigonometric Toeplitz $(n+1)$-tuple $\mathbf{S} = (\mathbf{T}, T_\varphi)$? Again, Theorem 4.7 provides a solution when $n = 1$.

Problem 5.7. Let $M(n)$ be a positive moment matrix and let $\mathbf{T}$ be a trigonometric Toeplitz n-tuple such that $M(n) = [\mathbf{T}^*, \mathbf{T}]_0$. Find a necessary and sufficient condition for the existence of a Toeplitz extension $M(n+1) = [\mathbf{S}^*, \mathbf{S}]_0 \geq 0$, where $\mathbf{S} = (\mathbf{T}, T_\varphi)$ for some trigonometric Toeplitz operator T_φ.

REFERENCES

[Ab] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[AIW] I. Amemiya, T. Ito, and T.K. Wong, *On quasinormal Toeplitz operators*, Proc. Amer. Math. Soc. **50** (1975), 254–258.

[At] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

[Br] J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75–94.

[BH] A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.

[Con] J.B. Conway, *The Theory of Subnormal Operators*, Math. Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

[CoS] J.B. Conway and W. Szymanski, *Linear combination of hyponormal operators*, Rocky Mountain J. Math. **18** (1988), 695–705.

[Cow] C.C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), 809–812.

[CoL] C.C. Cowen and J.J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351** (1984), 216–220.

[Cu1] R.E. Curto, *Quadratically hyponormal weighted shifts*, Integral Equations Operator Theory **13** (1990), 49–66.

[Cu2] R.E. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, Operator Theory: Operator Algebras and Applications (Durham, NH, 1988) (W.B. Arveson and R.G. Douglas, eds.), Proc. Sympos. Pure Math., vol. 51, Part II, Amer. Math. Soc., Providence, (1990), 69–91.

[CF1] R.E. Curto and L.A. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991), 603–635.

[CF2] R.E. Curto and L.A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory **17** (1993), 202–246.

[CF3] R.E. Curto and L.A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem, II*, Integral Equations Operator Theory **18** (1994), 369–426.

[CF4] R.E. Curto and L.A. Fialkow, *Solution of the truncated complex moment problem for flat data*, Mem. Amer. Math. Soc. vol. 119 (1996) no. 568.

[CMX] R.E. Curto, P.S. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Contributions to Operator Theory and Its Applications (Mesa, AZ, 1987) (I. Gohberg, J.W. Helton and L. Rodman, eds.), Operator Theory: Advances and Applications, vol. 35, Birkhäuser, Basel–Boston, (1988), 1–22.

[CP1] R.E. Curto and M. Putinar, *Existence of non-subnormal polynomially hyponormal operators*, Bull. Amer. Math. Soc. (N.S.) **25** (1991), 373–378.

[CP2] R.E. Curto and M. Putinar, *Nearly subnormal operators and moment problems*, J. Funct. Anal. **115** (1993), 480–497.

[DPY] R.G. Douglas, V.I. Paulsen, and K. Yan, *Operator theory and algebraic geometry*, Bull. Amer. Math. Soc. (N.S.) **20** (1989), 67–71.

[DY] R.G. Douglas and K. Yan, *A multi-variable Berger-Shaw theorem*, J. Operator Theory **27** (1992), 205–217.

[Fa1] P. Fan, *Remarks on hyponormal trigonometric Toeplitz operators*, Rocky Mountain J. Math. **13** (1983), 489–493.

[Fa2] P. Fan, *A note on hyponormal weighted shifts*, Proc. Amer. Math. Soc. **92** (1984), 271–272.

[Fa3] P. Fan, *Note on subnormal weighted shifts*, Proc. Amer. Math. Soc. **103** (1988), 801–802.

[FL1] D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348** (1996), 4153–4174.

[FL2] D.R. Farenick and W.Y. Lee, *On hyponormal Toeplitz operators with polynomial and circulant-type symbols*, Integral Equations Operator Theory **29** (1997), 202–210.

[FM] D.R. Farenick and R. McEachin, *Toeplitz operators hyponormal with the unilateral shift*, Integral Equations Operator Theory **22** (1995), 273–280.

[Gu] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124** (1994), 135–148.

[Hal1] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.

[Hal2] P.R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer, New York, 1982.

[Har] R.E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 109, Marcel Dekker, New York, 1988.

[IW] T. Ito and T.K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34** (1972), 157–164.

[Lu] A, Lubin, *Weighted shifts and commuting normal extension*, J. Austral. Math. Soc. Ser. A **27** (1979), 17–26.

[McCP] S. McCullough and V. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107** (1989), 187–195.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), 753–767.

[Sch] I. Schur, *Über Potenzreihen die im Innern des Einheitskreises beschränkt*, J. Reine Angew. Math. **147** (1917), 205–232.

[Sun] Sun Shunhua, *Bergman shift is not unitarily equivalent to a Toeplitz operator*, Kexue Tongbao (English Ed.) **28** (1983), 1027–1030.

[Smu] J.L. Smul'jan, *An operator Hellinger integral* (Russian), Mat. Sb. (N.S.) **91** (1959), 381–430.

[St] J. Stampfli, *Which weighted shifts are subnormal*, Pacific J. Math. **17** (1966), 367–379.

[Xi1] D. Xia, *Spectral Theory of Hyponormal Operators*, Operator Theory: Advances and Applications, vol. 10, Birkhäuser Verlag, Basel–Boston, 1983.

[Xi2] D. Xia, *On the semi-hyponormal n-tuple of operators*, Integral Equations Operator Theory **6** (1983), 879–898.

[Yo] T. Yoshino, *On the commuting extensions of nearly normal operators*, Tôhoku Math. J. (2) **25** (1973), 263–272.

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21** (1995), 376–381.

Department of Mathematics, University of Iowa, Iowa City, IA 52242
E-mail address: `curto@math.uiowa.edu`

Department of Mathematics, SungKyunKwan University, Suwon 440-746, Korea
E-mail address: `wylee@yurim.skku.ac.kr`

LIST OF SYMBOLS