REPRESENTATIONS OF POSITIVE DEFINITE SENARY INTEGRAL QUADRATIC FORMS BY A SUM OF SQUARES

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Abstract. As a generalization of the famous four square theorem of Lagrange, Mordell and Ko proved that every positive definite integral quadratic form of n variables is represented by the sum of $n+3$ squares for $1 \le n \le 5$. In this paper, we prove that every positive definite integral quadratic form of six variables that can be represented by a sum of squares is represented by the sum of ten squares, no less.

§1. Introduction

As a generalization of the famous Four Square Theorem of Lagrange [6], Mordell [7] proved that every positive definite integral quadratic form of two variables is represented by the sum of five squares. And later, Ko [4] proved that every positive definite integral quadratic form of three, four, five variables is represented by the sum of six, seven, eight squares, respectively. From these surprising results, they naturally expected that every positive definite integral quadratic form of n variables is represented by the sum of $n+3$ squares. This, however, turned out to be false. Indeed, the integral quadratic form associated to the Dynkin diagram E_6, which we denote also by E_6 by abuse of notations, cannot be represented by a sum of squares. After Mordell [8] found such an example, Ko [5] conjectured :

Every positive definite integral quadratic form of six variables except those exceptional ones (that cannot be represented by sums of squares) is represented by the sum of nine squares.

Recently, the authors found in [3] that the answer is again negative. But nothing is known yet about the minimum number of squares whose sum represents all such forms. In this paper, we will show the minimum number is ten, i.e., that every positive definite integral quadratic form of six variables that can be represented by a sum of squares is represented by a sum of ten squares, no less.

We adopt terminologies and notations from [10]. Let l be a positive definite $\mathbb{Z}$-lattice of rank n equipped with a symmetric bilinear form B and the corresponding quadratic map Q. Here, a $\mathbb{Z}$-lattice is a free $\mathbb{Z}$-module with $\mathfrak{s}(l) \subseteq \mathbb{Z}$, where $\mathfrak{s}(l)$ is the scale of l. We denote the corresponding quadratic form by $f_l(x_1, x_2, \cdots, x_n) = \sum_{i,j=1}^{n} f_{ij} x_i x_j$ and the corresponding matrix by $M_l = (f_{ij})$, where $f_{ij} = B(v_i, v_j) \in \mathbb{Z}$, for a fixed basis $\{v_1, v_2, \cdots, v_n\}$ of l. Let $I_N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_N$, where $\{e_1, e_2, \cdots, e_N\}$ is the standard basis of $\mathbb{Z}^N$ with $e_i \cdot e_j = \delta_{ij}$ for all $i, j = 1, 2, \cdots, N$. So, I_N is the $\mathbb{Z}$-lattice corresponding to the sum of N squares. We now define $g_{\mathbb{Z}}(n)$ to be the smallest positive integer g (if exists) for which $l \to I_g$ (meaning that l is represented by I_g) for every positive definite $\mathbb{Z}$-lattice l of rank n such that

$$l \to I_N \text{ for some } N = N(l). \tag{1.1}$$

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Of course, $g_{\mathbb{Z}}(n) = n+3$ holds for $1 \le n \le 5$ even without the condition (1.1) according to the above results of Lagrange, Mordell, and Ko. It is known (see [2], for example) that $g_{\mathbb{Z}}(n)$ exists for every positive integer n.

In section 2, we'll discuss the results of Lagrange, Mordell, and Ko in lattice theoretic language, and provide an example of positive definite $\mathbb{Z}$-lattice of rank six that can be represented by I_{10} but not by I_9 and thereby obtain $g_{\mathbb{Z}}(6) \ge 10$, as an immediate consequence. In section 3, we prove $g_{\mathbb{Z}}(6) \le 11$ and then in section 4, we improve this to conclude that $g_{\mathbb{Z}}(6) = 10$.

We want to mention here that our method is not applicable, unfortunately, to determine $g_{\mathbb{Z}}(n)$ for $n \ge 7$. We believe that the determination of $g_{\mathbb{Z}}(n)$ for $n \ge 7$ requires a totally different and more sophisticated approach. We want to express our gratitude to Professor J.S.Hsia of the Ohio State University for introducing this problem to us. We also want to say thanks to the referee for his valuable suggestions.

