2-UNIVERSAL POSITIVE DEFINITE INTEGRAL QUINARY QUADRATIC FORMS

Byeong Moon Kim[†], Myung-Hwan Kim[‡] and Byeong-Kweon Oh[‡]

† Dept. of Math., Kangnung Nat'l Univ., Kangwondo
210-702, Korea (kbm@knusun.kangnung.ac.kr)
‡ Dept. of Math., Seoul Nat'l Univ., Seoul 151-742, Korea
(mhkim@math.snu.ac.kr and oandhan@math.snu.ac.kr)

Abstract. As a generalization of the famous four square theorem of Lagrange, Ramanujan and Willerding found all positive definite integral quaternary quadratic forms that represent all positive integers. In this paper, we find all positive definite integral quinary quadratic forms that represent all positive definite integral binary quadratic forms. We also discuss recent results on positive definite integral quadratic forms which represent all positive integral quadratic forms in three or more variables.

§1. Introduction

The famous *four square theorem* of Lagrange [10] says that the quadratic form $x^2 + y^2 + z^2 + u^2$ represents all positive integers. In the early 20th century, Ramanujan [17] extended Lagrange's result by listing all 54 positive definite integral quaternary diagonal quadratic forms, up to equivalence, that represent all positive integers. Dickson [4] called such forms *universal* and confirmed Ramanujan's list. Willerding [18] proved that there are exactly 124 universal positive definite integral quaternary non-diagonal quadratic forms up to equivalence. It is not hard to show that 'quaternary' is the best possible in the sense that there is no universal positive definite integral ternary quadratic forms. Throughout this paper, by *integral* quadratic forms we mean *classic* ones, that is, quadratic forms with integer coefficients such that coefficients of non-diagonal terms are multiples of 2.

In 1930, Mordell [11] proved the *five square theorem* which says that the quadratic form $x^2+y^2+z^2+u^2+v^2$ represents all positive definite integral binary quadratic forms. This is a very interesting new direction of extending Lagrange's four square theorem.

1991 *Mathematics Subject Classification.* 11E20, 11E25, 11E41.

Key words and phrases. 2-universal, positive definite, quinary **Z**-lattices.

This work was partially supported by Korean Ministry of Education (BSRI 98-1414) and Seoul National University Development Research Fund.

(See [7],[9] for further development in this direction.) We call such a form *2-universal*. Then the following question arises naturally:

($*$) *How many 2-universal positive definite integral quinary quadratic forms are there?*

Recently, the authors together with Raghavan [6] provided a partial answer to the question by listing all five 2-universal positive definite integral quinary diagonal quadratic forms. In this paper, we provide a complete answer to the question by adding all non-diagonal ones to our previous list. In Sections 3 and 4, we prove that there are precisely six 2-universal positive definite integral quinary non-diagonal quadratic forms up to equivalence. So, there are eleven such forms altogether and they are :

$$x^2+y^2+z^2+u^2+v^2, \quad x^2+y^2+z^2+u^2+2v^2, \quad x^2+y^2+z^2+u^2+3v^2,$$
$$x^2+y^2+z^2+2u^2+2v^2, \quad x^2+y^2+z^2+2u^2+3v^2,$$
$$x^2+y^2+z^2+2u^2+2uv+2v^2, \quad x^2+y^2+z^2+2u^2+2uv+3v^2,$$
$$x^2+y^2+2z^2+2u^2+2uv+2v^2, \quad x^2+y^2+3z^2+2u^2+2uv+2v^2,$$
$$x^2+y^2+2z^2+2zu+2u^2+2uv+2v^2, \quad x^2+y^2+2z^2+2zu+2u^2+2uv+3v^2.$$

Remark 1. The five diagonal forms above coincide with the forms introduced by Peters in [16]. There, he proved the following interesting property : Let K be a real quadratic field and let $\mathcal{O}_f$ be an order of K with conductor f. Then each of the above five diagonal forms, over $\mathcal{O}_f$, represents all totally positive elements of $\mathcal{O}_f$, that can be represented as sums of squares of elements of $\mathcal{O}_f$.

In Sections 5 and 6, we provide a criterion for 2-universality of positive definite integral quadratic forms and introduce some of recent results concerning n-universal $\mathbf{Z}$-lattices for $n \geq 3$.

§2. A geometrical setting

We adopt lattice theoretic language. A $\mathbf{Z}$-*lattice* L is a finitely generated free $\mathbf{Z}$-module equipped with a non-degenerate symmetric bilinear form B such that $B(L,L) \subseteq \mathbf{Z}$. The corresponding quadratic map is denoted by Q. Note that $\mathbf{Z}$-lattices naturally corresponds to *(classic) integral* quadratic forms, that is, quadratic forms with integer coefficients such that coefficients of non-diagonal terms are multiples of 2.

