REPRESENTATIONS OF BINARY FORMS BY CERTAIN QUINARY POSITIVE INTEGRAL QUADRATIC FORMS

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Abstract. In this article, we provide the complete answer to a question raised by Kitaoka in his book [K1]. More precisely, we prove that $A_4 \perp \langle 4 \rangle$ represents all but one and $D_4 20[2\frac{1}{2}]$ represents all but three binary positive even $\mathbb{Z}$-lattices. We further investigate representations of the binary forms by quinary forms in certain positive even 2-universal genera of class number 2.

1. Introduction

In his book [K1], Kitaoka raised a question on representations of binary integral quadratic forms by the following two quinary positive even integral quadratic forms:

$$
\begin{aligned}
f &= 2(\sum_{i=1}^{4} x_i^2 + 2x_5^2 - x_1x_2 - x_2x_3 - x_3x_4) \qquad \text{and} \\
g &= 2(\sum_{i=1}^{4} x_i^2 + 3x_5^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_3x_5).
\end{aligned}
\tag{1.1}
$$

These two forms are indistinguishable locally everywhere but not isometric globally. In fact, the two classes form a genus, and the discriminant of this genus, which is 20, is the smallest among the genera of quinary positive even integral quadratic

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forms with class number bigger than 1. Since each form represents all binary positive even integral quadratic forms locally everywhere, the genus is even 2-universal, that is, every binary positive even integral quadratic form is represented by the genus and hence by either one of the forms globally. Kitaoka conjectured each of the two forms above represents all except finitely many binary positive even integral quadratic forms. In Section 2, we prove that his conjecture holds. More precisely, we prove that f (and g, respectively) represents all binary positive even integral quadratic forms with one (and three, respectively) exception(s). This is quite interesting in the sense that neither form is 2-universal and that the sets of binary forms represented by the quinary forms f and g, respectively, are explicitly demonstrated, which seems to be the first such example other than universal forms when the forms to be represented are not unary. (See [CKR], [CSc], [KKR], [KKO], [KO1,2], [O1,2] for recent works on universal forms and related topics.) In Section 3, we further investigate examples of positive even 2-universal genera of quinary forms of class number 2 whose classes have no exceptions, finitely many exceptions, or infinitely many exceptions. From these examples, we may conclude that all combinations are possible concerning representations of binary forms by such genera: both classes may represent all binary forms; both may have finitely many exceptions; one may represent all while the other has finitely or infinitely many exceptions; one may have finitely many exceptions while the other has infinitely many.

We shall adopt lattice theoretic language. A *$\mathbb{Z}$-lattice* L is a finitely generated free $\mathbb{Z}$-module in $\mathbb{R}^n$ equipped with a non-degenerate symmetric bilinear form B, such that $B(L,L) \subseteq \mathbb{Z}$. The corresponding quadratic map is denoted by Q.

For a $\mathbb{Z}$-lattice $L = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2 + \cdots + \mathbb{Z}\mathbf{e}_n$ with basis $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$, we write

$$L = (B(\mathbf{e}_i, \mathbf{e}_j)).$$

By $L = L_1 \perp L_2$ we mean $L = L_1 \oplus L_2$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$ for all $\mathbf{e}_1 \in L_1, \mathbf{e}_2 \in L_2$. We call L *diagonal* if it admits an orthogonal basis and in this case, we simply write

$$L = \langle Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n)\rangle,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is an orthogonal basis of L. We call L *non-diagonal* otherwise. L is called *positive definite* or simply *positive* if $Q(\mathbf{e}) > 0$ for any $\mathbf{e} \in L, \mathbf{e} \neq \mathbf{0}$. As usual, $dL := \det(B(\mathbf{e}_i, \mathbf{e}_j))$ is called the *discriminant* of L. We define $RL := R \otimes L$ for any commutative ring R containing $\mathbb{Z}$. For a $\mathbb{Z}$-lattice L and a prime p, we define $L_p := \mathbb{Z}_p L$ and call it the localization of L at p. If $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is an orthogonal basis of the quadratic space $V = \mathbb{Q}L$ or $\mathbb{Q}_p L$, we write

$$V = (Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n))$$

for convenience.

Let ℓ, L be $\mathbb{Z}$-lattices. We say L *represents* ℓ if there is an injective linear map from ℓ into L that preserves the bilinear forms, and write $\ell \to L$. Such a map will be called a *representation*. A representation is called an *isometry* if it is surjective. We say two $\mathbb{Z}$-lattices L, K are isometric if there is an isometry between them, and write $L \cong K$. The set of all $\mathbb{Z}$-lattices that are isometric to L is called the *class* of L, denoted by $\mathrm{cls}(L)$. We define $\ell_p \to L_p$ and $L_p \cong K_p$ in a similar manner over $\mathbb{Z}_p$. The set of all $\mathbb{Z}$-lattices K such that $L_p \cong K_p$ for all primes spots p (including ∞) is called the *genus* of L, denoted by $\mathrm{gen}(L)$. The number classes in a genus is called the *class number* of the genus (or of any $\mathbb{Z}$-lattice in the genus), which is known to be finite.

A positive $\mathbb{Z}$-lattice L is called *k-universal* if L represents all k-ary positive $\mathbb{Z}$-lattices. L is called *even* when $Q(L) \subset 2\mathbb{Z}$ and *odd* otherwise. A positive even $\mathbb{Z}$-lattice L is called *even k-universal* if L represents all k-ary positive even $\mathbb{Z}$-lattices. The genus of L is called (*even*) *k-universal* if for any k-ary positive (even) $\mathbb{Z}$-lattice the genus contains a $\mathbb{Z}$-lattice K that represents it. It is well known that $\mathrm{gen}(L)$ is (even) k-universal if and only if L_p represents all k-ary (even) $\mathbb{Z}_p$-lattices for all primes p.

Note that $\mathbb{Z}$-lattices naturally correspond to *classic integral* quadratic forms, i.e., quadratic forms with integer coefficients such that coefficients of non-diagonal terms are multiples of 2. A quadratic form with integer coefficient will be called *non-classic integral* or simply *integral*. In this paper, we always assume the following unless stated otherwise:

(1.2) *Every $\mathbb{Z}$-lattice is positive even.*

This, however, is not at all a restriction as far as the representation theory is concerned because representations by even $\mathbb{Z}$-lattices correspond to those by (non-classic) integral quadratic forms via scaling even $\mathbb{Z}$-lattices by $1/2$.