§2 Ko's result rewritten

We start with the following well-known result :

Theorem 2.1. *Every positive definite $\mathbb{Z}$-lattice l of rank n is represented by the genus of I_{n+3}, i.e., $l \to L$ for some $L \in gen(I_{n+3})$.*

Proof. Applying Theorems 1 and 2 in [9], one can easily obtain that $l_p \to (I_{n+3})_p$ for any finite prime p. Since l is positive definite, $l_\infty \to (I_{n+3})_\infty$, which completes the proof. □

Since $\mathrm{gen}(I_n) = \mathrm{cls}(I_n)$ for $n \le 8$, one can recapture the results of Lagrange [6], Mordell [7], and Ko [4] all at once in the following theorem :

Theorem 2.2. *Every positive definite $\mathbb{Z}$-lattice l of rank $n \le 5$ is represented by I_{n+3}.*

It is easy to check that $n+3$ is the minimum number of squares necessary to represent all such l, and thereby we obtain

$$g_{\mathbb{Z}}(n) = n+3 \quad \text{for } 1 \le n \le 5. \tag{2.1}$$

Observe that the condition (1.1) is not necessary for (2.1). See also [1] and [11] on Ko's results and more.

The conjecture "$g_{\mathbb{Z}}(n) = n+3$" is, however, broken even for $n = 6$. Indeed, if we let l be a positive definite $\mathbb{Z}$-lattice of rank 6 such that

$$l \simeq \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \tag{2.2}$$

then $l \nrightarrow I_9$ but $l \to I_{10}$ (see [3]). Therefore, we have

$$g_{\mathbb{Z}}(6) \ge 10. \tag{2.3}$$

We close this section with the following lemma, which will be used frequently later :

Lemma 2.3. *Let $M =$ diag $(f_1, f_2, \cdots, f_n)$ and $A = (\alpha_i \alpha_j)$ be $n \times n$ symmetric matrices. Then*

$$\det(M - A) = \prod_{i=1}^{n} f_i - \sum_{j=1}^{n} \alpha_j^2 \prod_{i \ne j} f_i. \tag{2.4}$$

Proof. We use induction on n. When $n = 1$, (2.4) is trivial. Let $n \geq 2$. If $\alpha_1 = 0$, then (2.4) follows immediately from induction hypothesis. So we may assume that $\alpha_1 \neq 0$. Let $N = (n_{ij}) = M - A$, and N_{ij} be the adjoint matrix of n_{ij}. Then

$$\det N = (f_1 - \alpha_1^2)\det(N_{11}) + \sum_{j=2}^{n}(-1)^{j+1}(-\alpha_1\alpha_j)\det(N_{1j}).$$

By induction hypothesis, we have

$$\det(N_{11}) = \prod_{i=2}^{n} f_i - \sum_{j=2}^{n}\alpha_j^2 \prod_{i\neq j,1} f_i.$$

Applying suitable elementary matrices to N_{1j} for $j \geq 2$, we may move the first column to the $(j-1)$st column. Then multiply the $(j-1)$st row by $\dfrac{\alpha_1}{\alpha_j}$ and denote the resulting matrix by $\tilde{N}_{1j}$. Then

$$\det(N_{1j}) = \det(\tilde{N}_{1j})(\frac{\alpha_j}{\alpha_1})(-1)^{j-2}.$$

Again by induction hypothesis, we obtain $\det(\tilde{N}_{1j}) = -\alpha_1^2\prod_{i\neq j,1} f_i$, and therefore,

$$\begin{aligned}\det N &= (f_1 - \alpha_1^2)\left(\prod_{i=2}^{n} f_i - \sum_{j=2}^{n}\alpha_j^2\prod_{i\neq j,1} f_i\right)\\ &+\sum_{j=2}^{n}(-1)^{j+1}(-\alpha_1\alpha_j)(\frac{\alpha_j}{\alpha_1})(-1)^{j-2}(-\alpha_1^2\prod_{i\neq j,1} f_i) = \prod_{i=1}^{n} f_i - \sum_{j=1}^{n}\alpha_j^2\prod_{i\neq j} f_i. \quad \square\end{aligned}$$

Remark. As the referee kindly pointed out to authors, the lemma above can be deduced from more general theorem of Pleskin [11, Corollary II.4].

