Invariant subspaces for operators whose spectra are Carathéodory regions

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Abstract. In this paper it is shown that if an operator T satisfies $||p(T)|| \leq ||p||_{\sigma(T)}$ for every polynomial p and the polynomially convex hull of $\sigma(T)$ is a Carathéodory region whose accessible boundary points lie in rectifiable Jordan arcs on its boundary, then T has a nontrivial invariant subspace. As a corollary, it is also shown that if T is a hyponormal operator and the outer boundary of $\sigma(T)$ has at most finitely many prime ends corresponding to singular points on $\partial\mathbb{D}$ and has a tangent at almost every point on each Jordan arc, then T has a nontrivial invariant subspace.

Keywords. invariant subspaces, spectral sets, Carathéodory regions, hyponormal operators.

1. Introduction

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. A closed subspace $\mathfrak{L} \subset \mathcal{H}$ is called an *invariant subspace* for the operator $T \in \mathcal{B}(\mathcal{H})$ if $T\mathfrak{L} \subset \mathfrak{L}$. The two trivial subspaces, the entire space and the space containing only the zero vector, are invariant for every operator. The invariant subspace problem (due to J. von Neumann) is stated as: *Does every operator in $\mathcal{B}(\mathcal{H})$ have a nontrivial invariant subspace?* This problem remains still open for separable infinite-dimensional complex Hilbert spaces. But there were significant accomplishments on the invariant subspace problem. In 1950, P. Halmos defined a *subnormal* operator as an operator having a normal extension to some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and asked whether subnormal operators have nontrivial invariant subspaces. For a long time many mathematicians have made many attempts towards this problem. Eventually, S. Brown [Br1] found an ingenious proof that subnormal operators do have nontrivial invariant subspaces. The proof relies upon an important paper of D. Sarason [Sa]. In 1979, S. Brown, B. Chevreau and C. Pearcy [BCP1] showed that every contraction T (i.e., $\|T\| \leq 1$) with rich spectrum has invariant subspaces. In 1980, J. Agler [Ag] showed that every von Neumann operator T (i.e., $\|f(T)\| \leq \|f\|_{\sigma(T)}$, where f is a rational function with poles off the spectrum $\sigma(T)$ and $\|f\|_{\sigma(T)}$ is the supremum of f on $\sigma(T)$) has a nontrivial invariant subspace. At the same year, J. Stampfli [St] proved that an operator T whose spectrum is a k-spectral set (i.e., $\|f(T)\| \leq k\|f\|_{\sigma(T)}$ for every rational function f with poles off $\sigma(T)$ and some $k > 0$) has a nontrivial invariant subspace. In 1987, S. Brown [Br2] showed that every *hyponormal* operator T (i.e., $T^*T - TT^* \geq 0$) whose spectrum has nonempty interior has a nontrivial invariant subspace. In 1988, S. Brown, B. Chevreau and C. Pearcy [BCP2] showed that every contraction operator whose spectrum contains the unit circle has a nontrivial invariant subspace. Very recently, C. Ambrozie and V. Müller [AM] showed that every polynomially bounded operator T (i.e., $\|p(T)\| \leq k\|p\|_{\mathbb{D}}$ for every polynomial p and some $k > 0$) whose spectrum contains the unit circle has a nontrivial invariant subspace.

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The result for the invariant subspaces of contraction operators T whose spectra contain the unit circle can be interpreted as the result for the cases where T satisfies $\|p(T)\| \le \|p\|_{\sigma(T)}$ for every polynomial p and the outer boundary of $\sigma(T)$ is the unit circle. However, if the outer boundary of $\sigma(T)$ is not a Jordan curve then the existence of nontrivial invariant subspaces for such operators seems to be very difficult even though it is interesting and challenging. This article is an attempt towards the invariant subspaces for such operators. Our main result concerns the invariant subspaces for an operator having the spectrum whose polynomially convex hull is a Carathéodory region.

2. The main result

To prove the main theorem we first review some definitions and auxiliary lemmas.