We set

$$[a, b, c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{1.3}$$

for convenience. For unexplained terminologies, notations, and basic facts about $\mathbb{Z}$-lattices, we refer the readers to O'Meara [O'M1] and Conway-Sloane [CS1,2].

2. On Kitaoka's quinary forms

Let L be a $\mathbb{Z}$-lattice and $\mathfrak{a}_{(p)} \subsetneqq \mathbb{Z}_p$ be an ideal. For convenience, we'll say that L is $\mathfrak{a}_{(p)}$-maximal if L_p is an $\mathfrak{a}_{(p)}$-maximal $\mathbb{Z}_p$-lattice. Note that for an ideal $\mathfrak{a} \subsetneqq \mathbb{Z}$, L is $\mathfrak{a}$-maximal if and only if L is $\mathfrak{a}_p$-maximal for all prime p. One can easily check that for every $2\mathbb{Z}_p$-maximal $\mathbb{Z}$-lattice L, $\mathrm{ord}_p(d(L)) \le 2$ if p is odd, and $\mathrm{ord}_2(d(L)) \le 3$ if $p = 2$. Let ℓ be a binary $2\mathbb{Z}_2$-maximal $\mathbb{Z}$-lattice. Then one can easily show by direct calculation that ℓ_2 is isometric to one of the following 15 binary $\mathbb{Z}_2$-lattices:

$$\langle 2, 2\gamma\rangle,\ \langle 6, 6\gamma\rangle,\ \langle 2, 4\alpha\rangle,\ \langle 2\beta, 4\gamma\rangle,\ [2,1,2],\ [0,1,0],\ [4,2,4], \tag{2.1}$$

where $\alpha = 1, 3, 5, 7,\ \beta = 3, 5,$ and $\gamma = 1, 5$.

In this section, we consider representations of $\mathbb{Z}$-lattices in the genus of $A_4 \perp \langle 4\rangle$, which corresponds to f in (1.1). (See [K1; Problem 9, p.257], [K2]). The genus has the smallest discriminant, which is 20, among the genera of quinary $\mathbb{Z}$-lattices of class number bigger than 1. The other class in the genus is that of $D(5)$, which corresponds to the class of g in (1.1). Here,

$$D(k) := D_4 4k[2\frac{1}{2}] = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & k+1 \end{pmatrix} \tag{2.2}$$

for positive integer k, following the notations in [CS1].

Theorem 2.1. *$A_4 \perp \langle 4\rangle$ represents all binary $\mathbb{Z}$-lattices except* $[4, 2, 4]$.

Proof. Let $\ell = [a, b, c]$ be a *Minkowski reduced* binary $\mathbb{Z}$-lattice, i.e., $0 \le 2b \le a \le c$ such that ℓ is $2\mathbb{Z}_2$-maximal. Let $L := A_4 \perp \langle 4\rangle$ and let $M := A_3 \perp \langle 4\rangle,\ N := A_4$ be sublattices of L. Both M and N are of class number 1. If $\ell_2 \not\simeq [0, 1, 0], [4, 2, 4]$, then $\ell \to M \subset L$. Hence we may assume that

$$\ell_2 \ \simeq\ [0, 1, 0],\ [4, 2, 4]. \tag{2.3}$$

Note that $\ell_2 \to N_2$ if $\ell_2 \simeq [0, 1, 0]$.

If p is an odd prime, then $\ell_p \to N_p$ except when $d\ell_p = -5$ and $S_p\ell = -1$, where S_p is the p-adic Hasse symbol. In particular, one can easily check that:

$$\ell_p \to N_p \ \text{ if } \begin{cases} p \equiv 1, 4 \pmod 5 \text{ or} \\ p \equiv 2, 3 \pmod 5 \text{ and } \gcd(p, a, b, c) = 1 \text{ or} \\ p = 5 \text{ and } \ell_5 \text{ represents a unit square in } \mathbb{Z}_5. \end{cases} \tag{2.4}$$

We put

$$\begin{aligned} \ell_s(t, r) :&= [a - 4t^2, as + b - 4tr, s^2a + 2sb + c - 4r^2] \\ &= \begin{pmatrix} a - 4t^2 & as + b - 4tr \\ as + b - 4tr & s^2a + 2sb + c - 4r^2 \end{pmatrix}, \end{aligned} \tag{2.5}$$

where s, t, r are integers. If $s = 0$, we simply write $\ell(t, r)$ instead of $\ell_0(t, r)$. Note that $\ell \cong \ell_s(0, 0)$ and that

$$\det(\ell_s(t, r)) = ac - b^2 - 4t^2(s^2a + 2sb + c) - 4r^2a + 8tr(as + b).$$

If $\ell_s(t, r) \to N$, then $\ell \to N \perp \langle 4\rangle = L$. Therefore, it suffices to find integers s, t, r for which $\ell_s(t, r)$ is positive such that $\ell_s(t, r) \to N$.

(a) Let $a \equiv 0, 3 \pmod 5$. Note that for any integer s, if $p \equiv 1, 4 \pmod 5$ or $p = 2, 5$, then $(\ell_s(1,0))_p \to N_p$. Let $p_1, p_2, \ldots, p_k$ be all the odd prime factors of $a - 4$ such that $p_j \equiv 2, 3 \pmod 5$. The possible such primes are $3, 7, 13, 17, 23, 37, 43, \cdots$. According to (2.4), it is enough to find an integer s such that

(i) $\gcd(p_1 p_2 \ldots p_k, sa + b) = 1$ and

(ii) $\ell_s(1,0)$ is positive definite.

Note that

$$\det(\ell_s(1,0)) = \frac{ac}{4} - b^2 + \frac{3ac}{4} - 4(s^2a + 2sb + c) \geq \frac{c}{4}(3a - 16(s^2 + |s| + 1)).$$

In fact, we want to find an integer σ for which

(iii) there exists an $s \in \Sigma := \{-\sigma, -\sigma + 1, \ldots, \sigma - 1, \sigma\}$ such that s satisfies (i).

