ON A QUESTION OF BROWN, DOUGLAS, AND FILLMORE

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Abstract. In this note we answer an old question of Brown, Douglas, and Fillmore [BDF].

Throughout this note, let $\mathcal{H}$ be a separable complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathbf{K}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$. If $K \in \mathbf{K}(\mathcal{H})$ has a polar decomposition $K = U\,|T|$, where $|T| := (T^*T)^{\frac{1}{2}}$ and U is a partial isometry, then $|T| \in \mathbf{K}(\mathcal{H})$ and so has a diagonal matrix $\text{diag}\,(\lambda_1, \lambda_2, \cdots)$ relative to some orthonormal basis fo $\mathcal{H}$. For $p \ge 1$ we define

$$\mathcal{C}_p(\mathcal{H}) := \left\{ K \in \mathbf{K}(\mathcal{H}) : \sum_{n=1}^{\infty} \lambda_n^p < \infty \right\},$$

which is called the *Schatten p-ideal*.

Two operators T_1 and T_2 are said to be *compalent* if T_1 is unitarily equivalent to T_2 modulo $\mathbf{K}(\mathcal{H})$ (notation: $T_1 \sim T_2$). An operator T is called *essentially normal* if $[T^*, T] \equiv T^*T - TT^* \in \mathbf{K}(\mathcal{H})$. The set of essentially normal operators will be denoted by $(EN)(\mathcal{H})$.

An operator $T \in B(\mathcal{H})$ is called *Fredholm* if T has closed range with finite dimensional null space and its range of finite co-dimension. The *index* of a Fredholm operator T is defined by the equality $\text{ind}(T) = \alpha(T) - \beta(T)$. An operator T is called *Weyl* if it is Fredholm of index zero. The essential spectrum, $\sigma_e(T)$, of T is the set of all complex numbers λ such that $T - \lambda$ is not Fredholm and the Weyl spectrum, $\omega(T)$, of T is the set of all complex numbers λ such that $T - \lambda$ is not Weyl. The *spectral picture* of T, denoted $\mathcal{SP}(T)$, is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes.

For a positive integer n, let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$. Let $\mathcal{E}(X)$ denote the set of all operators T such that T acts on one of the spaces $\mathcal{H}^{(n)}$, $n \ge 1$, and such that $T \in (EN)(\mathcal{H}_T)$ and satisfies $\sigma_e(T) = X$. Let $\text{Ext}(X)$ be the collection of equivalence classes into which $\mathcal{E}(X)$ is partitioned by the relation of compalence.

$\text{Ext}(X)$ was completely characterized by the following beautiful theorem due to Brown, Douglas, and Fillmore [BDF]:

Theorem 1. (BDF theorem) *If T_1 and T_2 are essentially normal operators then*

$$T_1 \sim T_2 \iff \mathcal{SP}(T_1) = \mathcal{SP}(T_2).$$

Let $E_p(X)$ denote the set of unitary equivalence classes modulo $\mathcal{C}_p$ of operators T such that $[T^*, T] \in \mathcal{C}_p$ and $\sigma_e(T) = X$. Then $E_p(X)$ is a commutative semigroup,

$$E_1(X) \subset E_2(X) \subset \cdots \subset E_\infty(X),$$

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and there is a natural homomorphism from $E_p(X)$ to $E_q(X)$ for $1 \le p \le q \le \infty$. Now define a notion of "stably equivalence" in $E_p(X)$. Call two operators T_1 and T_2 with $[T_1^*, T_1], [T_2^*, T_2] \in \mathcal{C}_p$ "stably equivalent" in $\widetilde{E}_p(X)$ if there exist normal operators N_1 and N_2 with essential spectra X such that $T_1 \oplus N_1$ is unitarily equivalent to some $\mathcal{C}_p$ perturbation of $T_2 \oplus N_2$. Then $\widetilde{E}_p(X)$ is a commutative semigroup,

$$\widetilde{E}_1(X) \subset \widetilde{E}_2(X) \subset \cdots \subset \widetilde{E}_\infty(X),$$

and there is a natural homomorphism from $\widetilde{E}_p(X)$ to $\widetilde{E}_q(X)$ for $1 \le p \le q \le \infty$. It was known [BDF] that $E_\infty(X) = \text{Ext}(X) = \widetilde{E}_\infty(X)$ and the natural map $E_1(X) \to \text{Ext}(X)$ is not generally injective. In [BDF, p.124], the following question was addressed:

Question 1. *Is the natural map $J : \widetilde{E}_p(X) \to \text{Ext}(X)$ injective for $p > 1$?*

Recall that an operator T is called *quasitriangular* if T is unitarily equivalent to a compact perturbation of a triangular operator and is called *biquasitriangular* if T and T^* are both quasitriangular and is denoted $T \in (BQT)(\mathcal{H})$.

We now answer Question 1 in the negative.

Proposition 1. *The natural map $J : \widetilde{E}_2(X) \to \text{Ext}(X)$ is not injective.*

Proof. Let X be a Cantor set with positive area density at each of its points and construct an irreducible hyponormal operator T with rank one self-commutator whose spectrum $\sigma(T)$ is the set X (its existence is guaranteed by the well-known result of Pincus [Pi]). If $\lambda \in \sigma(T) \setminus \sigma_e(T)$ then since $\sigma(T) = X$ has no interior, there exists a sequence (λ_n) in the resolvent such that $\lambda_n \to \lambda$, it follows from the punctured neighborhood theorem that λ is an isolated point of X, which contradicts the fact that X is perfect. Hence, $\sigma(T) = \sigma_e(T)$ and hence $\sigma_e(T) = \omega(T)$. Thus by the work of Apostol, Foias, and Voiculescu [AFV], which states that $T \in (BQT)(\mathcal{H}) \Longleftrightarrow \sigma_e(T) = \omega(T)$, we can see that T is biquasitriangular. But since T is essentially normal it follows from a consequence of the BDF theorem [BDF] that T should be of the form

$$T = N + K, \quad \text{where } N \text{ is normal and } K \text{ is compact.}$$

Evidently, $\sigma_e(N) = X$ and hence $[T]_\infty = [N]_\infty$ in $\text{Ext}(X)$. Assume to the contrary that $J : \widetilde{E}_2(X) \to \text{Ext}(X)$ is injective. Then $[T]_2 = [N]_2$ in $\widetilde{E}_2(X)$. Thus there exist normal operators N_1 and N_2 with essential spectra X such that $T \oplus N_1$ is unitarily equivalent to some $\mathcal{C}_2$ perturbation of $N \oplus N_2$. Thus

$$T \oplus N_1 = N_3 + K', \quad \text{where } N_3 \text{ is normal and } K' \in \mathcal{C}_2.$$

Recall ([LP]) that if $A = B + C$ (B is normal and $C \in \mathcal{C}_2$) is hyponormal then A is normal. Thus since $T \oplus N_1$ is hyponormal it follows at once that $T \oplus N_1$ should be normal. This is a contradiction because the self-commutator of $T \oplus N_1$ is of rank one. Therefore the map J is not injective. □

Corollary 1. *If T is a hyponormal operator whose essential spectrum X satisfies the property that the natural map $J : \widetilde{E}_2(X) \to \text{Ext}\,(X)$ is injective then T has a nontrivial invariant subspace.*

Proof. If $T \notin (BQT)(\mathcal{H})$ or T is not cyclic then evidently, T has a nontrivial invariant subspace. Thus we may suppose that $T \in (BQT)(\mathcal{H})$ and T is cyclic. By the Berger-Shaw theorem ([BS]) we have $[T^*, T] \in \mathcal{C}_1$. Thus by again a corollary of the BDF theorem, T is of the form $T = N + K$, where N is normal and K is compact. Then by the same argument as the proof of Proposition 1, T is normal. This proves the corollary. □

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