§3. An upper bound for $g_{\mathbb{Z}}(6)$

Firstly, we prove that every positive definite $\mathbb{Z}$-lattice of rank 6 with even discriminant is represented by I_9.

Proposition 3.1. *Let l be a positive definite $\mathbb{Z}$-lattice of rank 6 with even discriminant. Then $l \to I_9$.*

Proof. Let l be a positive definite $\mathbb{Z}$-lattice of rank 6 with even discriminant, that is, $dl \equiv 0 \pmod 2$. Then by a suitable change of basis, we may write

$$l = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_6, \quad B(v_1, l) \subseteq 2\mathbb{Z}.$$

Suppose $l \not\to I_9$. Then from Theorem 1.1 and the fact that

$$\text{gen}(I_9) = \text{cls}(I_9) \cup \text{cls}(\Phi_8 \perp \langle 1\rangle), \tag{3.1}$$

where the union is disjoint, we may conclude that $l \to \Phi_8 \perp \langle 1\rangle$. Then there exist $a_1, a_2, \cdots, a_6 \in \mathbb{Z}$ such that the $\mathbb{Z}$-lattice $\tilde{l}$ corresponding to the quadratic form $f_{\tilde{l}} := f_l - (a_1x_1 + a_2x_2 + \cdots + a_6x_6)^2$ is represented by Φ_8. Observe that $\tilde{l}$ is semi-positive definite. Since $B(v_1, l) \subseteq 2\mathbb{Z}$, all f_{1i}'s are even. In particular, f_{11} is even. From $Q(\Phi_8) \subseteq 2\mathbb{Z}$ follows that a_1 is also even. The entries in the first row of $M_{\tilde{l}}$, therefore, are all even and hence $d\tilde{l} \equiv 0 \pmod 2$. So, $\tilde{l}_2$ is not unimodular and hence $\tilde{l}_2 \to (I_8)_2$ (see [9, Theorem 2]). Since $(\Phi_8)_p \simeq (I_8)_p$ for all $p \neq 2$, $\tilde{l}_p \to (I_8)_p$ for all non 2-adic prime p (including ∞). This implies that $\tilde{l} \to I_8$ because gen $(I_8) = \text{cls}(I_8)$. But then again, we may conclude that $l \to I_9$, as desired. $\square$

Remark. Observe that we don't need the condition (1.1) in the above proposition.

Secondly, we prove that every positive definite $\mathbb{Z}$-lattice l of rank 6 with odd discriminant is represented by I_{11} if $l \to I_N$ for some N. We will prove in the next section that the representing lattice I_{11} can be replaced by I_{10}.

Proposition 3.2. *Let l be a positive definite $\mathbb{Z}$-lattice of rank 6 with odd discriminant such that $Q(l) \not\subseteq 2\mathbb{Z}$ and $l \to I_N$ for some N. Then $l \to I_{10}$.*

Proof. Let $l = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_6$ where $v_i = (a_{i1}, a_{i2}, \cdots, a_{iN}) \in \mathbb{Z}^N$ for $i = 1, 2, \cdots, 6$. We can assume $v_i \cdot v_j \equiv \delta_{ij} \pmod 2$ for $i, j = 1, 2, \cdots, 6$, by weak approximation theorem. Then $f_l = \sum_{i=1}^N (a_{1i}x_1 + a_{2i}x_2 + \cdots + a_{6i}x_6)^2$ and there exists at least one j for which $a_{1j} + a_{2j} + \cdots + a_{6j}$ is odd because dl is odd. Consider $\tilde{l}$ whose corresponding quadratic form $f_{\tilde{l}}$ is defined by $f_{\tilde{l}} = f_l - (a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{6j}x_6)^2$. Obviously, $\tilde{l}$ is semi-positive definite with even discriminant by Lemma 2.3. So, $\tilde{l} \to I_9$ by proposition 3.1 and hence $l \to I_{10}$. $\square$

Now, let l be a positive definite $\mathbb{Z}$-lattice with odd discriminant and that $Q(l) \subseteq 2\mathbb{Z}$. Then l_2 is even unimodular of rank 6 so that l_2 is isometric to one of the followings (see [9]) :

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{3.2}$$

Note that, in particular, $dl \equiv 7$ or $3 \pmod 8$, respectively.