If $k = 0$, then we put $\sigma = s = 0$. Since $a \equiv 0, 8 \pmod{10}$, $\det(l_0(1,0)) \geq c(3a - 16)/4 > 0$. If $k = 1$, then we put $\sigma = 1$ and $s = 0$ or -1 so that $\gcd(p_1, sa + b) = 1$. Clearly, $\ell_s(1,0)$ is positive definite. Assume that $2 \leq k \leq 4$. If we let $\sigma = k - 1$, then such an s exists in Σ. Note that

$$\det(\ell_s(1,0)) \geq \frac{c}{4}(3(p_1 p_2 \ldots p_k + 4) - 16(\sigma^2 + |\sigma| + 1)) > 0 \quad \text{for all } s \in \Sigma.$$

Lastly, assume that $k \geq 5$. By Lemma 3 in [KKO], we may take $\sigma = k2^{k-1}$ satisfying the condition (iii). If $s \in \Sigma$ satisfies the condition (i), then

$$\begin{aligned}\det(\ell_s(1,0)) &= \frac{ac}{4} - b^2 + \frac{c}{4}(3a - 16(k2^{k-1} + 1)^2) \\ &\geq \frac{c}{4}(3 \times 3 \times 7 \times 13 \times 17 \times 23^{k-4} - 16(k2^{k-1} + 1)^2) > 0.\end{aligned}$$

(b) Let $a \equiv 4 \pmod 5$. Note that $(\ell_s(1,0))_2 \to N_2$ and that if $sa+b$ is not divisible by 5, then $(\ell_s(1,0))_5 \to N_5$. Let $p_1, p_2, \ldots, p_k$ be all the odd prime factors of $a - 4$ such that $p_j \equiv 2, 3 \pmod 5$ and let $p_0 = 5$. We replace the condition (i) by

(i′) $\gcd(p_0 p_1 \ldots p_k, sa + b) = 1$.

Then for each $k = 0, 1, 2, 3, 4, 5$, we may take $\sigma = 1, 1, 2, 3, 6, 11$, respectively, satisfying the condition (iii). If $a \geq 24$, then $\ell_s(1,0)$ is always positive definite and hence $\ell_s(1,0) \to N$. Let $a = 4, 14$.

$$\text{(2.6)} \qquad \begin{aligned} &\ell \to M \quad \text{if} \begin{cases} a = 4, b = 0 \text{ or} \\ a = 4, b = 2, \text{ and } c \equiv 2 \pmod 4 \text{ or} \\ a = 14, b \equiv 0 \pmod 2, \end{cases} \\ &\ell \to N \quad \text{if} \begin{cases} a = 4, b = 1 \text{ or} \\ a = 14, b = 1, 3, 5. \end{cases} \end{aligned}$$

Furthermore, $\ell(1,0) \to N$ if $a=14, b=7$. If $a=4$ and $b=2$, then $c \equiv 4 \pmod 8$ because ℓ_2 is $2\mathbb{Z}_2$-maximal $\mathbb{Z}_2$-lattice. Hence $\ell(0,1) \to N$ if $c \neq 4$. If $c=4$, then $[4,2,4]$ is not represented by $A_4 \perp \langle 4 \rangle$. When $k \geq 6$, the same argument as in case (a) works.

(c) Let $a \equiv 1 \pmod 5$. This case can also be proved in a quite similar manner as in case (a) by replacing $\ell_s(1,0)$ by $\ell_s(5,0)$. Note that if $p=2,5$, then $(\ell_s(5,0))_p \to N_p$. Let's assume that $a \geq 134$ and let $p_1, p_2, \dots, p_k$ be all the odd prime factors of $a-100$ such that $p_j \equiv 2,3 \pmod 5$. For each k, we take σ satisfying the condition (iii) as follows: $\sigma=0$ if $k=0$, $\sigma=k-1$ if $1 \leq k \leq 5$, and $\sigma=k2^{k-1}$ if $k \geq 6$. In this case, however, we have to take care separately of the cases when the positive definiteness of $\ell_s(5,0)$ is not guaranteed. Such cases are:

$$a = \begin{cases} 146, 156, 166, 186, 196, 206, 236, 246, 266 & \text{if } k=1 \\ 226, 256, 376 & \text{if } k=2 \\ 646 & \text{if } k=3. \end{cases} \tag{2.7}$$

Let $a = 646 = 2 \times 17 \times 19$. Then $\ell \to N$ if c is not divisible by 17 and $\ell(0,1) \to N$ otherwise. For the remaining cases, we only provide a proof of the case when $a = 156 = 3 \times 4 \times 13$ and $k=1$ as a sample, because proofs of the other cases are all alike. If $\gcd(7,b,c)=1$, then $\ell(5,0) \to N$. Hence we may assume that b and c are divisible by 7. Then $\ell_{-1}(5,0) \to N$ if it is positive definite, i.e., $c \geq 280$. Therefore it suffices to check the following cases:

$$c = 168,\ 182,\ 196,\ 210,\ 224,\ 238,\ 252,\ 266. \tag{2.8}$$

By brute force computation, one can show that:

$$\begin{aligned} &\ell \to N && \text{if } c = 224, 238, 266, \\ &\ell(1,0) \to N && \text{if } c = 196, \\ &\ell(0,1) \to N && \text{if } c = 168, 182, 210, 252. \end{aligned} \tag{2.9}$$

(d) Let $a \equiv 2 \pmod 5$. If $\ell_2 \simeq [0,1,0]$, then $(\ell_s(2,0))_p \to N_p$ for $p=2,5$. So, this case can also be proved in a similar manner as in case (a) by replacing $\ell_s(1,0)$ by $\ell_s(2,0)$. Let $p_1, p_2, \dots, p_k$ be all the odd prime factors of $a-16$ such that $p_j \equiv 2,3 \pmod 5$. The rest resembles the case (c) with a few exceptional cases that can easily be taken care of. The exceptional cases are when $a=12$ and when $a=42,62$ with $k=1$. Let $\ell_2 \simeq [4,2,4]$, i.e., $a, c \equiv 4 \pmod 8$ and $b \equiv 2 \pmod 4$. Note that $a \equiv 12 \pmod{40}$. Since $d(\ell_s(1,0)) = ac - b^2 - 4c - 4(s^2a + 2sb)$, there exists an integer $s_0 \in \{-1,0,1\}$ such that $(l_{s_0}(1,0))_5$ is a unimodular $\mathbb{Z}_5$-lattice. Therefore for all integers i, $(\ell_{5i+s_0}(1,0))_5 \to N_5$. Let $p_1, p_2, \dots, p_k$ be all the odd prime factors of $a-4$ such that $p_j \equiv 2,3 \pmod 5$. The proof is similar to that of case (a) if we replace the condition (iii) by (iii′) there exists an $s \in \Sigma' = \{-5\sigma + s_0, -5(\sigma-1)+s_0, \dots, s_0, \dots, 5\sigma + s_0\}$ such that s satisfies (i).