Let K be a compact subset of $\mathbb{C}$. Write ηK for the *polynomially convex hull* of K. The *outer boundary* of K means $\partial(\eta K)$, i.e., the boundary of ηK. If Γ is a Jordan curve then $\operatorname{int}\Gamma$ means the bounded component of $\mathbb{C}\backslash\Gamma$. If K is a compact subset of $\mathbb{C}$ then $C(K)$ denotes the set of all complex-valued continuous functions on K; $P(K)$ for the uniform closure of all polynomials in $C(K)$; $R(K)$ for the uniform closure of all rational functions with poles off K in $C(K)$; and $A(K)$ for the set of all functions on K which are analytic on $\operatorname{int} K$ and continuous on K. A compact set K is called a *spectral set* for an operator T if $\sigma(T) \subset K$ and $\|f(T)\| \le \|f\|_K$ for any $f \in R(K)$ and is called a *k-spectral set* for an operator T if $\sigma(T) \subset K$ and there exists a constant $k > 0$ such that

$$\|f(T)\| \le k\|f\|_K \quad \text{for any } f \in R(K).$$

A *function algebra* on a compact space K is a closed subalgebra $\mathcal{A}$ of $C(K)$ that contains the constant functions and separates the points of K. A function algebra $\mathcal{A}$ on a set K is called a *Dirichlet algebra* on K if $\operatorname{Re}\mathcal{A} \equiv \{\operatorname{Re} f : f \in \mathcal{A}\}$ is dense in $C_{\mathbb{R}}(K)$ which is the set of all real-valued continuous functions on K.

The following lemma will be used for proving our main theorem.

Lemma 2.1. [Ag, Proposition 1] *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose that K is a spectral set for T and $R(K)$ is a Dirichlet algebra. If T has no nontrivial reducing subspaces then there exists a norm contractive algebra homomorphism $\varphi : H^\infty(\operatorname{int} K) \to \mathcal{B}(\mathcal{H})$ such that $\varphi(z) = T$. Furthermore, φ is continuous when domain and range have their weak* topologies.*

We recall [Co] that a *Carathéodory domain* is an open connected subset of $\mathbb{C}$ whose boundary coincides with its outer boundary. We can easily show that a Carathéodory domain G is a component of $\operatorname{int}\eta G$ and hence is simply connected. The notion of a Carathéodory domain was much focused in giving an exact description of the functions in $P^2(G) \equiv$ the closure of the polynomials in $L^2(G)$: for example, $P^2(G)$ is exactly the Bergman space $L^2_a(G)$ if G is a bounded Carathéodory domain (cf. [Co, Theorem 8.15]). Throughout this paper, a *Carathéodory region* means a closed set in $\mathbb{C}$ whose interior is a Carathéodory domain.

We note that the boundary of a bounded Carathéodory domain need not be a Jordan arc. A simple example is a *Cornucopia*, which is an open ribbon G that winds about the unit circle so that each point of $\partial\mathbb{D}$ belongs to ∂G. In this case, ∂G is not a Jordan curve because every point c of $\partial\mathbb{D}$ is not an *accessible* boundary point, in the sense that it cannot be joined with an arbitrary point of the domain G by a continuous curve that entirely lies in G except for the end point c. Of course, $\partial G \setminus \partial\mathbb{D}$ is a Jordan arc. In particular, $\partial\mathbb{D}$ is called a *prime end* of a Cornucopia G (for the definition of prime ends, see [Go, p.39]). We note that if φ is a conformal map from $\mathbb{D}$ onto G then φ can be extended to a homeomorphism from $\operatorname{cl}\mathbb{D} \setminus \{\text{one point on } \partial\mathbb{D}\}$ onto $G \cup (\partial G \setminus \partial\mathbb{D})$ (cf. [Go, pp.40-44]).