Proposition 3.3. *Let l be a positive $\mathbb{Z}$-lattice of rank 6 such that $dl \equiv 7 \pmod 8$ such that $Q(l) \subseteq 2\mathbb{Z}$. Then $l \to I_9$.*

Proof. As in proposition 3.1, let $l = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_6$. We may assume that

$$M_l = \begin{pmatrix} A & a & 0 & 0 & 0 & 0 \\ a & B & b & 0 & 0 & 0 \\ 0 & b & C & c & 0 & 0 \\ 0 & 0 & c & D & d & 0 \\ 0 & 0 & 0 & d & E & e \\ 0 & 0 & 0 & 0 & e & F \end{pmatrix}.$$

Suppose $l \nrightarrow I_9$. Then $l \to \Phi_8 \perp \langle 1 \rangle$ and hence we have $\tilde{l}$, whose corresponding quadratic form is $f_{\tilde{l}} = f_l - (a_1x_1 + a_2x_2 + \cdots + a_6x_6)^2$ for some $a_1, a_2, \cdots, a_6 \in \mathbb{Z}$, such that $\tilde{l} \to \Phi_8$. Observe that a_i's are all even for $i = 1, 2, \cdots, 6$, because $Q(\Phi_8) \subseteq 2\mathbb{Z}$.

A direct computation leads us to $d\tilde{l} \equiv dl \equiv 7 \pmod 8$. So, by (3.2) and [9, Theorem 2],

$$\tilde{l}_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to (I_8)_2.$$

Since $\tilde{l}_p \to (\Phi_8)_p \simeq (I_8)_p$ for all non 2-adic p (including ∞), we have $\tilde{l} \to I_8$, which implies $l \to I_9$. $\square$

Remark. Observe that the condition (1.1) is not necessary in the above proposition.

Proposition 3.4. *Let l be a positive definite $\mathbb{Z}$-lattice of rank 6 with $dl \equiv 3 \pmod 8$ such that $Q(l) \subseteq 2\mathbb{Z}$ and $l \to I_N$ for some N. Then $l \to I_{11}$.*

Proof. As in proposition 3.2, let $l = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_6$, where $v_i = (a_{i1}, a_{i2}, \cdots, a_{iN}) \in \mathbb{Z}^N$ for $i = 1, 2, \cdots, 6$. Then $f_l = \sum_{i=1}^N (a_{1i}x_1 + a_{2i}x_2 + \cdots + a_{6i}x_6)^2$ and there exists at least one j for which a_{1j} is odd. Let $\tilde{l}$ be the $\mathbb{Z}$-lattice corresponding to $f_{\tilde{l}} = f_l - (a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{6j}x_6)^2$. From Corollary II.4 of [11] follows $d\tilde{l}$ is odd. So, $\tilde{l} \to I_{10}$ by Proposition 3.2. This is because $Q(\tilde{l}) \not\subseteq 2\mathbb{Z}$. Therefore, $l \to I_{11}$ as desired. $\square$

Combining (2.3) and previous propositions, we obtain.

$$10 \le g_{\mathbb{Z}}(6) \le 11. \tag{3.3}$$

§4. Determination of $g_{\mathbb{Z}}(6)$

In this section, we will improve the upper bound 11 for $g_{\mathbb{Z}}(6)$ in (3.3) to 10, and thereby obtain $g_{\mathbb{Z}}(6) = 10$. To do this, we need to discuss $\mathbb{Z}$-lattice with odd discriminant in more detail.