For a $\mathbf{Z}$-lattice $L = \mathbf{Z}\mathbf{e}_1 + \mathbf{Z}\mathbf{e}_2 + \cdots + \mathbf{Z}\mathbf{e}_n$ where $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ is a fixed basis, we write $L \cong (B(\mathbf{e}_i, \mathbf{e}_j))$. For $\mathbf{Z}$-sublattices L_1, L_2 of L, we write $L = L_1 \perp L_2$ when $L = L_1 \oplus L_2$ and $B(\mathbf{v}_1, \mathbf{v}_2) = 0$ for all $\mathbf{v}_1 \in L_1, \mathbf{v}_2 \in L_2$. If L admits an orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, we call L *diagonal* and simply write $L \cong \langle Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n)\rangle$. We call L *non-diagonal* otherwise. L is called *positive definite* or simply *positive* if $Q(\mathbf{v}) > 0$ for any $\mathbf{v} \in L, \mathbf{v} \neq \mathbf{0}$. As usual, $dL = \det(B(\mathbf{e}_i, \mathbf{e}_j))$ is called the *discriminant* of L.

We define $RL := R \otimes L$ for any ring R containing $\mathbf{Z}$. If $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is an orthogonal basis of the quadratic space $V = \mathbf{Q}L$ or $\mathbf{Q}_pL$, we write $V \cong (Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n))$ for convenience.

Let ℓ, L be $\mathbf{Z}$-lattices. We say that L *represents* ℓ if there is a representation σ from ℓ into L, and write $\ell \to L$. A *representation* is an injective $\mathbf{Z}$-linear map preserving the bilinear form B. A positive $\mathbf{Z}$-lattice L is called *n-universal* if L represents all n-ary positive $\mathbf{Z}$-lattices ℓ.

So, each of Ramanujan's 54 forms above corresponds to a 1-universal positive quaternary diagonal $\mathbf{Z}$-lattice. Mordell's *five square theorem* is nothing but the 2-universality of $I_5 \cong \langle 1,1,1,1,1\rangle$ while Lagrange's *four square theorem* is precisely the 1-universality of I_4. In fact, it is known [9] that I_n is $(n-3)$-universal if $4 \le n \le 8$.

We will denote for convenience

$$[a,b,c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad [a,b,c,d,e] \cong \begin{pmatrix} a & b & 0 \\ b & c & d \\ 0 & d & e \end{pmatrix}.$$

For any unexplained terminology and basic facts about $\mathbf{Z}$-lattices, we refer the readers to O'Meara's book [15].

§3. Main theorem

In this section, we eliminate all others except eleven candidates for 2-universal positive quinary $\mathbf{Z}$-lattices. Of course, the eleven candidates are the positive quinary $\mathbf{Z}$-lattices corresponding to the quadratic forms listed in Section 1.

(Diagonal case)

Let $L \cong \langle a_1, a_2, a_3, a_4, a_5\rangle$ be a 2-universal positive quinary diagonal $\mathbf{Z}$-lattice with $0 < a_1 \le a_2 \le a_3 \le a_4 \le a_5$. Since L represents $\langle 1,1\rangle$ which is unimodular, $\langle 1,1\rangle$ splits L (see [15;82:15]). So, we obtain $a_1 = a_2 = 1$. If $a_3 \ge 2$, then L cannot represent $[2,1,2]$. So, we obtain $a_3 = 1$. Similarly, if $a_4 \ge 3$, then L cannot represent $[3,2,3]$. So, $a_4 = 1$ or 2. Now suppose $a_5 \ge 4$. If $a_4 = 1$, then $[4,1,4]$ cannot be represented by L, and if $a_4 = 2$, then $\langle 3,3\rangle$ cannot be represented by L. So, we obtain $a_5 = 1, 2$, or 3. In summary, only five candidates for 2-universal positive quinary diagonal $\mathbf{Z}$-lattices L survive and they are

$$L_{a,b} \cong \langle 1,1,1,a,b\rangle \text{ with } (a,b) = (1,1),(1,2),(1,3),(2,2),(2,3). \tag{3.1}$$

(Non-diagonal case)

Now, we assume L is non-diagonal. Still $\langle 1,1\rangle$ splits L and hence $L \cong \langle 1,1\rangle \perp L_0$ for some non-diagonal ternary $\mathbf{Z}$-lattice L_0.