If f is a conformal mapping of $\mathbb{D}$ onto the inside of a Jordan curve Γ, then f has a continuous one-to-one extension up to $\partial\,\mathbb{D}$ and when thus extended takes $\partial\,\mathbb{D}$ onto Γ. If Γ has a tangent at a point, we have:

Lemma 2.2. (Lindelöf theorem)[Ko, p.40] *Let G be a simply connected domain bounded by a Jordan curve Γ and $0 \in \Gamma$. Suppose that f maps $\mathbb{D}$ conformally onto G and $f(1) = 0$. If Γ has a tangent at 0, then for a constant c,*

$$\arg f(z) - \arg(1-z) \to c \quad for\ |z| < 1,\ z \to 1.$$

Note that Lemma 2.2 says that the conformal images of sectors in $\mathbb{D}$ with their vertices at 1 are asymptotically like sectors in G of the same opening with their vertices at 0.

We can extend Lemma 2.2 slightly.

Lemma 2.3. (An extension of Lindelöf theorem) *Let G be a simply connected domain and suppose a conformal map $\varphi : \mathbb{D} \to G$ can be extended to a homeomorphism*

$$\widetilde{\varphi} : \mathrm{cl}\,\mathbb{D} \setminus \{z_i \in \partial\mathbb{D} : i \in \mathbb{N}\} \to G \cup \{J_i : i \in \mathbb{N}\},$$

where the J_i are Jordan arcs on $\partial\,G$. If $0 \in J_1$, $\widetilde{\varphi}^{-1}(0) = 1 \notin \mathrm{cl}\,\{z_i : i \in \mathbb{N}\}$, and J_1 has a tangent at 0, then for a constant c,

$$\arg \varphi(z) - \arg(1-z) \to c \quad for\ |z| < 1,\ z \to 1.$$

Proof. Consider open disks $D_i = D_i(z_i, r_i)$ $(i = 1, 2, \cdots)$, where r_i is chosen so that $1 \notin \mathrm{cl}\,D_i$. Let $D = \mathbb{D} \setminus \cup_{i=1}^{\infty} \mathrm{cl}\,D_i$. Then D is simply connected. So by Riemann's mapping theorem there exists a conformal map ψ from $\mathbb{D}$ onto D such that $\psi(0) = 0$ and $\psi(1) = 1$. Then $\varphi \circ \psi$ is a conformal map from $\mathbb{D}$ onto a simply connected domain bounded by a Jordan curve. Clearly, the Jordan curve has a tangent at $\varphi \circ \psi(1) = \varphi(1) = 0$. Note that $1 - \psi$ is a conformal map from $\mathbb{D}$ onto $1 - D$. Also $\partial(1 - D)$ is a Jordan curve and $\partial(1 - D)$ has a tangent at $1 - \psi(1) = 0$. Now applying Lemma 2.2 with $\varphi \circ \psi$ and $1 - \psi$ gives the result. □

Applying Lemma 2.2, we can show the following geometric property of a bounded Carathéodory domain whose accessible boundary points lie in rectifiable Jordan arcs on its boundary. The following property was proved for the open unit disk in [Ber]. But our case is little subtle. The following lemma plays a key role in proving our main theorem.

Lemma 2.4. *Let G be a bounded Carathéodory domain whose accessible boundary points lie in rectifiable Jordan arcs on its boundary. If a subset $\Lambda \subset G$ is not dominating for G, i.e., there exists $h \in H^\infty(G)$ such that $\|h\|_G > \sup_{\lambda \in \Lambda} |h(\lambda)|$, then we can construct two rectifiable simple closed curves Γ and Γ' satisfying*

(i) *Γ and Γ' are exterior to each other;*
(ii) *Γ (resp. Γ') meets a Jordan arc J (resp. J') at two points, where $J \subset \partial G$ (resp. $J' \subset \partial G$);*
(iii) *Γ and Γ' cross Jordan arcs along line segments which are orthogonal to the tangent lines of the Jordan arcs;*
(iv) *$\Gamma \cap \Lambda = \phi$ and $\Gamma' \cap \Lambda = \phi$.*