We may take $\sigma = 0,1,1,2,3,4,5$ for $k = 0,1,2,3,4,5,6$, respectively, and take $\sigma = k2^{k-1}$ if $k \geq 7$. For each $s \in \Sigma'$ satisfying the condition (i), $\ell_s(1,0)$ is positive except the following cases:

$$(2.10) \qquad a = 12 \text{ or } 52 \leq a \leq 228 \text{ with } k = 1,2 \text{ or } 110 \leq a \leq 708 \text{ with } k = 3.$$

Note that for any integer a such that $110 \leq a \leq 708$ and $a \equiv 2 \pmod{10}$, $a-4$ cannot have three distinct odd prime factors that are non-square units in $\mathbb{Z}_5$. The rest resembles the case (c).

It only remains to consider sublattices of $[4,2,4]$ of even index, which are not $2\mathbb{Z}_2$-maximal anymore. But none of these can be an exception because $\langle 4, 12\rangle$, the unique sublattice of $[4,2,4]$ of index 2, is represented by L. This completes the proof. □

Theorem 2.2. *$D(5)$ represents all binary $\mathbb{Z}$-lattices except three. The exceptions are $[2,1,4], [4,1,4]$, and $[8,1,8]$.*

Proof. Let $\ell = [a,b,c]$ be a Minkowski's reduced binary $2\mathbb{Z}_2$-maximal and let $I(5) := I_4 \perp \langle 5 \rangle$. Then

$$(2.11) \qquad D(5) = I(5)^e := \{x \in I(5) \mid Q(x) \equiv 0 \pmod 2\}.$$

Therefore, $\ell \to D(5)$ if and only if $\ell \to I(5)$ for a binary $\mathbb{Z}$-lattice ℓ. If $\ell \otimes \mathbb{Q}_2$ is not a hyperbolic plane, then $\ell \to I_4 \to I(5)$. Hence we may assume that $\ell_2 \simeq [0,1,0]$. We put

$$(2.12) \qquad \ell(t,r) = [a-5t^2, b-5tr, c-5r^2] = \begin{pmatrix} a-5t^2 & b-5tr \\ b-5tr & c-5r^2 \end{pmatrix}.$$

If we can choose integers t, r such that $d(\ell(t,r)) \equiv 1,3,5 \pmod 8$ and $\ell(t,r)$ is positive, then $\ell(t,r) \to I_4$ and hence $\ell \to I_4 \perp \langle 5 \rangle$.

If $c \not\equiv 0 \pmod 8$ and $a \geq 8$, then $\ell(1,0) \to I_4$. Similarly, if $a \not\equiv 0 \pmod 8$ and $a \geq 8$, then $\ell(0,1) \to I_4$. When $a = 2,4,6$, one can easily check that $\ell(0,1) \to I_4$ provided that $c \geq 8$. The first two exceptions are obtained from brute force computation among the remaining possibilities.

We now assume that $a \equiv c \equiv 0 \pmod 8$. Note that $a - 2b + c \equiv 2 \pmod 4$. If

$$(2.13) \qquad [a-5, -a+b, a-2b+c] = \begin{pmatrix} a-5 & -a+b \\ -a+b & a-2b+c \end{pmatrix}$$

is positive definite, then $\ell \to I_4 \perp \langle 5 \rangle$. It is easy to check that the $\mathbb{Z}$-lattice (2.13) is always positive definite except when $a = c = 8$ and $b = 1$, which is the third exception.

It only remains to consider sublattices of $[2,1,4],[4,1,4]$, and $[8,1,8]$ of even index. But none of them can be an exception because

$$\langle 2,14\rangle,\quad [4,2,8],\quad [4,2,16],\quad \langle 6,10\rangle,\quad \langle 14,18\rangle,\quad [8,2,32],$$

the only sublattices of index 2, are all represented by $D(5)$. This proves the theorem. $\square$

Remark 2.3. Theorems 2.1 and 2.2 provide an example of an even 2-universal genus of quinary $\mathbb{Z}$-lattices of class number 2 such that both classes in the genus represent all but finitely many binary $\mathbb{Z}$-lattices.

3. OTHER EXAMPLES

In this section, we investigate some other even 2-universal genera of quinary $\mathbb{Z}$-lattices of class number 2 concerning representations of binary $\mathbb{Z}$-lattices.

For a positive integer k, let

$$I(k) := I_4 \perp \langle k\rangle \tag{3.1}$$

and define

$$I(k)^e := \{x \in I_4 \perp \langle k\rangle \mid Q(x) \equiv 0 \pmod 2\}. \tag{3.2}$$

It is easy to check that

$$I(k)^e = \begin{cases} D_4 \perp \langle k\rangle & \text{if } k \text{ is even,} \\ D(k) & \text{if } k \text{ is odd.} \end{cases} \tag{3.3}$$

The class number of $I(k)^e$ is 1 for $k = 1,2,3$ (see [N1]). From this and local representation theory follows that $I(1)^e, I(2)^e, I(3)^e$ are all even 2-universal. If k is divisible by 4, then $(I(k)^e)_2$ is not locally even 2-universal at the dyadic prime 2 because it cannot represent $[0,1,0]$ over $\mathbb{Z}_2$. We now know that $I(5)^e = D(5)$ has three exceptions by Theorem 2.2. (See (2.2) for $D(5)$.) Note that

$$I(7)^e = D(7). \tag{3.4}$$

Proposition 3.1. *(a) $I(7)^e$ represents all binary $\mathbb{Z}$-lattices except eleven. The eleven exceptions are as follows:*

$$\begin{gathered} [2,1,4],\ [4,1,4],\ [4,1,6],\ [6,3,8],\ [8,1,8],\ [8,1,16], \\ [8,1,24],\ [8,1,32],\ [8,1,40],\ [8,3,8],\ [8,3,16]. \end{gathered} \tag{3.5}$$

(b) $I(6)^e$ represents all binary $\mathbb{Z}$-lattices except the following:

$$[4,1,c],\ [8,1,8],\ [8,1,12],\ [8,1,16],\ [8,3,8], \tag{3.6}$$

where c is any positive even integer.

