Abstract. In this note we answer an old question of Brown, Douglas, and Fillmore [BDF].

Throughout this note, let \( \mathcal{H} \) be a separable complex Hilbert space and \( B(\mathcal{H}) \) be the algebra of all bounded linear operators on \( \mathcal{H} \). Let \( K(\mathcal{H}) \) be the set of compact operators on \( \mathcal{H} \). If \( K \in K(\mathcal{H}) \) has a polar decomposition \( K = U |T| \), where \( |T| = (T^*T)^{\frac{1}{2}} \) and \( U \) is a partial isometry, then \( |T| \in K(\mathcal{H}) \) and so has a diagonal matrix \( \text{diag}(\lambda_1, \lambda_2, \cdots) \) relative to some orthonormal basis for \( \mathcal{H} \). For \( p \geq 1 \) we define
\[
C_p(\mathcal{H}) := \left\{ K \in K(\mathcal{H}) : \sum_{n=1}^{\infty} \lambda_n^p < \infty \right\},
\]
which is called the Schatten \( p \)-ideal.

Two operators \( T_1 \) and \( T_2 \) are said to be compalent if \( T_1 \) is unitarily equivalent to \( T_2 \) modulo \( K(\mathcal{H}) \) (notation: \( T_1 \sim T_2 \)). An operator \( T \) is called essentially normal if \( [T^*, T] \equiv T^*T - TT^* \in K(\mathcal{H}) \). The set of essentially normal operators will be denoted by \( (EN)(\mathcal{H}) \).

An operator \( T \in B(\mathcal{H}) \) is called Fredholm if \( T \) has closed range with finite dimensional null space and its range of finite co-dimension. The index of a Fredholm operator \( T \) is defined by the equality \( \text{ind}(T) = \alpha(T) - \beta(T) \). An operator \( T \) is called Weyl if it is Fredholm of index zero. The essential spectrum, \( \sigma_e(T) \), of \( T \) is the set of all complex numbers \( \lambda \) such that \( T - \lambda \) is not Fredholm and the Weyl spectrum, \( \omega(T) \), of \( T \) is the set of all complex numbers \( \lambda \) such that \( T - \lambda \) is not Weyl. The spectral picture of \( T \), denoted \( \mathcal{SP}(T) \), is the structure consisting of the set \( \sigma_e(T) \), the collection of holes and pseudoholes in \( \sigma_e(T) \), and the indices associated with these holes and pseudoholes.

For a positive integer \( n \), let \( \mathcal{H}^{(n)} \) denote the direct sum of \( n \) copies of \( \mathcal{H} \). Let \( \mathcal{E}(X) \) denote the set of all operators \( T \) such that \( T \) acts on one of the spaces \( \mathcal{H}^{(n)} \), \( n \geq 1 \), and such that \( T \in (EN)(\mathcal{H}_T) \) and satisfies \( \sigma_e(T) = X \). Let \( \text{Ext}(X) \) be the collection of equivalence classes into which \( \mathcal{E}(X) \) is partitioned by the relation of compalance.

\( \text{Ext}(X) \) was completely characterized by the following beautiful theorem due to Brown, Douglas, and Fillmore [BDF]:

**Theorem 1.** (BDF theorem) If \( T_1 \) and \( T_2 \) are essentially normal operators then
\[
T_1 \sim T_2 \iff \mathcal{SP}(T_1) = \mathcal{SP}(T_2).
\]

Let \( E_p(X) \) denote the set of unitary equivalence classes modulo \( C_p \) of operators \( T \) such that \( [T^*, T] \in C_p \) and \( \sigma_e(T) = X \). Then \( E_p(X) \) is a commutative semigroup,
\[
E_1(X) \subset E_2(X) \subset \cdots \subset E_\infty(X),
\]