Proposition 4.1. *Let l be a positive definite $\mathbb{Z}$-lattice of rank 6 with odd discriminant such that $Q(l) \nsubseteq 2\mathbb{Z}$. Then $l \to I_9$ if either $dl \equiv 1, 5 \pmod 8$ with $S_2(l) = -1$ or $dl \equiv 3, 7 \pmod 8$ with $S_2(l) = 1$, where $S_2(l)$ is the 2-adic Hasse symbol of l.*

Proof. We provide a proof only for the case $dl \equiv 1 \pmod 8$ with $S_2(l) = -1$ because all other cases can be proved in a similar manner. If $dl \equiv 1 \pmod 8$ with $S_2(l) = -1$, one can easily obtain that

$$l_2 \simeq I_4 \perp \langle 3 \rangle \perp \langle 3 \rangle.$$

By the weak approximation theorem for rotations [10, 101:7], we may assume

$$M_l \equiv \text{ diag } (1,1,1,1,3,3) \pmod 8.$$

If we suppose $l \nrightarrow I_9$, then $l \to \Phi_8 \perp \langle 1 \rangle$ and hence we have $\tilde{l}$ corresponding to $f_{\tilde{l}} = f_l - (a_1x_1 + a_2x_2 + \cdots + a_6x_6)^2$ for some $a_1, a_2, \cdots, a_6 \in \mathbb{Z}$ such that $\tilde{l} \to \Phi_8$. Since f_{ii} is odd for every $i = 1, 2, \cdots, 6$, and $Q(\Phi_8) \subseteq 2\mathbb{Z}$, we have a_i is odd for every $i = 1, 2, \cdots, 6$. Then by Lemma 2.3, $d\tilde{l} \equiv 7 \pmod 8$. So, we obtain

$$\tilde{l}_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence $\tilde{l}_2 \to (I_8)_2$ as in Proposition 3.3. Since $\tilde{l}_p \to (\Phi_8)_p \simeq (I_8)_p$ for all non 2-adic prime p (including ∞), $\tilde{l} \to I_8$ and hence $l \to I_9$ as desired. □

Remark. Observe that the condition (1.1) is not necessary in the above proposition.

In order to improve Proposition 3.4, we need the following technical lemma :

Lemma 4.2. *Let l be a positive definite $\mathbb{Z}$-lattice of rank 6 with $dl \equiv 3 \pmod 8$ such that $Q(l) \subseteq 2\mathbb{Z}$. Let $\tilde{l}$ be the positive definite $\mathbb{Z}$-lattice corresponding to $f_{\tilde{l}} = f_l - (a_1x_1 + a_2x_2 + \cdots + a_6x_6)^2$ for some $a_1, a_2, \cdots, a_6 \in \mathbb{Z}$ such that $d\tilde{l} \equiv 1, 5 \pmod 8$. Then $l \to I_{10}$.*

Proof. Obviously $Q(\tilde{l}) \nsubseteq 2\mathbb{Z}$ and $\tilde{l}_2$ is proper unimodular. More precisely, if $d\tilde{l} \equiv 1 \pmod 8$, then $\tilde{l}_2 \simeq I_6$ or $I_4 \perp \langle 3 \rangle \perp \langle 3 \rangle$, and if $d\tilde{l} \equiv 5 \pmod 8$, then $\tilde{l}_2 \simeq I_5 \perp \langle 5 \rangle$ or $I_3 \perp \langle 3 \rangle \perp \langle 3 \rangle \perp \langle 5 \rangle$. Suppose $(\tilde{l})_2 \simeq (I_6)_2$. Then $l_2 \to (I_7)_2$ and hence $l_2 \perp \langle 3 \rangle \simeq (I_7)_2$ as space. But this cannot happen because their Hasse symbols do not match. So, if $d\tilde{l} \equiv 1 \pmod 8$, then $\tilde{l}_2 \simeq I_4 \perp \langle 3 \rangle \perp \langle 3 \rangle$. Similarly, if $d\tilde{l} \equiv 5 \pmod 8$, then $\tilde{l}_2 \simeq I_3 \perp \langle 3 \rangle \perp \langle 3 \rangle \perp \langle 5 \rangle$. Since $S_2(\tilde{l}) = -1$ in both cases, we have $\tilde{l} \to I_9$ by Proposition 4.1 and hence $l \to I_{10}$. □

