1 Isometric Embedding

Let $M^n$ be an $n$-dimensional Riemannian manifold with metric locally given by

$$ds^2 = g_{ij}(x)dx^idx^j,$$

where $x = (x^1, \ldots, x^n)$ are local coordinates on $M$.

Isometric embedding means a one-to-one $C^\infty$-mapping

$$u : M^n \to \mathbb{R}^N$$

such that

$$< du, du > = ds^2$$

or in local coordinates

$$\sum_{\lambda=1}^{N} \frac{\partial u^\lambda}{\partial x^i} \frac{\partial u^\lambda}{\partial x^j} = g_{ij}, \quad i, j = 1, \ldots, n. \quad (1)$$

So a local isometric embedding problem is reduced to a PDE system. There are three different cases to deal with according to the number of equations and the number of unknowns. The number of equations of (1) is $\frac{n(n+1)}{2}$ and the number of unknowns is $N = n + d$. The system (1) is

(i) underdetermined if $\frac{n(n+1)}{2} < N$,

(ii) determined if $\frac{n(n+1)}{2} = N$,

(iii) overdetermined if $\frac{n(n+1)}{2} > N$.

In the determined case, there exists an analytic embedding by the following theorem.

**Theorem 1.1 (Cartan-Janet,[3]).** If $N = \frac{1}{2}n(n + 1)$ and $g_{ij} \in C^\omega$, then there exists a $C^\omega$-solution $u = (u^1, \ldots, u^{\frac{1}{2}n(n+1)})$.

Some of the results on the underdetermined case were obtained by J. Nash[4].
Theorem 1.2. Any Riemannian $n$-manifold with $C^k$ positive metric, where $3 \leq k \leq \infty$, has a $C^k$ isometric embedding in $(\frac{3}{2}n^3 + 7n^2 + \frac{11}{2}n)$-space, in fact in any small portion of this space.

In overdetermined case, we consider the case of codimension one.

2 Isometric Embedding of Codimension One

Isometric embedding of codimension one is an isometric embedding

$$u : M^n \to \mathbb{R}^{n+1}. \quad (2)$$

This is determined if $n = 2$ and overdetermined if $n > 2$. The question of finding a necessary and sufficient condition for the existence of local isometric embedding (2) is reduced to the problem of solving the Gauss and Codazzi equations.

Let $(e_1, \ldots, e_{n+1})$ be an adapted orthonormal frame and $\theta = (\theta^1, \ldots, \theta^n)^t$ be a dual frame of $(e_1, \ldots, e_n)$. For any 1-forms $\eta$ and $\psi$, the symmetric product is defined by

$$\eta \circ \psi = \frac{1}{2}(\eta \otimes \psi + \psi \otimes \eta).$$

Let $I = \sum_{i=1}^{n} \theta^i \circ \theta^i = \sum_{i=1}^{n} (\theta^i)^2$ be the first fundamental form of $M$.

Let $X$ be a tangent vector field on $M$ and $Y = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x^i}$ a vector field on $M$ which is not necessarily tangent to $M$. Define

$$\nabla_X Y = \sum_{i=1}^{n+1} (Xa_i) \frac{\partial}{\partial x_i}.$$ 

Proposition 2.1. If $X$ and $Y$ are tangent vector fields to $M$, then $\nabla_X Y - \nabla_Y X = [X, Y]$. So $[X, Y]$ is also a tangent vector field to $M$. 

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Proof. If $X$ and $Y$ are tangent vector fields, then $X = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$ and $Y = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$. Thus

$$[X, Y] = XY - YX = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \left( \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \right) - \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} \right)$$

$$= \sum_{i,j} b_j \frac{\partial}{\partial x_j} \left( a_i \frac{\partial}{\partial x_i} \right) + \sum_{i,j} a_i b_j \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} - \sum_{i,j} a_i b_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$$

$$= \nabla_X Y - \nabla_Y X.$$ 

Since $\nabla_X Y$ and $\nabla_Y X$ are tangent to $M$, so is $[X, Y]$. \qed

**Definition 2.2.** For tangent vector fields $X$, $Y$ and normal vector field $N$, the second fundamental form $\Pi$ of $M$ is defined by

$$\Pi(X, Y) = -< \nabla_X N, Y >.$$ 

**Proposition 2.3.** For tangent vector fields $X$, $Y$ and normal vector field $N$, the second fundamental form has the properties:

$$\Pi(X, Y) = < \nabla_X Y, N >, \quad \Pi(X, Y) = \Pi(Y, X).$$

Proof. Since $N$ is a normal vector field, $< Y, N > = 0$. Thus

$$X < Y, N > = < \nabla_X Y, N > + < Y, \nabla_X N > = 0.$$ 

By definition 2.2, we have

$$\Pi(X, Y) = -< \nabla_X N, Y >$$

$$= < Y, \nabla_X N >$$

$$= < \nabla_X Y, N >.$$ 

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Using the previous proposition, we get
\[
\Pi(X, Y) - \Pi(Y, X) = \langle \nabla_X Y, N \rangle - \langle \nabla_Y X, N \rangle = \langle \nabla_X Y - \nabla_Y X, N \rangle = \langle [X, Y], N \rangle = 0.
\]

Since \( \Pi \) is a symmetric 2-form on \( M \), we write
\[
\Pi = \sum_{i,j=1}^{n} h_{ij} \theta^i \otimes \theta^j,
\]
where \( h_{ij} = h_{ji} \). Then \( h_{ij} = \Pi(e_i, e_j) \). Since \( (h_{ij}) \) is symmetric, its eigenvalues are real. Let \( k_1, \ldots, k_n \) be eigenvalues. We call them the principal curvatures.

**Theorem 2.4.** Let \( (\omega^{ij}) = A^{-1}dA \), where \( A = (e_1, \ldots, e_{n+1}) \). On \( M \),
\[
\omega^{n+1}_i = \sum_{\lambda=1}^{n} h_{i\lambda} \theta^\lambda.
\]

**Proof.** We know that \( \nabla_X e_i = \sum_{j=1}^{n+1} \omega^{ij}_i(X)e_j \). Since \( \omega \) is generated by \( \theta^1, \ldots, \theta^n \), it is enough to show that \( \omega^{n+1}_i = h_{i\lambda} \). Since \( (h_{ij}) \) is symmetric, \( h_{i\lambda} = h_{\lambda i} \) and since \( (e_1, \ldots, e_n, e_{n+1}) \) is the adapted orthonormal frame, \( e_{n+1} \) is a normal vector. Therefore, we have
\[
\begin{align*}
\Pi(e_{\lambda}, e_i) &= h_{i\lambda} = -\langle \nabla_{e_{\lambda}} e_{n+1}, e_i \rangle = -\langle \sum_{j=1}^{n+1} \omega^{ij}_n(e_{\lambda}) e_j, e_i \rangle = -\omega^{n+1}_i(e_{\lambda}) = \omega^{n+1}_i(e_{\lambda}).
\end{align*}
\]
The last equality follows from the fact that $\omega$ is skew-symmetric as shown below. Since $\langle e_i, e_j \rangle = \delta_{ij}$, we have

\[
0 = d \langle e_i, e_j \rangle = \langle de_i, e_j \rangle + \langle e_i, de_j \rangle = \langle \sum_{\lambda=1}^{n+1} \omega^i_\lambda e_\lambda, e_j \rangle + \langle e_i, \sum_{\lambda=1}^{n+1} \omega^j_\lambda e_\lambda \rangle = \omega^j_i + \omega^i_j.
\]

\[\Box\]

From now on we consider the case of $n = 3$. In order to obtain the structure equations, consider $E(4) \hookrightarrow GL(5, \mathbb{R})$ with Maurer-Cartan form $\gamma = g^{-1}dg$ of $E(4)$. $E(4)$ is the set of all matrices $\begin{bmatrix} 1 & 0 \\ x & A \end{bmatrix}$ with $^tAA = I$.