(c) $I(10)^e$ represents all binary $\mathbb{Z}$-lattices except the following:

$$\begin{aligned} &[16,1,16],\ [16,3,16],\ [16,1,20],\ [12,1,36],\ [12,1,40],\ [12,1,44],\\ &[12,1,48],\ [10,5,12],\ [8,1,c_1],\ [8,3,c_1],\ [6,1,8],\ [6,3,8],\\ &[4,1,c_2],\ [2,1,8],\ [12,b_1,12],\ [12,b_1,16],\ [12,b_1,20],\ [12,b_2,24],\\ &[12,b_2,28],\ [12,b_2,32],\ [4,2,8]^*,\ [8,2,8]^*,\ [16,2,16]^*, \end{aligned} \tag{3.7}$$

where c_1, c_2 are any positive even integers satisfying $c_1 \geq 8$, $b_1 = 1,3,5$ and $b_2 = 1,3$. The $*$-marked ones at the end are those which are not $2\mathbb{Z}_2$-maximal while the others are all $2\mathbb{Z}_2$-maximal.

Proof. (a) One can apply a similar argument to that of Theorem 2.2 to conclude that $I(7)^e$ represents all but the listed eleven exceptions.

(b)-(c) Let $k = 6$ or 10. Assume that $\ell = [a,b,c]$ is a Minkowski's reduced binary $2\mathbb{Z}_2$-maximal $\mathbb{Z}$-lattice. From the local representation theory [O'M2] and the fact that the class number of D_4 is 1, it follows $\ell \to D_4 \to I(k)^e$ provided that $\ell \otimes \mathbb{Q}_2$ is not isometric to a hyperbolic plane. So we may assume that $\ell_2 \simeq [0,1,0]$. Since one of the following $\mathbb{Z}$-lattices

$$[a-k,b,c],\ [a,b,c-k],\ [a-k,b-k,c-k] \tag{3.8}$$

is isometric to $[2,1,2]$ over $\mathbb{Z}_2$, we may conclude $\ell \to I(k)^e$ provided that $a > 8k/3$.

The rest of the proof for the case when $k = 6$ is almost identical to that for the case when $k = 10$. So, we only prove the latter case, which involves more subcases. By brute force computation, one can obtain those exceptions listed without $*$-mark from $a \leq 26 < 80/3$.

We now assume that ℓ is not $2\mathbb{Z}_2$-maximal. Let ℓ be one of the sublattices with index 2 of the listed exceptions without $*$-mark. We only provide a proof for the case when ℓ is a sublattice of $[4,1,c]$ with index 2, as a sample, because proofs for the other cases are all alike. Observe that ℓ is isometric to one of the following $\mathbb{Z}$-lattices:

$$[16,2,c],\ [4,2,4c],\ [16,10,c+6]. \tag{3.9}$$

In the first case, by comparing the discriminant of each lattice, we may conclude that at least one of the $\mathbb{Z}$-lattices $[6,2,c], [6,-8,c-10]$ is not contained in the hyperbolic plane. Therefore, $\ell \to I(10)^e$ provided that $c \geq 22$. The $*$-marked exception $[16,2,16]$ is deduced when $c \le 20$. For the second case, since $[4,2,4c-10]$ is represented by D_4, we obtain $\ell \to I(10)^e$ provided that $c \ge 4$. The $*$-marked exception $[4,2,8]$ is deduced when $c = 2$. We now consider the third case. Observe that $\ell \simeq [16,10,c+6] \simeq [16,6,c+2]$. So, at least one of $\mathbb{Z}$-lattices $[6,6,c+2], [6,-4,c-8]$ is represented by D_4 and hence by $I(10)^e$ provided that $c \ge 12$. There is no exception, however, can be deduced when $c \le 10$. Instead, the remaining $*$-marked exception $[8,2,8]$ is deduced as a sublattice of $[2,1,8]$ with index 2 in a similar manner. So far, we have obtained three $*$-marked exceptions as follows:

$$(3.10) \qquad [4,2,8] \subset [4,1,2], \quad [8,2,8] \subset [2,1,8], \quad [16,2,16] \subset [4,1,16].$$

In fact, these are the only exceptions that are sublattices of index 2 of the listed exceptions without $*$-mark.

In order to finish the proof, it suffices to show that there is no other exception ℓ for which ℓ_2 is not $2\mathbb{Z}_2$-maximal. To this end, we suppose that ℓ is an exception other than the $*$-marked ones such that ℓ_2 is not $2\mathbb{Z}_2$-maximal. Then we may regard ℓ as a sublattice of finite index of a $*$-marked exception. If ℓ is of even index, then ℓ is contained in a sublattice of index 2. The possible sublattices of index 2 of the $*$-marked exceptions are:

$$(3.11) \quad [16,4,8], [4,4,32], [16,12,16], [32,4,8], [32,20,20], [64,4,16], [64,36,36].$$

However, these are all represented by $I(10)^e$ and hence $\ell \to I(10)^e$, which is absurd. If ℓ is of odd index, then $a \equiv c \equiv b-2 \equiv 0 \pmod 4$. So one of the $\mathbb{Z}$-lattices in (3.8) for $k = 10$ is represented by D_4 and hence $\ell \to I(10)^e$ provided that $a \ge 28$, which is again absurd. By brute force computation, one can show that no such ℓ is possible when $a \le 24$. $\square$

Observe that both $I(6)^e$ and $I(10)^e$ have infinitely many exceptions. Let us consider the genus of $A_2 \perp \langle 2,2,2\rangle$. The genus has the second smallest discriminant, which is 24, among the genera of quinary $\mathbb{Z}$-lattices of class number bigger than 1.

Theorem 3.2. *(a) The genus of $A_2 \perp \langle 2,2,2\rangle$ has class number 2, where the other class in the genus is that of $I(6)^e = D_4 \perp \langle 6\rangle$. Furthermore, the genus is even 2-universal.*

(b) $A_2 \perp \langle 2,2,2\rangle$ is even 2-universal.