Proposition 4.3. *Let l be a positive definite $\mathbb{Z}$-lattice of rank 6 with $dl \equiv 3 \pmod 8$ such that $Q(l) \subseteq 2\mathbb{Z}$ and $l \to I_N$ for some N. Then $l \to I_{10}$.*

Proof. We may assume that $l \to I_{11}$ according to Proposition 3.4 and that

$$M_l \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \pmod 8. \tag{4.1}$$

Let $l = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_6$, with $v_i = (a_{i1}, a_{i2}, \cdots, a_{i(11)}) \in \mathbb{Z}^{11}$ for $i = 1, 2, \cdots, 6$. Then $f_l = \sum_{j=1}^{11}(a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{6j}x_6)^2$. We define, for each $j = 1, 2, \cdots, 11$, $\tilde{l}(j)$ to be the $\mathbb{Z}$-lattice corresponding to the quadratic form $f_{\tilde{l}(j)} = f_l - (a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{6j}x_6)^2$. So, $\tilde{l}(j) = \mathbb{Z}v_1(j) + \mathbb{Z}v_2(j) + \cdots + \mathbb{Z}v_6(j)$, where $v_i(j) = (a_{i1}, \cdots, a_{i(j-1)}, a_{i(j+1)}, \cdots, a_{i(11)}) \in \mathbb{Z}^{10}, i = 1, 2, \cdots, 6$. Corollary II.4 of [11] imlies

$$(4.2) \qquad d\tilde{l}(j) \equiv 2(a_{1j}a_{2j} + a_{3j}a_{4j} + a_{5j}a_{6j} - a_{5j}^2 - a_{6j}^2) + 3 \pmod 8.$$

We claim :

(4.3) *There exists* $j_0 \in \{1, 2, \cdots, 11\}$ *such that* $a_{1j_0}a_{2j_0} + a_{3j_0}a_{4j_0} + a_{5j_0}a_{6j_0} - a_{5j_0}^2 - a_{6j_0}^2$ *is odd.*

We'll provide a proof of the claim (4.3) in the Appendix. Assuming the claim, we let $\tilde{l} = \tilde{l}(j_0)$. Then by (4.2), $d\tilde{l} \equiv 1, 5 \pmod 8$. Therefore, by Lemma 4.2, we obtain $l \to I_{10}$. $\square$

We summarize all the previous results in the following theorem :

Theorem 4.4. *Let l be a positive definite $\mathbb{Z}$-lattice of rank 6. Then we have :*
(1) $l \to I_9$ if dl is even or $dl \equiv 7 (mod\ 8)$ with $Q(l) \subseteq 2\mathbb{Z}$.
(2) $l \to I_{10}$ if $l \to I_N$ for some N and if dl is odd with $Q(l) \nsubseteq 2\mathbb{Z}$ or $dl \equiv 3 (mod\ 8)$ with $Q(l) \subseteq 2\mathbb{Z}$.

Corollary 4.5. $g_{\mathbb{Z}}(6) = 10$.

Remark. As we saw in the proofs, our main result depends heavily on (3.1) while $\text{gen}(I_8) = \text{cls}(I_8)$, $\text{gen}(\Phi_8) = \text{cls}(\Phi_8)$, and $\text{gen}(I_8) \neq \text{gen}(\Phi_8)$. Unfortunately, this property is no longer applicable when we discuss $g_{\mathbb{Z}}(n)$ for $n \geq 7$.

Appendix

We now prove the claim (4.3) in the proof of Proposition 4.3. We keep the setting of the proposition. We call $w_j = {}^t(a_{1j}, a_{2j}, \cdots, a_{6j})$ a good column if $a_{1j}a_{2j} + a_{3j}a_{4j} + a_{5j}a_{6j} - a_{5j}^2 - a_{6j}^2$ is odd, and call it a bad column otherwise. We have to show that their exists at least one good column for any given bases $\{v_1, v_2, \cdots, v_6\}$ of l.