(Subcase 1) 1 is represented by L_0, i.e., $1 \in Q(L_0)$: In this case, $L_0 \cong \langle 1\rangle \perp L_1$, where $L_1 \cong [a,b,c]$. We may assume that $2 \le a \le c$, $0 < 2b \le a$, i.e., $[a,b,c]$ is Minkowski reduced. If $a \ge 3$, then L cannot represent $[2,1,3]$. So, $a = 2$ and $b = 1$. $c = 2$ or 3 in order for L to represent $\langle 3,3\rangle$. Therefore,

$$L \cong \langle 1,1,1\rangle \perp L_1 \text{ with } L_1 \cong [2,1,2],\quad [2,1,3]. \tag{3.2}$$

(Subcase 2) 1 is not represented by L_0, i.e., $1 \notin Q(L_0)$: Since $[2,1,2]$ should be represented by L_0, we may easily conclude

$$L_0 \cong [2,1,2] \perp \langle c\rangle,\quad [2,1,2,1,c].$$

In order for L to represent $\langle 2,3\rangle$, it is necessary to require $c = 2$ or 3 and hence

$$L \cong \langle 1,1\rangle \perp L_0 \text{ with } L_0 \cong [2,1,2] \perp \langle 2\rangle,\ [2,1,2] \perp \langle 3\rangle,\ [2,1,2,1,2],\ [2,1,2,1,3]. \tag{3.3}$$

So far, we reduced the number of candidates for 2-universal positive quinary $\mathbf{Z}$-lattices down to eleven and they are the ones listed in (3.1), (3.2) and (3.3). We prove that these eleven survivors are indeed 2-universal.

Remark 2. A $\mathbf{Z}$-lattice L is called *2-regular* if L represents all positive binary $\mathbf{Z}$-lattices which are represented by L locally everywhere. Recently, Earnest [5] proved that there are only finitely many 2-regular primitive positive quaternary $\mathbf{Z}$-lattices, up to isometry. Observe from the above that no positive quaternary $\mathbf{Z}$-lattice can be 2-universal. So, 'quinary' is the smallest possible rank for 2-universal positive $\mathbf{Z}$-lattice.

Theorem 1. *There are exactly eleven 2-universal positive quinary $\mathbf{Z}$-lattices up to isometry, and they are*

$$\begin{gathered}\langle 1,1,1,1,1\rangle,\ \langle 1,1,1,1,2\rangle,\ \langle 1,1,1,1,3\rangle,\ \langle 1,1,1,2,2\rangle,\ \langle 1,1,1,2,3\rangle,\\ \langle 1,1,1\rangle \perp [2,1,2],\quad \langle 1,1,1\rangle \perp [2,1,3],\\ \langle 1,1,2\rangle \perp [2,1,2],\quad \langle 1,1,3\rangle \perp [2,1,2],\\ \langle 1,1\rangle \perp [2,1,2,1,2],\quad \langle 1,1\rangle \perp [2,1,2,1,3].\end{gathered}$$

Proof. **(Diagonal case)** The five diagonal forms $L_{a,b}$ with $(a,b) = (1,1),(1,2),(1,3),(2,2),(2,3)$ are all 2-universal because each of them is 2-universal over p-adic integer ring $\mathbf{Z}_p$ at every prime p (including ∞) and has class number 1. For the details, see [6].

(Non-diagonal case) Let's denote

$$K_c^a := \langle 1,1,a\rangle \perp [2,1,c],\quad M_e := \langle 1,1\rangle \perp [2,1,2,1,e].$$

where $(a,c) = (1,2),(1,3),(2,2),(3,2); e = 2,3$.

By using Theorems 1 and 3 of [14], one can show that the above six non-diagonal $\mathbf{Z}$-lattices are all 2-universal locally at every prime p (including ∞). Furthermore, one can easily check that all of $K_2^1, K_3^1, K_2^3, M_2, M_3$ have class number 1 (see [3] for example) and hence they are all 2-universal (over $\mathbf{Z}$) as was to be shown. Although K_2^2 has class number 2, it is still 2-universal. We provide its proof in the next section. $\square$

§4. Supplementary proof of the main theorem

In this section, we prove the 2-universality of K_2^2, which completes the proof of Theorem 1. Although the local 2-universality of K_2^2 at every prime p (including ∞) has been proved in the previous section, we cannot immediately conclude its 2-universality because its class number is 2. The other class in the genus of K_2^2 is the class of $\langle 1,1,1,1,6\rangle$, which we denote by K_6.

Let $\ell \cong [a,b,c]$ be a positive definite binary $\mathbf{Z}$-lattice. We may assume that:

$$[a,b,c] \text{ is Minkiwski reduced, i.e., } 0 \le 2b \le a \le c. \tag{4.1}$$

Lemma 1. *(1) If ℓ is not even unimodular over $\mathbf{Z}_2$, then $\ell \to K_2^2$ over $\mathbf{Z}$.*
(2) If $d\ell \not\equiv 1 \pmod 3$, then $\ell \to K_2^2$ over $\mathbf{Z}$.