(c) $I(6)^e$ has infinitely many exceptions. More precisely, $I(6)^e$ represents all binary $\mathbb{Z}$-lattices except those listed in (3.6).

Proof. (a) follows from [N1].

(b) Since $A_2 \perp \langle 1, 1, 2\rangle$, which is positive but not even, is 2-universal (see [KKO]), $A_2 \perp \langle 2, 2, 2\rangle$ is even 2-universal.

(c) See Proposition 3.1-(b). $\square$

Next, we consider the genus $A_3 \perp [2, 1, 4]$ whose discriminant is 28.

Theorem 3.3. *(a) The genus of $A_3 \perp [2, 1, 4]$ has class number 2, where the other class is that of $I(7)^e = D(7)$. Furthermore, the genus is even 2-universal.*

(b) $A_3 \perp [2, 1, 4]$ is even 2-universal.

(c) $I(7)^e$ has exactly eleven exceptions as listed in (3.5).

Proof. (a) follows from (3.4) and [N1].

(b) Let $L := A_3 \perp [2, 1, 4]$. L contains a sublattice $A_3 \perp \langle 4\rangle$ of class number 1. Let ℓ be any binary $\mathbb{Z}$-lattice. In order to show $\ell \to L$, we may assume that ℓ is a $2\mathbb{Z}$-maximal $\mathbb{Z}$-lattice. We also assume that $\ell = [a, b, c]$ is Minkowski's reduced. If $\ell_2 \not\simeq [0, 1, 0],\ [4, 2, 4]$, then $\ell \to A_3 \perp \langle 4\rangle \to L$ by Theorems 2 and 3 in [O'M2]. Hence we may assume that

$$\ell_2 \simeq [0, 1, 0],\ [4, 2, 4]. \tag{3.12}$$

We now take another sublattice $A_3 \perp \langle 2\rangle$ of L of class number 1. If $\ell_2 \not\simeq \langle 2, -4\rangle$, then $\ell_2 \to (A_3 \perp \langle 2\rangle)_2$. For an odd prime p, $\ell_p \to (A_3 \perp \langle 2\rangle)_p$ except when $d\ell_p = -2$ and $S_p(\ell) = -1$. In particular, if either $2 \in \mathbb{Z}_p^{*2}$ or $2 \notin \mathbb{Z}_p^{*2}$ with $p \nmid \gcd(a, b, c)$, then $\ell_p \to (A_3 \perp \langle 2\rangle)_p$. We proceed as in Theorem 2.1.

(b-1) Let $a \le 18$. If $a = 2, 4, 8, 14, 16$, then one can easily check that $\ell \to A_3 \perp \langle 4\rangle$ or $\ell \to A_3 \perp \langle 2\rangle$. The same conclusion follows if $\gcd(a, b, c)$ is not divisible by any odd prime. Furthermore, if $b = 0$, then $\ell \to A_3 \perp \langle 4\rangle$. Therefore the remaining possibilities for ℓ are:

$$[6, 3, 6c], [10, 5, 10c], [12, 3, 6c], [12, 6, 10c], [18, 3, 6c], [18, 6, 10c], [18, 9, 6c]. \tag{3.13}$$

We provide a proof only for $\ell = [12, 6, 6c]$, for the other ℓ's can be treated similarly. Note that $c \equiv 2 \pmod 4$ because ℓ is primitive. If $c \ge 6$, then $[12, 6, 6c - 14] \to A_3 \perp \langle 2\rangle$. If $c = 2$, then $[12, 6, 12] \to A_3 \perp \langle 2\rangle$.

(b-2) Let $a \ge 20$. For integers s and t, we put

$$\ell_s(t) := [a - 14t^2, sa + b, s^2a + 2sb + c] = \begin{pmatrix} a - 14t^2 & sa + b \\ sa + b & s^2a + 2sb + c \end{pmatrix}. \tag{3.14}$$

Since $\ell_2 \simeq [0, 1, 0]$ or $[4, 2, 4]$, $(\ell_s(1))_2 \to (A_3 \perp \langle 2\rangle)_2$ for any integer s. Hence if we find an s for which $\ell_s(1)$ is positive definite and $(\ell_s(1))_p$ represents a unit for every odd prime p, then the theorem follows.

Let $p_1, p_2, \ldots, p_k$ be all odd prime factors of $a-14$ such that 2 is a non-square unit in $\mathbb{Z}_{p_j}$ for all $j = 1, 2, \ldots, k$. If either no such prime exists or p_j cannot divide $\gcd(b,c)$ for all j, then $\ell_0(1) \to A_3 \perp \langle 2 \rangle$ and hence $\ell \to A_3 \perp \langle 2, 14 \rangle \to L$. Note that $\ell_0(1)$ is positive for $a \geq 20$. If $k = 1$ and p_1 divides $\gcd(b,c)$, then $\ell_{-1}(1) \to A_3 \perp \langle 2 \rangle$. Let $2 \leq k \leq 5$. Then there exists an s, $-k+1 \leq s \leq k-1$, such that $sa+b$ is not divisible by $p_1 p_2 \cdots p_k$. Hence for this s, $\ell_s(1) \to A_3 \perp \langle 2 \rangle$ except when $k = 2$ and $a = 44$. The exceptional case can be treated in a similar manner as above. We omit the proof of the case when $k \geq 6$, which is almost identical to that of the case (a) of Theorem 2.1.

(c) See Proposition 3.1-(a). $\square$

We now consider the genus of $A_3 \perp \langle 2, 6 \rangle$ whose discriminant is 48.

Theorem 3.4. *(a) The genus of $A_3 \perp \langle 2, 6 \rangle$ has class number 2, where the other class is that of $A_2 \perp \langle 2, 2, 4 \rangle$. Furthermore, the genus is even 2-universal.*

(b) Both $A_3 \perp \langle 2, 6 \rangle$ and $A_2 \perp \langle 2, 2, 4 \rangle$ are even 2-universal.

Proof. (a) follows from [N1].