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and there is a natural homomorphism from $E_p(X)$ to $E_q(X)$ for $1 \leq p \leq q \leq \infty$. Now define a notion of “stably equivalence” in $E_p(X)$. Call two operators $T_1$ and $T_2$ with $[T_1^*, T_1], [T_2^*, T_2] \in C_p$ “stably equivalent” in $E_p(X)$ if there exist normal operators $N_1$ and $N_2$ with essential spectra $X$ such that $T_1 \oplus N_1$ is unitarily equivalent to some $C_p$ perturbation of $T_2 \oplus N_2$. Then $E_p(X)$ is a commutative semigroup,

$$E_1(X) \subset E_2(X) \subset \cdots \subset E_\infty(X),$$

and there is a natural homomorphism from $E_p(X)$ to $E_q(X)$ for $1 \leq p \leq q \leq \infty$. It was known [BDF] that $E_\infty(X) = \text{Ext}(X) = E_\infty(X)$ and the natural map $E_1(X) \to \text{Ext}(X)$ is not generally injective. In [BDF, p.124], the following question was addressed:

**Question 1.** Is the natural map $J : E_p(X) \to \text{Ext}(X)$ injective for $p > 1$?

Recall that an operator $T$ is called *quasitriangular* if $T$ is unitarily equivalent to a compact perturbation of a triangular operator and is called *biquasitriangular* if $T$ and $T^*$ are both quasitriangular and is denoted $T \in (BQT)(\mathcal{H})$.

We now answer Question 1 in the negative.

**Proposition 1.** The natural map $J : \tilde{E}_2(X) \to \text{Ext}(X)$ is not injective.

**Proof.** Let $X$ be a Cantor set with positive area density at each of its points and construct an irreducible hyponormal operator $T$ with rank one self-commutator whose spectrum $\sigma(T)$ is the set $X$ (its existence is guaranteed by the well-known result of Pincus [Pl]). If $\lambda \in \sigma(T) \setminus \sigma_e(T)$ then since $\sigma(T) = X$ has no interior, there exists a sequence $(\lambda_n)$ in the resolvent such that $\lambda_n \to \lambda$, it follows from the punctured neighborhood theorem that $\lambda$ is an isolated point of $X$, which contradicts the fact that $X$ is perfect. Hence, $\sigma(T) = \sigma_e(T)$ and hence $\sigma_e(T) = \omega(T)$. Thus by the work of Apostol, Foias, and Voiculescu [AFV], which states that $T \in (BQT)(\mathcal{H}) \iff \sigma_e(T) = \omega(T)$, we see that $T$ is biquasitriangular. But since $T$ is essentially normal it follows from a consequence of the BDF theorem [BDF] that $T$ should be of the form

$$T = N + K, \quad \text{where } N \text{ is normal and } K \text{ is compact.}$$

Evidently, $\sigma_e(N) = X$ and hence $[T]_\infty = [N]_\infty$ in $\text{Ext}(X)$. Assume to the contrary that $J : \tilde{E}_2(X) \to \text{Ext}(X)$ is injective. Then $[T]_2 = [N]_2$ in $\tilde{E}_2(X)$. Thus there exist normal operators $N_1$ and $N_2$ with essential spectra $X$ such that $T \oplus N_1$ is unitarily equivalent to some $C_2$ perturbation of $N \oplus N_2$. Thus

$$T \oplus N_1 = N_3 + K', \quad \text{where } N_3 \text{ is normal and } K' \in C_2.$$

Recall ([LP]) that if $A = B + C$ ($B$ is normal and $C \in C_2$) is hyponormal then $A$ is normal. Thus since $T \oplus N_1$ is hyponormal it follows at once that $T \oplus N_1$ should be normal. This is a contradiction because the self-commutator of $T \oplus N_1$ is of rank one. Therefore the map $J$ is not injective.

**Corollary 1.** If $T$ is a hyponormal operator whose essential spectrum $X$ satisfies the property that the natural map $J : \tilde{E}_2(X) \to \text{Ext}(X)$ is injective then $T$ has a nontrivial invariant subspace.

**Proof.** If $T \notin (BQT)(\mathcal{H})$ or $T$ is not cyclic then evidently, $T$ has a nontrivial invariant subspace. Thus we may suppose that $T \in (BQT)(\mathcal{H})$ and $T$ is cyclic. By the Berger-Shaw theorem ([BS]) we have $[T^*, T] \in C_1$. Thus by again a corollary of the BDF theorem, $T$ is of the form $T = N + K$, where $N$ is normal and $K$ is compact. Then by the same argument as the proof of Proposition 1, $T$ is normal. This proves the corollary.
REFERENCES


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