Proof. For a sublattice ℓ of $L \perp M$ of the form

$$\ell = \mathbf{Z}(x_1+y_1) + \mathbf{Z}(x_2+y_2) + \cdots + \mathbf{Z}(x_n+y_n),$$

for $x_i \in L$, $y_i \in M$, we define sublattices

$$\ell(L) := \mathbf{Z}x_1 + \mathbf{Z}x_2 + \cdots + \mathbf{Z}x_n, \quad \ell(M) := \mathbf{Z}y_1 + \mathbf{Z}y_2 + \cdots + \mathbf{Z}y_n.$$

We also simply write $\ell(L)$ instead of $\sigma(\ell)(L)$ if it causes no confusion, where $\sigma : \ell \to L \perp M$ is a representation.

(1) Let ℓ be not even unimodular over $\mathbf{Z}_2$. Assume that $\ell \to K_6 = I_4 \perp \langle 6\rangle$ over $\mathbf{Z}$. Note that over $\mathbf{Z}_2$ every binary lattice which is not isometric to $[2,1,2]$ is represented by $\langle 1,1,2,2\rangle$ if it is represented by $\langle 1,1,1,1\rangle$. So, $\ell(I_4) \to \langle 1,1,2,2\rangle$ over $\mathbf{Z}_2$ and hence over $\mathbf{Z}$

$$\ell \to \ell(I_4) \perp \langle 6\rangle \to \langle 1,1,2,2,6\rangle \to K_2^2.$$

(2) Let $d\ell \not\equiv 1 \pmod 3$. Note that $\langle 1,3\rangle \perp [2,1,2] \perp \langle 6\rangle$ is a sublattice of K_2^2. The result follows in a similar manner as above after replacing $\langle 1,1,2,2\rangle$ by $\langle 1,3\rangle \perp [2,1,2]$. $\square$

By Lemma 1, we may assume that $\ell \cong [a,b,c]$ satisfies:

$$Q(\ell) \subseteq 2\mathbf{Z}, \quad d\ell \equiv 1 \pmod 6. \tag{4.2}$$

Note that $a \equiv 2, 4 \pmod 6$ and b is odd. We define ℓ_s and $\ell(\alpha, \beta)$ by

$$\ell_s \cong [a, sa+b, s^2a+2sb+c], \quad \ell(\alpha,\beta) \cong [a-2\alpha^2, b-2\alpha\beta, c-2\beta^2],$$

where s, α, β are integers. Note that $\ell_s \cong \ell = \ell_0$. If $\ell_s(\alpha,\beta)$ is represented by $N :\cong \langle 1,1\rangle \perp [2,1,2]$ over $\mathbf{Z}$, then ℓ is represented by $N \perp \langle 2\rangle = K_2^2$. In the following, we prove that there exists an s for which $\ell_s(3,0)$ is positive definite and $\ell_s(3,0) \to N$ over $\mathbf{Z}$ under the assumption that ℓ is not represented by N and that the minimum a of ℓ is large. The case when a is not large enough to make $\ell_s(3,0)$ positive definite can be proved by simple computation. Note that:

$$\ell_s(3,0) \to N \text{ over } \mathbf{Z}_p \text{ if } \begin{cases} p = 2,3 \ \text{ by (4.2) or} \\ \left(\dfrac{3}{p}\right) = 1 \ \text{ or} \\ \left(\dfrac{3}{p}\right) = -1 \ \text{ and } \mathfrak{s}(\ell_s(3,0)) \not\subseteq p\mathbf{Z}_p. \end{cases}$$

Since N is of class number 1, it is enough to show that for every $p \ge 5$ satisfying $\left(\dfrac{3}{p}\right) = -1$ (for example $p = 5, 7, 17, 19, 29, 31, \cdots$), there exists an s such that $\ell_s(3,0)$ is positive definite and $\ell_s(3,0) \to N$ over $\mathbf{Z}_p$.

Lemma 2. *If $a > 24(s^2+|s|+1)$, then $\ell_s(3,0)$ is positive definite.*

Proof. Note that $d\ell_s(3,0) = ac - b^2 - 18(s^2a+2sb+c)$. Hence

$$d\ell_s(3,0) = \frac{ac}{4} - b^2 + \frac{3ac}{4} - 18(s^2a+2sb+c) \ge \frac{3c}{4}\left(a - 24(s^2+|s|+1)\right) > 0. \quad \square$$

Lemma 3. *For $t \ge 2$, let $p_1 < p_2 < \cdots < p_t$ be primes and let a be an integer such that $\gcd(a, p_1p_2\ldots p_t) = 1$. If $n \ge \dfrac{p_1+t-1}{p_1-1}2^t$, then there is a number in the set $\{d, a+d, \cdots, (n-1)a+d\}$, which is relatively prime to $p_1p_2\ldots p_t$ for any integer d.*