Proof. Let φ be a conformal map from $\mathbb{D}$ onto the domain G. Then it is well known (cf. [Go, pp. 41-42]) that there exists a one-one correspondence between points on $\partial\mathbb{D}$ and the prime ends of the domain G and that every prime end of G contains no more than one accessible boundary point of G. Since G is a simply connected domain, the map φ^{-1} can be extended to a homeomorphism which maps a Jordan arc γ on ∂G, no interior point of which is a cluster point for $\partial G \setminus \gamma$, onto an arc on $\partial\mathbb{D}$ (cf. [Go, p.44, Theorem 4']). But since by our assumption, every accessible boundary point of ∂G lies in a Jordan arc of ∂G and the set of all points on $\partial\mathbb{D}$ corresponding to accessible boundary points of ∂G is dense in $\partial\mathbb{D}$ (cf. [Go, p.37, Theorem 1]), it follows that every prime end which contains no accessible boundary point of ∂G must be corresponded to an end point of an arc on $\partial\mathbb{D}$ corresponding to a Jordan arc on ∂G or a limit point of a sequence of disjoint Jordan arcs on $\partial\mathbb{D}$. Thus the points on $\partial\mathbb{D}$ corresponding to the prime ends which contain no accessible boundary points of ∂G form a countable set. Now let V be the set of 'singular' points, that is, points on $\partial\mathbb{D}$ corresponding to the prime ends which contain no accessible boundary points of ∂G. Then V is countable and the map φ can be extended to a homeomorphism from $\mathrm{cl}\,\mathbb{D}\setminus V$ onto $G\cup\{J_i : i=1,2,\cdots\}$, where the J_i are rectifiable Jordan arcs on ∂G. We denote this homeomorphism by still φ. Then we claim that

$$\Lambda'=\varphi^{-1}(\Lambda) \text{ is not dominating for } \mathbb{D}. \tag{2.1}$$

Indeed, by our assumption, $\|h\|_G>\sup_{\lambda\in\Lambda}|h(\lambda)|$ for some $h\in H^\infty(G)$. Since $\|h\|_G=\|h\circ\varphi\|_{\mathbb{D}}$ and φ is conformal on $\mathbb{D}$, we have that $h\circ\varphi\in H^\infty(\mathbb{D})$. Also, since

$$\sup_{\lambda\in\Lambda}|h(\lambda)|=\sup_{\lambda\in\Lambda'}|h(\varphi(\lambda))|=\sup_{\lambda\in\Lambda'}|(h\circ\varphi)(\lambda)|,$$

it follows that

$$\|h\circ\varphi\|_{\mathbb{D}}>\sup_{\lambda\in\Lambda'}|(h\circ\varphi)(\lambda)|,$$

giving (2.1). Write

$$\omega:=\Big\{\lambda\in\partial\mathbb{D}:\ \lambda \text{ is not approached nontangentially by points in } \Lambda'\Big\}.$$

Remember that $S\equiv\{\alpha_n\}\subset\mathbb{D}$ is dominating for $\mathbb{D}$ if and only if almost every point on $\partial\mathbb{D}$ is approached nontangentially by points of S (cf. [BSZ, Theorem 3]). It thus follows that ω has a positive measure. We put

$$W:=\big\{x\in J_i:\ J_i \text{ does not have a tangent at } x \text{ for } i=1,2,\cdots\big\}.$$

Then W has measure zero since the J_i are rectifiable and every rectifiable Jordan arc has a tangent almost everywhere. Now let $W'=\varphi^{-1}(W)$. Also W' has measure zero. Let θ be a fixed angle with $\frac{3}{4}\pi<\theta<\pi$ and let A_λ be the sector whose vertex is λ and whose radius is r_λ, of opening θ. Then for each $\lambda\in\omega$ we can find a rational number $r_\lambda\in(0,1)$ such that the sector A_λ contains no point in Λ'. Write

$$\widetilde{\omega}\equiv\omega\setminus(V\cup W').$$

Since $\widetilde{\omega}$ has a positive measure and hence it is uncountable, there exist a rational number $r\in(0,1)$ and an uncountable set $\omega'\subset\widetilde{\omega}$ such that $r=r_\lambda$ for all $\lambda\in\omega'$. Clearly, we can find distinct points $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ in ω' such that

$$A_{\lambda_1}\cap A_{\lambda_2}\neq\phi \ \text{ and } \ A_{\lambda_3}\cap A_{\lambda_4}\neq\phi.$$