Let $\overline{w}_j = {}^t(a_{1j}, a_{2j}, a_{3j}, a_{4j})$ be called a O_1-type if it matches one of the ten leftmost columns, and a O_2-type if it matches one of the six rightmost columns in the following table (mod 2) :

$$(A.1)$$

$$\begin{array}{ccccccccccccccccccc}
a_{1j} & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & : & 1 & 0 & 1 & 1 & 1 & 0 \\
a_{2j} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & : & 1 & 0 & 1 & 1 & 0 & 1 \\
a_{3j} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & : & 0 & 1 & 1 & 0 & 1 & 1 \\
a_{4j} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & : & 0 & 1 & 0 & 1 & 1 & 1.
\end{array}$$

Let $\underline{w}_j = \begin{pmatrix} a_{5j} \\ a_{6j} \end{pmatrix}$ be called a K_1-type if $\underline{w}_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and a K_2-type if $\underline{w}_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. It is easy to check that w_j is a good column if and only if either $\overline{w}_j$ is O_1-type and $\underline{w}_j$ is K_1-type or $\overline{w}_j$ is O_2-type and $\underline{w}_j$ is K_2-type. In order to match the lower left 2×2 block of (4.1) (mod 2), $\begin{pmatrix} v_5 \\ v_6 \end{pmatrix}$ or $\begin{pmatrix} v_6 \\ v_5 \end{pmatrix}$ should be one of the following forms (mod 2) up to permutation on columns :

$$(A.2)$$

$$\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&0\\0&1&1&1&1&1&1&1&1&1&1\end{pmatrix},\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&0\\0&0&0&0&0&1&1&1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&0\\0&0&0&0&0&0&0&0&0&1&1\end{pmatrix},\begin{pmatrix}1&1&1&1&1&1&0&0&0&0&0\\0&1&1&1&1&1&1&0&0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&1&1&0&0&0&0&0\\0&0&0&1&1&1&1&1&1&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1&1&1&0&0&0&0&0\\0&0&0&0&0&1&1&1&1&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&1&1&0&0&0&0&0\\0&0&0&0&0&1&1&0&0&0&0\end{pmatrix},\begin{pmatrix}1&1&0&0&0&0&0&0&0&0&0\\0&1&1&0&0&0&0&0&0&0&0\end{pmatrix}.$$

Reading columns, the numbers of K_1-types and K_2-types in each $\begin{pmatrix}v_5\\v_6\end{pmatrix}$ of (A.2) are

(A.3) (11 and 0) or (9 and 2) or (7 and 4) or (3 and 8).

Checking all the possible ${}^t(v_1, v_2, v_3, v_4) \pmod 2$ matching the upper left 4×4 block of (4.1) (mod 2), one can find by reading their columns that the number of O_1-types and O_2-types in each ${}^t(v_1, v_2, v_3, v_4) \pmod 2$ are

(A.4) (9 and 2) or (7 and 4) or (5 and 6) or (3 and 8) or (1 and 10).

There are 159 possible such ${}^t(v_1, v_2, v_3, v_4) \pmod 2$ up to permutation on columns and rows, but instead of providing the list of all 159 such possibilities, we only provide one example of the possible ${}^t(v_1, v_2, v_3, v_4) \pmod 2$ for each case of (A.4) in the following :

$$(A.5)$$

$$\begin{pmatrix}1&1&1&1&1&1&1&1&0&0&0\\0&0&0&1&1&1&1&1&1&1&1\\0&1&1&1&1&1&1&0&1&1&0\\0&0&0&0&1&1&1&1&0&0&0\end{pmatrix},\begin{pmatrix}1&1&1&1&1&1&1&1&0&0&0\\0&1&1&1&1&1&1&1&1&0&0\\0&1&1&1&1&1&1&0&0&1&1\\0&0&1&1&1&1&1&1&0&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&1&1&1&1&0&0&0\\0&1&1&1&1&1&1&1&1&0&0\\1&1&1&1&1&1&0&0&1&1&0\\1&0&0&1&1&1&1&1&1&0&1\end{pmatrix},\begin{pmatrix}1&1&1&1&1&1&1&1&0&0&0\\0&1&1&1&1&1&1&1&1&0&0\\0&1&1&1&1&1&1&0&0&1&1\\0&0&0&0&0&0&1&1&0&1&1\end{pmatrix}$$

$$\begin{pmatrix}1&1&1&1&1&1&1&1&0&0&0\\0&1&1&1&1&1&1&1&1&0&0\\1&1&1&1&1&1&0&0&1&1&0\\1&0&0&0&0&0&1&0&1&1&0\end{pmatrix}.$$

Now, combining $(A.3)$ and $(A.4)$, we may conclude that there always exists at least one good column w_{j_0} for any given basis $\{v_1, v_2, \cdots, v_6\}$ of l.