Proof. In the set $T := \{d, a+d, \cdots, (n-1)a+d\}$, the number of multiples $p_{j_1}p_{j_2}\cdots p_{j_k}$ is smaller than or equal to $\dfrac{n}{p_{j_1}p_{j_2}\cdots p_{j_k}} + 1$ and is larger than $\dfrac{n}{p_{j_1}p_{j_2}\cdots p_{j_k}} - 1$. So, the number of elements of T which is relatively prime to $p_1p_2\cdots p_t$ is larger than

$$\begin{aligned} n &- \left(\frac{n}{p_1} + 1 + \frac{n}{p_2} + 1 + \cdots + \frac{n}{p_t} + 1\right) + \left(\frac{n}{p_1p_2} - 1 + \cdots + \frac{n}{p_{t-1}p_t} - 1\right) \\ &+ \cdots + (-1)^t\left(\frac{n}{p_1p_2\ldots p_t} + (-1)^{t-1}\right) \\ &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_t}\right) - 2^t + 1 \ge n\frac{p_1-1}{p_1+t-1} - 2^t + 1 \ge 1. \quad \square \end{aligned}$$

If a is not divisible by any prime $p \geq 5$ satisfying $\left(\frac{3}{p}\right) = -1$, then $\ell \to N$. Hence we assume that

$$a \text{ is divisible by at least one prime } p \geq 5 \text{ satisfying } \left(\frac{3}{p}\right) = -1. \tag{4.3}$$

Let $p_1, \cdots, p_t \geq 5$ be the odd prime factors of $a - 18$ satisfying $\left(\frac{3}{p_i}\right) = -1$ for all i. We want to choose a suitable integer s such that $sa + b$ is relatively prime to $p_1 p_2 \ldots p_t$.

(Step 1) $t = 0$:

If $a \geq 24$, then $\ell(3,0) = \ell_0(3,0)$ is positive definite and $\ell(3,0) \to N$ over $\mathbf{Z}$. Let $a < 24$. If $c \geq 24$, than either ℓ or $\ell(0,3)$ is represented by N over $\mathbf{Z}$. The remaining cases are trivial.

(Step 2) $t = 1$:

Since either $-a + b$ or $a + b$ is not divisible by p_1, we may choose s from $\{-1, 1\}$. Then $\ell_s(3,0) \to N$ over $\mathbf{Z}_p$ for every finite prime p. Hence by Lemma 2, if $a \geq 74$, then $\ell_s(3,0)$ is positive definite and $\ell_s(3,0) \to N$ over $\mathbf{Z}$. If $a \leq 72$, $a = 28, 38, 56, 58, 68$ because ℓ satisfies (4.1)-(4.3). Now we only consider the case $a = 68$. The other cases can be treated in a similar manner. If c is not divisible by 17, then $\ell \to N$ over $\mathbf{Z}$. If c is divisible by 17, then $\ell(0,3) \cong [68, b, c - 18] \to N$.

(Step 3) $t = 2$:

We may choose s from $\{-1, 0, 1\}$. If $a \geq 74$, then $\ell_s(3,0)$ is positive definite and $\ell_s(3,0) \to N$ over $\mathbf{Z}$. Note that $a \leq 72$ is not possible.

(Step 4) $t = 3, 4$:

In these cases, we may choose s from $\{-2, -1, 0, 1, 2\}$. Since $a - 18 \geq 5 \times 7 \times 11 = 385$, $\ell_s(3,0)$ is positive definite and $\ell_s(3,0) \to N$.

(Step 5) $t \geq 5$:

By Lemma 3, there exists an integer $s \in \{-t2^{t-1}, -t2^{t-1} + 1, \ldots, t2^{t-1} - 1, t2^{t-1}\}$ such that $as + b$ is relatively prime to $p_1 p_2 \ldots p_t$. Since

$$a \geq p_1 p_2 \ldots p_t + 18 \geq 5 \times 7 \times 17 \times 19 \times 29^{t-4} + 18,$$

$\ell_s(3,0)$ is positive definite and $\ell_s(3,0) \to N$ over $\mathbf{Z}$. $\square$

§5. Characterization of 2-universality

Recently, Conway and Schneeberger [2] announced the so called '15-Theorem', which characterizes the (1-)universality by the representability of a finite set of numbers, namely, 1, 2, 3, 5, 6, 7, 10, 14, and 15. In this section, we provide a 2-universal analogue of the 15-Theorem.

Theorem 2. *A positive* $\mathbf{Z}$*-lattice is 2-universal if and only if it represents the following six positive binary* $\mathbf{Z}$*-lattices:*

$$\langle 1,1\rangle,\quad \langle 2,3\rangle,\quad \langle 3,3\rangle,\quad [2,1,2],\quad [2,1,3],\quad [2,1,4].$$

Moreover, this is a minimal set, that is, for any ℓ *among the six* $\mathbf{Z}$*-lattices above, there is a positive* $\mathbf{Z}$*-lattice that represents the other five except* ℓ.