We can thus construct two rectifiable arcs Γ_1° and Γ_2° in $\mathrm{cl}\,\mathbb{D}$ such that

$$\Gamma_1^\circ\cap\mathbb{D}\subset A_{\lambda_1}\cup A_{\lambda_2},\quad \Gamma_1^\circ\cap\mathbb{T}=\{\lambda_1,\lambda_2\}$$

and

$$\Gamma_2^\circ\cap\mathbb{D}\subset A_{\lambda_3}\cup A_{\lambda_4},\quad \Gamma_2^\circ\cap\mathbb{T}=\{\lambda_3,\lambda_4\}.$$

Let $\eta_i:=\varphi(\lambda_i)$ for $i=1,\dots,4$. Then, since φ is a homeomorphism, η_i's are distinct. Also, each η_i is contained in a Jordan arc of ∂G. Let $B_i:=\varphi(A_{\lambda_i})$. Then, since $\frac{3}{4}\pi<\theta<\pi$, we can, by Lemma 2.3, find a line segment $l_i\subset B_i$ which is orthogonal to the tangent line at η_i. Let $L_i:=\varphi^{-1}(l_i)$. Then, by cutting off the end parts of Γ_1° and Γ_2° and joining L_i's, we can construct two new rectifiable arcs $\widetilde{\Gamma}_1^\circ$ and $\widetilde{\Gamma}_2^\circ$. Let $\widetilde{\Gamma}:=\varphi(\widetilde{\Gamma_1^\circ})$ and $\widetilde{\Gamma}':=\varphi(\widetilde{\Gamma_2^\circ})$. Since G is a Carathéodory domain and the end parts of $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ are line segments, by extending straightly the end parts of $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ in the unbounded component of $\mathbb{C}\setminus\mathrm{cl}\,G$, we can construct two Jordan curves $\widehat{\Gamma}$ and $\widehat{\Gamma}'$ whose end parts cross the boundary of G through line segments. Therefore, by joining end points of $\widehat{\Gamma}$ (resp., the end points of $\widehat{\Gamma}'$) by a rectifiable arc in the unbounded component, we can find a simple closed rectifiable curve Γ (resp., Γ') satisfying the given conditions. $\square$

We are ready for proving the main theorem.

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be such that $\|p(T)\| \le \|p\|_{\sigma(T)}$ for every polynomial p. If $\eta\sigma(T)$ is a Carathéodory region whose accessible boundary points lie in rectifiable Jordan arcs on its boundary, then T has a nontrivial invariant subspace.*

Proof. To investigate the invariant subspaces, we may assume that T has no nontrivial reducing subspace and $\sigma(T) = \sigma_{ap}(T)$, where $\sigma_{ap}(T)$ denotes the approximate point spectrum of T. Since the complement of $\eta\sigma(T)$ is connected, we have that by Mergelyan's theorem, $R(\eta\sigma(T)) = P(\eta\sigma(T))$. We thus have

$$\|f(T)\| \le \|f\|_{\eta\sigma(T)} \quad \text{for any } f \in R(\eta\sigma(T)),$$

which says that $\eta\sigma(T)$ is a spectral set for T. On the other hand, we note that $R(\eta\sigma(T))$ $(= P(\eta\sigma(T)))$ is a Dirichlet algebra. Thus if $\sigma(T) \not\subset \mathrm{cl}\,(\mathrm{int}\,\eta\sigma(T))$, then it follows from a theorem of J. Stampfli [St, Proposition 1] that T has a nontrivial invariant subspace. So we may, without loss of generality, assume that $\sigma(T) \subset \mathrm{cl}\,(\mathrm{int}\,\eta\sigma(T))$. In this case we have that $\eta\sigma(T) = \mathrm{cl}\,(\mathrm{int}\,\eta\sigma(T))$. Hence $\mathrm{int}\,\eta\sigma(T)$ is a Carathéodory domain.