Remark. Instead of obtaining (A.4) by checking all 159 possible ${}^t(v_1, v_2, v_3, v_4)$ (mod 2), one can directly show that if ${}^t(v_1, v_2, v_3, v_4)$ (mod 2) matches the upper left 4×4 block of (4.1), then the number of O_1-type columns is odd and the number of O_2-type columns is even. From this and (A.3) the claim (4.3) follows.

Proof. If a lattice $\mathbb{Z}v_1 + \mathbb{Z}v_2 \subset I_{11}$ satisfies

$$(B(v_i, v_j)) \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 8 , \tag{A.6}$$

then $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ or $\begin{pmatrix} v_2 \\ v_1 \end{pmatrix}$ should look like one of the followings (mod 2) up to permutation on columns :

$$(A.7)$$

$$\begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&1&1&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix}$$
$$\begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0 \\ 0&0&0&0&0&1&1&1&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}$$
$$\begin{pmatrix} 1&1&1&1&0&0&0&0&0&0&0 \\ 0&1&1&1&1&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&1&1&1&0&0&0&0&0&0&0 \\ 0&0&0&1&1&1&1&0&0&0&0 \end{pmatrix}.$$

Observe that the number of the columns of the form $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is even and the numbers of the columns of the forms $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are all odd in each of (A.7).

Now assume that a lattice $\mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3 + \mathbb{Z}v_4 \subset I_{11}$ satisfies

$$(B(v_i, v_j)) \equiv \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix} \pmod 8$$

which is the upper left 4×4 block of (4.1).

Then both $\mathbb{Z}v_1 + \mathbb{Z}v_2$ and $\mathbb{Z}v_3 + \mathbb{Z}v_4$ satisfy (A.6). Let the columns ${}^t(0,0,0,0), {}^t(0,0,0,1), {}^t(0,0,1,0), {}^t(0,0,1,1), {}^t(0,1,0,0), \cdots, {}^t(1,0,1,1), {}^t(1,1,0,0), {}^t(1,1,0,1), {}^t(1,1,1,0), {}^t(1,1,1,1)$ be called $(1,1), (1,2), (1,3), (1,4), (2,1), \cdots, (3,4), (4,1), (4,2), (4,3), (4,4)$-type columns, respectively. Let's define a_{ij} for each $i, j = 1,2,3,4$ to be the number of (i,j)-type columns in ${}^t(v_1, v_2, v_3, v_4)$ (mod 2). Then the following relations are obvious from the above observation :

$$\begin{aligned}
&a_{11} + a_{12} + a_{13} + a_{14} \equiv 0 \pmod 2, \\
&a_{i1} + a_{i2} + a_{i3} + a_{i4} \equiv 1 \pmod 2, \text{ if } i = 2,3,4, \\
&a_{11} + a_{21} + a_{31} + a_{41} \equiv 0 \pmod 2, \\
&a_{1i} + a_{2i} + a_{3i} + a_{4i} \equiv 1 \pmod 2, \text{ if } i = 2,3,4.
\end{aligned}$$

By using these, we obtain

$$\begin{aligned}
&\text{The number of } O_1\text{-type columns} \\
&\equiv a_{11} + a_{12} + a_{13} + a_{21} + a_{22} + a_{23} + a_{31} + a_{32} + a_{33} + a_{44} \\
&\equiv a_{14} + 1 + a_{24} + 1 + a_{34} + a_{44} \equiv 1 \pmod 2.
\end{aligned}$$

Therefore, the number of O_1-type columns is odd, and hence the number of O_2-type columns is even.

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