Proof. For the first assertion, it is enough to show the sufficiency. Since L represents $\langle 1,1\rangle$, L is isometric to $\langle 1,1\rangle \perp L_0$ for some sublattice L_0 of L. Since L represents $[2,1,2]$, L_0 represents 1 or $[2,1,2]$.

(i) Let L_0 represent 1:

Then L is isometric to $\langle 1,1,1\rangle \perp L_1$ for some sublattice L_1 of L_0. Since L represents $[2,1,3]$, L_1 represent 1 or 2.

If L_1 represent 1, L is isometric to $\langle 1,1,1,1\rangle \perp L_2$ for some sublattice L_2 of L_1. Since L represents $[2,1,4]$, L_2 represents $1,2$ or 3. So L has a sublattice isometric to $\langle 1,1,1,1,1\rangle$, $\langle 1,1,1,1,2\rangle$ or $\langle 1,1,1,1,3\rangle$, all of which are 2-universal, whence L is 2-universal.

If L_1 does not represent 1, then L_1 represents 2. Since L represents $\langle 3,3\rangle$, L has a sublattice isometric to $\langle 1,1,1,2,2\rangle$, $\langle 1,1,1,2,3\rangle$, $\langle 1,1,1\rangle \perp [2,1,2]$ or $\langle 1,1,1\rangle \perp [2,1,3]$, all of which are 2-universal, whence L is 2-universal.

(ii) Let L_0 do not represent 1:

Then L_0 represents $[2,1,2]$. Since L represents $\langle 2,3\rangle$, L_0 represents $\langle 2\rangle \perp [2,1,2]$, $\langle 3\rangle \perp [2,1,2]$, $[2,1,2,1,2]$ or $[2,1,2,1,3]$. So L has a sublattice isometric to $\langle 1,1,2\rangle \perp [2,1,2]$, $\langle 1,1,3\rangle \perp [2,1,2]$, $\langle 1,1\rangle \perp [2,1,2,1,2]$ or $\langle 1,1\rangle \perp [2,1,2,1,3]$, all of which are 2-universal, whence L is 2-universal. This proves the sufficiency of the first part.

For the second assertion, we let S be the set of six positive binary $\mathbf{Z}$-lattices listed in the theorem. One can easily show that $\langle 1,2\rangle \perp [2,1,2]$ represents all elements of S except $\langle 1,1\rangle$ and that $\langle 1,1\rangle \perp [2,1,2]$ represents all of the elements of S except $(2,3)$. Furthermore, each of positive quaternary $\mathbf{Z}$-lattices $\langle 1,1,1,2\rangle$, $\langle 1,1,2,2\rangle$, $\langle 1,1,1,3\rangle$ and $\langle 1,1,1,1\rangle$ represents all elements of S except $\langle 3,3\rangle$, $[2,1,2]$, $[2,1,3]$ and $[2,1,4]$, respectively. Therefore, the set S is a minimal set as desired. $\square$

§6. Universal Z-lattices of higher ranks

In this section, we introduce some of recent results concerning n-universal positive $\mathbf{Z}$-lattices for $n \geq 3$. We adopt notations from [3] for $\mathbf{Z}$-lattices. We define:

$$u_{\mathbf{Z}}(n) = \min \{ \operatorname{rank}(L) \mid L \text{ is } n\text{-universal} \}. \tag{6.1}$$

Note that $u_{\mathbf{Z}}(n)$ exists for every $n \geq 1$. We also define $u_{\mathbf{Q}}(n)$ to be the minimal dimension of positive definite quadratic spaces over $\mathbf{Q}$ which represent all positive definite quadratic spaces of rank n. Then it is well known that

$$u_{\mathbf{Q}}(n) = n + 3 \quad \text{for all} \quad n.$$

Proposition 1. *Let $u_p(n) := u_{\mathbf{Z}_p}(n)$ be the minimal rank of $\mathbf{Z}_p$-lattices which represent all $\mathbf{Z}_p$-lattices of rank n. Then:*

$$u_p(1) = \begin{cases} 2 & \text{if } p \neq 2, \\ 3 & \text{if } p = 2, \end{cases} \qquad u_p(2) = \begin{cases} 4 & \text{if } p \neq 2, \\ 5 & \text{if } p = 2, \end{cases}$$

and $u_p(n) = n + 3$ for all $n \geq 3$. $\square$

Proof. We say that a $\mathbf{Z}_p$-lattice ℓ is *n-universal* if it represents all n-ary $\mathbf{Z}_p$-lattices. We only provide a proof of $u_2(2) = 5$ for the other cases can be treated in a similar and easier manner. Since I_5 is 2-universal over $\mathbf{Z}_2$, $u_2(2) \leq 5$. Suppose that there exist a 2-universal $\mathbf{Z}_2$-lattice, say N, of rank 4. Since $N \otimes Q_2$ is 2-universal Q_2-space, $dN \equiv 1 \pmod 2$ and $S_2(N \otimes Q_2) = -1$, where $S_2(N \otimes Q_2)$ is the 2-adic Hasse symbol of $N \otimes Q_2$. Furthermore, since $I_2 \to N$, we have $N \to I_2 \perp \langle -1 \rangle \perp \langle -1 \rangle$. But $A_2 \not\to I_2 \perp \langle -1 \rangle \perp \langle -1 \rangle$. Therefore, $u_2(2) = 5$. $\square$

There are infinitely many n-universal $\mathbf{Z}$-lattices of rank $u_{\mathbf{Z}}(n) + 1$. The following proposition says, however, that there are only finitely many n-universal $\mathbf{Z}$-lattices of rank $u_{\mathbf{Z}}(n)$.