Now since $\eta\sigma(T)$ is a spectral set, $R(\eta\sigma(T))$ is a Dirichlet algebra, and T has no nontrivial reducing subspaces, it follows from Lemma 2.1 that there exists an extension of the functional calculus of T to a norm contractive algebra homomorphism

$$\phi : H^\infty(\mathrm{int}\,\eta\sigma(T)) \to \mathcal{B}(\mathcal{H}). \tag{2.2}$$

Moreover, ϕ is weak*-weak* continuous. Let $0 < \varepsilon < \frac{1}{2}$. Consider the following set:

$$\Lambda(\varepsilon) = \Big\{\lambda \in \mathrm{int}\,\eta\sigma(T) : \ \exists \text{ a unit vector } x \text{ such that } \ \|(T-\lambda)x\| < \varepsilon\,\mathrm{dist}\,\big(\lambda, \partial(\eta\sigma(T))\big)\Big\}.$$

There are two cases to consider.

Case 1: $\Lambda(\varepsilon)$ is not dominating for $\mathrm{int}\,\eta\sigma(T)$. Since $\mathrm{int}\,\eta\sigma(T)$ is a Carathéodory domain, we can find two rectifiable simple closed curves Γ and Γ' satisfying the conditions given in Lemma 2.4; in particular, $\Gamma \cap \Lambda(\epsilon) = \emptyset$ and $\Gamma' \cap \Lambda(\epsilon) = \emptyset$. Let

$$\Gamma \cap \partial(\eta\sigma(T)) = \{\lambda_1, \lambda_2\} \quad \text{and} \quad \Gamma' \cap \partial(\eta\sigma(T)) = \{\lambda_3, \lambda_4\}.$$

Since $\sigma(T) = \sigma_{ap}(T)$, it is clear that $\Lambda(\varepsilon) \supset \mathrm{int}\,\eta\sigma(T) \cap \sigma(T)$. So $T - \lambda$ is invertible for any λ in $\Gamma\backslash\{\lambda_1, \lambda_2\}$ and $\Gamma'\backslash\{\lambda_3, \lambda_4\}$. If $\lambda \in \Gamma\backslash\eta\sigma(T)$, then since the functional calculus in (2.2) is contractive, we have

$$\|(\lambda - T)^{-1}\| \le \sup\left\{\frac{1}{|\lambda - \mu|} : \mu \in \mathrm{int}\,\eta\sigma(T)\right\} = \frac{1}{\mathrm{dist}(\lambda,\ \partial\,(\eta\sigma(T))}.$$

Let $\lambda \in \Gamma\, \cap\, \mathrm{int}\,\eta\sigma(T)$. Since $\Gamma \cap \Lambda(\varepsilon) = \emptyset$, we have that for any unit vector x,

$$\|(T-\lambda)x\| \ \ge\ \varepsilon\,\mathrm{dist}\,(\lambda, \partial(\eta\sigma(T))),$$

which implies that

$$\|(\lambda - T)^{-1}\| \le \frac{1}{\varepsilon\,\mathrm{dist}\,(\lambda,\ \partial(\eta\sigma(T)))}.$$