(b) Let $\ell = [a, b, c]$ be a Minkowski's reduced binary $2\mathbb{Z}$-maximal $\mathbb{Z}$-lattice. Firstly, we prove the even 2-universality of $L := A_2 \perp \langle 2, 2, 4 \rangle$. Since $I_3 \perp \langle 2, 3 \rangle$ is 2-universal (see [KKO]), every binary $\mathbb{Z}$-lattice ℓ with $\mathfrak{s}(l) \subseteq 2\mathbb{Z}$ can be represented by $L' := \langle 2, 2, 2, 4, 6 \rangle \subset L$. So we may assume that b is odd. We now consider a quaternary sublattice $M := A_2 \perp \langle 2, 2 \rangle$ of L, whose class number is 1. Note that

$$\text{(3.15)} \qquad \ell_p \to M_p \quad \text{if} \begin{cases} p = 2 \text{ or} \\ p \not\equiv \pm 5 \pmod{12} \text{ or} \\ p \equiv \pm 5 \pmod{12} \text{ and } \gcd(p, a, b, c) = 1 \text{ or} \\ p = 3 \text{ and } l_3 \text{ represents a unit square in } \mathbb{Z}_3. \end{cases}$$

Define for integers s and t,

$$\text{(3.16)} \qquad \ell_s(t) := [a - 4t^2, sa+b, s^2a + 2sb + c] = \begin{pmatrix} a - 4t^2 & sa+b \\ sa+b & s^2a+2sb+c \end{pmatrix}.$$

It is easy to check that $(\ell_s(1))_3 \to M_3$ if $a \equiv 2 \pmod 3$ and that $(\ell_s(3))_3 \to M_3$ if $a \equiv 1 \pmod 3$ for all s. Furthermore, if $a \equiv 0 \pmod 3$ and either b or c is not divisible by 3, then there exists an $s_0 \in \{0, 1\}$ such that $(\ell_{3i+s_0}(1))_3 \to M_3$. Then we may take a suitable s such that $(\ell_s(t))_p \to (L_1)_p$ for all $p \equiv \pm 5 \pmod{12}$ by using an almost same argument as in Theorem 2.1. We now assume that $a \equiv b \equiv c \equiv 0 \pmod 3$. Since ℓ is primitive, $\ell_3 \simeq \langle 3, 3 \rangle$. From the fact that the class number of $N := A_2 \perp \langle 2, 6 \rangle$ is 1 follows that $\ell \to N \to L$. Therefore, L is even 2-universal.

Secondly, we prove the 2-universality of $K := A_3 \perp \langle 2, 6\rangle$. Note that both L' and N are contained in K. So we may assume that b is odd and that at least one of a, b, c is not divisible by 3. We consider the sublattice $H := A_3 \perp A_1$ of K, whose class number is 1. Observe that

$$\ell_p \to H_p \quad \text{if} \begin{cases} p = 2, 3 \text{ or} \\ p \not\equiv \pm 3 \pmod 8 \text{ or} \\ p \text{ is not a prime given above and } \gcd(p, a, b, c) = 1. \end{cases} \tag{3.17}$$

Define for integer s

$$\ell_s(1) := [a-6, sa+b, s^2a+2sb+c] = \begin{pmatrix} a-6 & sa+b \\ sa+b & s^2a+2sb+c \end{pmatrix}. \tag{3.18}$$

Then it is easy to check that $(\ell_s(1))_p \to K_p$ for $p = 2, 3$ and $p \not\equiv \pm 3 \pmod 8$. The rest resembles the above. $\square$

Finally, we consider the genus of $J \perp \langle 2\rangle$ whose discriminant is 40, where

$$J := A_1A_1A_1 10[111\frac{1}{2}] = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}. \tag{3.19}$$

following the notations in [CS1].

Theorem 3.5. *(a) The genus of $J \perp \langle 2\rangle$ has class number 2, where the other class is that of $I(10)^e = D_4 \perp \langle 10\rangle$. Furthermore, the genus is even 2-universal.*

(b) $J \perp \langle 2\rangle$ represents all binary $\mathbb{Z}$-lattices but one, which is $[2, 1, 2]$.

(c) $I(10)^e$ has infinitely many exceptions. More precisely, $I(10)^e$ represents all binary $\mathbb{Z}$-lattices except those listed in (3.7).

Proof. Proof (a) follows from [N1].

(b) Let $L := J \perp \langle 2\rangle$ and let $\ell = [a, b, c]$ be a Minkowski's reduced $2\mathbb{Z}_2$-maximal binary $\mathbb{Z}$-lattice. Note that $\langle 2, 2, 2, 2\rangle \to L$. So if $\ell_2 \not\cong [2, 1, 2], [0, 1, 0]$, then $\ell \to L$. Hence we may assume that

$$\ell_2 \cong [2, 1, 2] \text{ or } [0, 1, 0]. \tag{3.20}$$

If we define $K := I_2 \perp [2, 1, 3] \perp \langle 2\rangle$, then

$$L = K^e := \{x \in K \mid Q(x) \equiv 0 \pmod 2\}. \tag{3.21}$$

So $\ell \to L$ if and only if $\ell \to K$. We consider the genus of $\langle 1,1,5,5\rangle$ that consists of $\mathrm{cls}(\langle 1,1,5,5\rangle)$ and $\mathrm{cls}([2,1,3] \perp [2,1,3])$ (see [N2]). Note that both $\langle 1,1,5,5,10\rangle$ and $[2,1,3] \perp [2,1,3] \perp \langle 10\rangle$ are represented by K.

If $\ell_2 \simeq [2,1,2]$ and $\ell_5 \not\simeq \langle 1,2\rangle$, then ℓ is represented by $\mathrm{gen}(\langle 1,1,5,5\rangle)$. So, $\ell \to K$ and hence $\ell \to L$.

If $\ell_2 \simeq [0,1,0]$ and $\ell_5 \not\simeq \langle 1,2\rangle$, then at least one of

$$[a-10,b,c],\ [a-10,b-10,c-10],\ [a,b,c-10] \tag{3.22}$$

is represented by $\mathrm{gen}(\langle 1,1,5,5\rangle)$ provided that $a \geq 28$. This implies that $\ell \to L$. By brute force computation, one can show that there is no exception when $a \leq 26$.