Proposition 2. *The number of n-universal positive $\mathbf{Z}$-lattices of rank $u_{\mathbf{Z}}(n)$ is always finite up to isometry.*

Proof. Suppose that there are infinitely many pairwise non-isometric n-universal $\mathbf{Z}$-lattices, say $L_1, L_2, \ldots$, of rank $u_{\mathbf{Z}}(n)$. Define

$$\mathfrak{L} = \{ \ \ell \mid \ell \xrightarrow{\sigma_i} L_i \ \text{ for infinitely many } i \ \},$$

where σ_i is a representation from ℓ to L_i.

Let ℓ be any $\mathbf{Z}$-lattice which has a maximal rank among the lattices in $\mathfrak{L}$. Then $n \le \text{rank}(\ell) < u_{\mathbf{Z}}(n)$. Define $\ell_i = (\sigma_i(\ell) \otimes Q) \cap L_i$. Then ℓ_i is a primitive sublattice of L_i. Since the number of $\mathbf{Z}$-sublattices of ℓ_i containing $\sigma_i(\ell)$ is finite up to isometry, we may assume that $\sigma_i(\ell)$ is a primitive sublattice of L_i by deleting L_i's and by replacing ℓ's if necessary. Since $\text{rank}(\ell) < u_{\mathbf{Z}}(n)$, ℓ cannot be n-universal. Therefore, there exist a lattice ℓ' of rank n such that

$$\ell' \not\to \ell \ \text{ but } \ \ell' \xrightarrow{\sigma_i'} L_i \ \text{ for all } i.$$

Let N_i be the $\mathbf{Z}$-sublattice of L_i generated by $\sigma_i(\ell)$ and $\sigma_i'(\ell')$. Since $\sigma_i(\ell)$ is a primitive sublattice,

$$\text{rank}(\ell) < \text{rank}(N_i) \le u_{\mathbf{Z}}(n),$$

and the number of N_i is finite up to isometry because N_i has a bounded discriminant and a bounded rank. Therefore, we can take a $\mathbf{Z}$-lattice with rank greater than the rank of ℓ and which is represented by L_i for infinitely many i. But this is absurd and the proposition follows. $\square$

Note that $u_{\mathbf{Z}}(n) \ge n + 3$ for all positive integer n. The following theorem lists the value of $u_{\mathbf{Z}}(n)$ for $1 \le n \le 10$. For a detailed proof and more, see [12] and [13].

Theorem 3. *Let $u_{\mathbf{Z}}(n)$ be as in (6.1). Then for $1 \le n \le 10$, we have:*

$$u_{\mathbf{Z}}(n) = \begin{cases} n+3 & \text{if } 1 \le n \le 5, \\ 13 & \text{if } n = 6, \\ 15 & \text{if } n = 7, \\ 16 & \text{if } n = 8, \\ 28 & \text{if } n = 9, \\ 30 & \text{if } n = 10. \end{cases}$$

We list all n-universal $\mathbf{Z}$-lattices of rank $u_{\mathbf{Z}}(n)$ for $3 \le n \le 9$ in the following proposition. We also include a few $\mathbf{Z}$-lattices with $*$-mark, which are not yet determined to be n-universal.

Proposition 3. *(1) There are nine 3-universal $\mathbf{Z}$-lattices of rank $u_{\mathbf{Z}}(3) = 6$ found so far, and there can be at most two more. They are:*

$$I_6,\quad I_5 \perp A_1,\quad I_5 \perp \langle 3\rangle,\quad I_4 \perp A_1 \perp A_1,\quad I_4 \perp A_2,\quad I_4 \perp A_1 10[1\frac{1}{2}],$$
$$I_3 \perp A_3,\quad I_3 \perp A_2 21[1\frac{1}{3}],\quad I_4 \perp A_1 \perp \langle 3\rangle,\quad I_3 \perp A_1 \perp A_2{}^*,\quad I_3 \perp A_2 \perp \langle 3\rangle^*.$$