On the other hand, Since $\partial(\eta\sigma(T))$ has a tangent at λ_i, it follows that in a sufficiently small neighborhood N_i of λ_i, $\partial(\eta\sigma(T))$ lies in a double-sector A_i of opening $2\theta_i$ $(0 < \theta_i < \frac{\pi}{2})$ for each $i = 1, 2$. But since Γ is a line segment in a sufficiently small neighborhood of each λ_i $(i = 1, 2)$, it follows that if $\lambda \in N_i \cap \Gamma$, then

$$\frac{|\lambda - \lambda_i|}{\mathrm{dist}\,(\lambda,\ \partial(\eta\sigma(T)))} \ \le\ \frac{|\lambda - \lambda_i|}{\mathrm{dist}\,(\lambda,\ A_i)} = \frac{1}{\cos\,\theta_i} =: c.$$

We thus have

$$\|(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - T)^{-1}\| \ \le\ \frac{c}{\varepsilon}|\lambda - \lambda_2| \le M \quad \text{on } N_1 \cap \Gamma,$$

which says that $S_\lambda \equiv (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-T)^{-1}$ is bounded on $N_1\cap\Gamma$. Also S_λ has at most two discontinuities on Γ. So the following operator A is well-defined ([Ap]):

$$A := \frac{1}{2\pi i}\int_\Gamma (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-T)^{-1}d\lambda.$$

Now, using the argument of [Ber, Lemma 3.1]), we can conclude that $\ker(A)$ is a nontrivial invariant subspace for T.

Case 2: $\Lambda(\varepsilon)$ *is dominating for* $\operatorname{int}\eta\sigma(T)$. In this case, we can show that ϕ is isometric, i.e.,

$$\|h(T)\| = \|h\|_{\operatorname{int}\eta\sigma(T)} \quad \text{for all } h\in H^\infty(\operatorname{int}\eta\sigma(T)),$$

by using the same argument as the well-known method due to Apostol (cf. [Ap]), in which it was shown that the Sz.-Nagy-Foias calculus is isometric. Now consider a conformal map $\varphi:\mathbb{D}\to \operatorname{int}\eta\sigma(T)$ and then define the function ψ by

$$\psi=\varphi^{-1}:\operatorname{int}\eta\sigma(T)\to\mathbb{D}.$$

Then $\psi\in H^\infty(\operatorname{int}\eta\sigma(T))$. Define $A:=\psi(T)$. Then A is an absolutely continuous contraction with norm 1. Thus we can easily show that

$$\|h(A)\| = \|h\|_{\mathbb{D}} \quad \text{for any } h\in H^\infty(\mathbb{D}).$$

Thus if $\lambda_0\in\mathbb{T}$, then

$$\lim_{\lambda\to\lambda_0,|\lambda|>1}||(A-\lambda)^{-1}|| = \lim_{\lambda\to\lambda_0,|\lambda|>1}||(z-\lambda)^{-1}||_{\mathbb{D}}=\infty,$$

which implies that $A-\lambda_0$ is not invertible, so that we get $\mathbb{T}\subset\sigma(A)$. Since every contraction whose spectrum contains the unit circle has a nontrivial invariant subspace ([BCP2]), A has a nontrivial invariant subspace. On the other hand, since $T\in \text{weak}^*\text{-cl}\,\{p(A): p \text{ is a polynomial}\}$, we can conclude that T has a nontrivial invariant subspace. $\square$

A simple example for the set satisfying $||p(T)||\le||p||_{\sigma(T)}$ for every polynomial p is the set of 'polynomially normaloid' operators, in the sense that $p(T)$ is normaloid (i.e., norm equals spectral radius) for every polynomial p. Indeed if $p(T)$ is normaloid then $||p(T)|| = \sup_{\lambda\in\sigma(p(T))}|\lambda| = ||p||_{\sigma(T)}$ by the spectral mapping theorem.