We now assume that $\ell_5 \simeq \langle 1,2\rangle$. The class number of J is 1 and $J_2 \simeq [0,1,0] \perp [4,2,4]$ and $J_5 \simeq \langle 1,1,2,10\rangle$. Note that if $\langle 2\rangle \to \ell_5$, then $\ell_5 \to J_5$. For the primes $p \neq 2,5$, if $p \equiv \pm 1 \pmod 5$, then $\ell_p \to J_p$, and if $p \equiv \pm 2 \pmod 5$ and $\gcd(p,a,b,c) = 1$, then $\ell_p \to J_p$.

Firstly, we assume that $\ell_2 \simeq [2,1,2]$. Since $\ell_5 \simeq \langle 1,2\rangle$, a is not divisible by 5 and $a \equiv 2 \pmod 4$. We let

$$\ell_s(t) := [a-2t^2, sa+b, s^2+2sb+c]. \tag{3.23}$$

If $(\ell_s(5))_p \to J_p$ for all p (including ∞), then $\ell \to L$. Note that for $p = 2,5$, $(\ell_s(5))_p \to J_p$. Let $p_1, p_2, \ldots p_k$ be odd prime factors of $a-50$ such that $p_j \equiv \pm 2 \pmod 5$. As in the proof of Theorem 2.1, we can find an integer s such that $\gcd(sa+b, p_1p_1\ldots p_k) = 1$ for large a. The rest resembles the proof of Theorem 2.1, which we omit. For small a, we ought to check whether $\ell \to L$ or not by brute force computation as we did in Theorem 2.1. The only exception $[2,1,2]$ comes from this case.

Secondly, assume that $\ell_2 \simeq [0,1,0]$. Note that $(\ell_s(2))_2 \to J_2$. Let $p_1, p_2, \ldots p_k$ be odd prime factors of $a-8$ such that $p_j \equiv \pm 2 \pmod 5$. Note that there exists at least one $s, -1 \leq s \leq 1$ such that $d(\ell_s(2))$ is not divisible by 5 and that there exist at least three s's, $-2 \leq s \leq 2$ such that $d(\ell_s(2))$ is not divisible by 5. So, if $k = 0$ and $a \geq 34$ or $k = 1$ and $a \geq 76$, then $\ell_s(2) \to J$ for some s. If $k = 2$ and $a \geq 140$, then $\ell_s(2) \to J$ for some $s, -3 \leq s \leq 3$. If $k = 3$, then $\ell_s(2) \to J$ for some $s, -5 \leq s \leq 5$. If $k = 4$, $\ell_s(2) \to J$ for some $s \in \{-15+s_0, -10+s_0, \ldots, s_0, \ldots, 15+s_0\}$. The case when $k \geq 5$ can be treated in a similar manner as in Theorem 2.1 and is omitted. For the remaining cases with bounded a, one can show that there are no exceptions by brute force computation.

The only sublattice of $[2,1,2]$ of index 2 is $\langle 2,6\rangle$, which is represented by L. This completes the proof.

(c) See Proposition 3.1-(c). $\square$

We know that there are only finitely many quinary positive odd $\mathbb{Z}$-lattices that represent all binary positive $\mathbb{Z}$-lattices (not necessarily even) except only finitely many (see [O2]). Such $\mathbb{Z}$-lattices are said to be *almost 2-universal.* Observe that this implies that almost all 2-universal genera have the property that all classes in the genera have infinitely many exceptions. One such example is the following:

$$gen(\langle 1,1,1,2,6\rangle) = \{cls(\langle 1,1,1,2,6\rangle), \quad cls(\langle 1,2,2\rangle \perp [2,1,2])\}.$$

Note that

$$\langle 3, 3^{2t+1}\rangle \not\to \langle 1,1,1,2,6\rangle \quad \text{and} \quad \langle 3, 3^{2t}\rangle \not\to \langle 1,2,2\rangle \perp [2,1,2],$$

for all nonnegative integers t.

In this vein, although we couldn't find an even 2-universal genus of quinary $\mathbb{Z}$-lattices of class number 2 such that both classes in the genus have infinitely many exceptions, we believe that there are only finitely many quinary almost even 2-universal $\mathbb{Z}$-lattices.

References

[CKR] W.-K. Chan, M.-H. Kim, S. Raghavan, *Ternary universal integral quadratic forms over real quadratic fields*, Japanese J. Math. **22** (1996), 263–273.

[CS1] J. H. Conway, N. J. A. Sloane, *Sphere packings, lattices and groups, Third Edition*, Springer-Verlag, 1999.

[CS2] ______, *Low dimensional lattices. I. Quadratic forms of small determinant*, Proc. Royal. Soc. London Ser. A **418** (1988), 17-41.

[CSc] J. H. Conway and W. Schneeberger, *A 15-theorem for universal quadratic forms*, preprint.

[K1] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge Univ. Press, reprint with corrections, 1999.

[K2] ______, *Modular forms of degree n and representation by quadratic forms V*, Nagoya Math. J. **111** (1988), 173-179.

[KKO] B. M. Kim, M.-H. Kim, B.-K. Oh, *2-universal positive definite integral quinary quadratic forms*, Contemporary Math. **249** (1999), 51-62.

[KKR] B. M. Kim, M.-H. Kim and S. Raghavan, *2-universal positive definite integral quinary diagonal quadratic forms*, Ramanujan J. **1** (1997), 333-337.

[KO1] M.-H. Kim and B.-K. Oh, *A lower bound for the number of squares whose sum represents integral quadratic forms*, J. Korean Math, Soc. **33** (1996), 651-655.

[KO2] ______, *Representations of positive definite senary integral quadratic forms by a sum of squares*, J. Number Theory **63** (1997), 89-100.

[N1] G. L. Nipp, *http://www.research.att.com/~njas/lattices/nipp5.html.*

[N2] ______, *Quaternary quadratic forms*, Springer-Verlag, 1991.

[O1] B.-K. Oh, *Universal $\mathbb{Z}$-lattices of minimal ranks*, Proc. Amer. Math. Soc. **128**, 683-689.

[O2] ______, *Extension of a problem of Kloosterman to lower ranks*, preprint.

[O'M1] O. T. O'Meara, *Introduction to quadratic forms*, Springer-Verlag, 1973.

[O'M2] ______, *The integral representations of quadratic forms over local fields*, Amer. J. Math. **80** (1958), 843-878.