(2) There are exactly three 4-universal $\mathbf{Z}$*-lattices of rank* $u_{\mathbf{Z}}(4) = 7$*. They are:*

$$I_7, \quad I_6 \perp A_1, \quad I_5 \perp A_2.$$

(3) There are two 5-universal $\mathbf{Z}$*-lattices of rank* $u_{\mathbf{Z}}(5) = 8$ *found so far, and there can be at most one more. They are:*

$$I_8, \quad I_7 \perp A_1, \quad I_6 \perp A_2{}^*.$$

(4) There are exactly two 6-universal $\mathbf{Z}$*-lattices of rank* $u_{\mathbf{Z}}(6) = 13$*. They are:*

$$I_6 \perp E_7, \quad I_7 \perp E_6.$$

(5) There are exactly three 7-universal $\mathbf{Z}$*-lattices of rank* $u_{\mathbf{Z}}(7) = 15$*. They are:*

$$I_7 \perp E_8, \quad I_8 \perp E_7, \quad I_7 \perp E_7 6[1\frac{1}{2}].$$

(6) $I_8 \perp E_8$ *is the only 8-universal* $\mathbf{Z}$*-lattice of rank* $u_{\mathbf{Z}}(8) = 16$*. Furthermore, a positive* $\mathbf{Z}$*-lattice is 8-universal if and only if it represents both* I_8 *and* E_8*.*

Proof. We provide only sketchy proofs for (1)-(3) here. For (4)-(6), see [12] and [13]. So we assume $3 \le n \le 5$. Let L be an n-universal $\mathbf{Z}$-lattice of rank $n+3$. Since $I_{n-2} \perp \ell \to L$ for any positive binary $\mathbf{Z}$-lattice ℓ, $L \simeq I_{n-2} \perp N$ for some 2-universal $\mathbf{Z}$-lattice N of rank 5. If a lattice K is not n-universal, then $K \perp \langle 1 \rangle$ is not $(n+1)$-universal. Hence the following observations lead us to the candidates in (1)-(3).

$$A_2 \perp A_1 10[1\frac{1}{2}] \not\to I_6 \perp \langle 3 \rangle, \ A_2 \perp A_2 \not\to I_5 \perp A_1 \perp A_1,$$
$$A_2 \perp A_2 \not\to I_5 \perp A_1 \perp \langle 3 \rangle, \ A_2 \perp A_2 \not\to I_5 \perp A_1 10[1\frac{1}{2}], \ A_4 \not\to I_4 \perp A_2 \perp A_1,$$
$$A_4 \not\to I_4 \perp A_2 \perp \langle 3 \rangle, \ A_4 \not\to I_4 \perp A_2 21[1\frac{1}{3}], \ A_4 \not\to I_4 \perp A_3.$$

All $\mathbf{Z}$-lattices in (1), (2), (3) except $I_4 \perp A_1 \perp \langle 3 \rangle, I_7 \perp A_1$ and those with $(*)$-mark have class number 1 and are locally $3, 4, 5$-universal, respectively. Therefore, they are all $3, 4, 5$-universal, respectively.

Observe that

$$gen(I_4 \perp A_1 \perp \langle 3 \rangle) = \{I_4 \perp A_1 \perp \langle 3 \rangle, I_1 \perp A_5\},$$
$$gen(I_7 \perp A_1) = \{I_7 \perp A_1, I_1 \perp E_7\}.$$

In [7], it was proved that every $\mathbf{Z}$-lattice of rank 6 with even discriminant is represented by I_9, whose class number is 2. Therefore, for any positive $\mathbf{Z}$-lattice ℓ of rank 3,

$$\ell \perp A_1 \perp A_2 \to I_9.$$

The sublattice $(A_1 \perp A_2)^\perp$ in I_9 is isometric to $I_4 \perp A_1 \perp \langle 3\rangle$ and hence $\ell \to I_4 \perp A_1 \perp \langle 3\rangle$. This proves the 3-universality of $I_4 \perp A_1 \perp \langle 3\rangle$. The 6-universality of $I_7 \perp A_1$ can be proved in a very similar manner after replacing $A_1 \perp A_2$ by A_1 in the above. $\square$

Note that the **Z**-lattices with $(*)$-mark in (1) and (3) are 3- and 5-universal, respectively, over $\mathbf{Z}_p$ for all p.

Remark 3. See [8] for nonclassic n-universal **Z**-lattices. A positive even **Z**-lattice is said to b e *even n-universal* if it represents all positive even **Z**-lattices of rank n. Then nonclassic n-universal **Z**-lattices correspond to even n-universal **Z**-lattices via scaling by 2 and *vice versa* via scaling by 1/2. See also [1] for 1-universal positive ternary $\mathcal{O}$-lattices over the ring $\mathcal{O} = \mathcal{O}_F$ of algebraic integers of real quadratic fields F. There, it is proved that $u_{\mathcal{O}}(1) = 3$ if and only if $F = \mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{3})$ or $\mathbf{Q}(\sqrt{5})$.

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