Remark. We were unable to decide whether in Theorem 2.5, the condition "$||p(T)||\le||p||_{\sigma(T)}$" can be relaxed to the condition "$||p(T)||\le k\,||p||_{\sigma(T)}$ for some $k>0$". However we can prove that *if $T\in\mathcal{B}(\mathcal{H})$ is such that $||p(T)||\le k\,||p||_{\sigma(T)}$ for every polynomial p and some $k>0$ and if the outer boundary of $\sigma(T)$ is a Jordan curve then T has a nontrivial invariant subspace.* This is a corollary of the theorem of C. Ambrozie and V. Müller [AM, Theorem A]. The proof goes as follows. Since $\partial\,(\eta\sigma(T))$ is a Jordan curve, then by Carathéodory's theorem on extensions of the conformal representations, a conformal map $\varphi:\operatorname{int}\eta\sigma(T)\to\mathbb{D}$ can be extended to a homeomorphism $\psi:\eta\sigma(T)\to\operatorname{cl}\mathbb{D}$. Since $\mathbb{C}\setminus\eta\sigma(T)$ is connected, we can find polynomials p_n such that $p_n\to\psi$ uniformly on $\eta\sigma(T)$. Since the spectrum function $\sigma:\mathcal{B}(\mathcal{H})\to\mathbb{C}$ is upper semi-continuous, it follows that

$$\psi(\partial(\eta\sigma(T)))\subset\psi(\sigma(T))=\limsup p_n(\sigma(T))=\limsup\sigma(p_n(T))\subset\sigma(\psi(T)).$$

But since ψ is a homeomorphism we have that $\partial\mathbb{D}\subset\sigma(\psi(T))$. By our assumption we can also see that

$$||(p\circ\psi)(T)||\le k\,||p\circ\psi||_{\operatorname{int}\eta\sigma(T)} = k\,||p||_{\mathbb{D}} \quad \text{for every polynomail } p,$$

which says that $\psi(T)$ is a polynomially bounded operator. Therefore by the theorem of C. Ambrozie and V. Müller[AM], $\psi(T)$ has a nontrivial invariant subspace. Hence we can conclude that T has a nontrivial invariant subspace.

We conclude with a result on the invariant subspaces for hyponormal operators (this applies, in particular, to the case when $\eta\sigma(T)$ is the closure of a Cornucopia).

Corollary 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator. If the outer boundary of $\sigma(T)$ has at most finitely many prime ends corresponding to singular points on $\partial\mathbb{D}$ and has a tangent at almost every point on each Jordan arc, with respect to a conformal map from $\mathbb{D}$ onto $\operatorname{int}\eta\sigma(T)$, then T has a nontrivial invariant subspace.*

Proof. Suppose that

$$\|h\|_{\operatorname{int}\eta\sigma(T)} = \sup\bigl\{|h(\lambda)| : \lambda \in \sigma(T) \cap \operatorname{int}\eta\sigma(T)\bigr\}$$

for all $h \in H^\infty(\operatorname{int}\eta\sigma(T))$. Then $\sigma(T) \cap \operatorname{int}\eta\sigma(T)$ is dominating for $\operatorname{int}\eta\sigma(T)$. Thus by the well-known theorem due to S. Brown [Br2, Theorem 2], T has a nontrivial invariant subspace. Suppose instead that

$$\|h\|_{\operatorname{int}\eta\sigma(T)} > \sup\bigl\{|h(\lambda)| : \lambda \in \sigma(T) \cap \operatorname{int}\eta\sigma(T)\bigr\}$$

for some $h \in H^\infty(\operatorname{int}\eta\sigma(T))$. By an analysis of the proof of Lemma 2.4, we can construct two rectifiable curves Γ and Γ' satisfying the conditions (i) - (iv). Let $\Gamma \cap \partial\eta\sigma(T) = \{\lambda_1, \lambda_2\}$ and $\Gamma' \cap \partial\eta\sigma(T) = \{\lambda_3, \lambda_4\}$. Since T is a hyponormal operator, we have

$$\|(\lambda - T)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))} \quad \text{on} \quad \lambda \in \bigl(\Gamma\backslash\{\lambda_1, \lambda_2\}\bigr) \cup \bigl(\Gamma'\backslash\{\lambda_3, \lambda_4\}\bigr).$$

Now the same argument as in Case 1 of the proof of Theorem 2.5 shows that T has a nontrivial invariant subspace. □

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