Let $\sigma : M \rightarrow E(4)$ be an adapted frame $\sigma(x) = (e_1, e_2, e_3, e_4)_x$. Then it follows that

\[
\sigma^*(\gamma) = \begin{bmatrix}
0 & 0 \\
^tAdX & ^tAdA \\
0 & 0 & 0 & 0 & 0 & \theta^1 & 0 & -\omega^2_1 & -\omega^3_1 & -\eta_1 \\
0 & \theta^2 & \omega^2_1 & 0 & -\omega^3_2 & -\eta_2 \\
0 & \theta^3 & \omega^3_1 & \omega^3_2 & 0 & -\eta_3 \\
0 & \eta_1 & \eta_2 & \eta_3 & 0 & 0 & 0 & 0 & \theta & \omega & -^t\eta & 0 & \eta & 0
\end{bmatrix}
\]

where $\eta_i = \omega^i_4$, $A = (e_1, \ldots, e_4)$, $\eta = (\eta_1, \eta_2, \eta_3)$, $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ and
\[ \omega = \begin{bmatrix} 0 & -\omega_1^2 & -\omega_1^3 \\ \omega_1^2 & 0 & -\omega_2^3 \\ \omega_1^3 & \omega_2^3 & 0 \end{bmatrix}. \]

Maurer-Cartan equation \( d\gamma = -\gamma \wedge \gamma \) implies that
\[
d(\sigma^*\gamma) = (-\sigma^*\gamma) \wedge (\sigma^*\gamma).
\]

Thus
\[
d \begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -t \eta \\ 0 & \eta & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -t \eta \\ 0 & \eta & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 0 & 0 \\ \theta & \omega & -t \eta \\ 0 & \eta & 0 \end{bmatrix},
\]
\[
d \begin{bmatrix} 0 & 0 & 0 \\ d\theta & d\omega & -t(d\eta) \\ 0 & d\eta & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 \\ \omega \wedge \theta & \omega \wedge \omega - t \eta \wedge \eta & -\omega \wedge t \eta \wedge \eta \\ \eta \wedge \theta & \eta \wedge \omega & -\eta \wedge t \eta \end{bmatrix}.
\]

By Theorem 2.4, we have \( \eta_i = \omega_i^4 = \sum_{\lambda=1}^3 h_{i\lambda} \theta^\lambda \), that is, \( \eta = t \theta H \).

Thus we obtain
\[
d\theta = -(\omega \wedge \theta),
\]
\[
d\omega = -\omega \wedge \omega + t \eta \wedge \eta \quad \text{(Gauss equation)},
\]
\[
d\eta = -\eta \wedge \omega \quad \text{(Codazzi equation)},
\]
\[
\eta \wedge \theta = 0,
\]
\[
\eta \wedge t \eta = 0,
\]
\[
\eta = t \theta H.
\]

It is enough to show that there exists the second fundamental form \( \Pi = (h_{ij}) = H \) by the following theorem

**Theorem 2.5 (Bonnet, [5]).** Suppose that two hypersurfaces \( M \) and \( \tilde{M} \subset \mathbb{R}^{n+1} \) have the same first and second fundamental forms. Then they are congruent.

Let us summarize the process of solving the case of \( n = 3 \) as follows:
(i) Start with metric \( g = I \).

(ii) Find orthonormal frame \( \theta \) such that \( I = \sum_{i=1}^{3} (\theta_i)^2 \).

(iii) Find Levi-Civita connection for \( (\omega^i_j) \) \( i, j = 1, 2, 3 \) such that \( d\theta = -\omega \wedge \theta \) and \( ^t\omega = -\omega \). Then compute curvature \( \Phi = d\omega + \omega \wedge \omega = ^t\eta \wedge \eta \).

(iv) Solve the algebraic equation \( \Phi = H \theta ^t \theta H \) for \( H \). Compute \( ^t\eta \wedge \eta = H \theta \wedge ^t \theta H \). Let \( \Phi = (\Phi^i_j) \). Compare both sides of \( \Phi = H \theta \wedge ^t \theta H \). Both sides are skew-symmetric. Then we obtain the following three equations.

\[
(h_{22}h_{33} - h_{23}^2)\theta^2 \wedge \theta^3 + (h_{23}h_{13} - h_{12}h_{33})\theta^3 \wedge \theta^1 + (h_{12}h_{23} - h_{22}h_{13})\theta^1 \wedge \theta^2 = \Phi^3_2,
\]

\[
(h_{13}h_{23} - h_{12}h_{33})\theta^2 \wedge \theta^3 + (h_{11}h_{33} - h_{13}^2)\theta^3 \wedge \theta^1 + (h_{12}h_{13} - h_{11}h_{23})\theta^1 \wedge \theta^2 = -\Phi^3_1,
\]

\[
(h_{12}h_{23} - h_{13}h_{22})\theta^2 \wedge \theta^3 + (h_{13}h_{12} - h_{11}h_{23})\theta^3 \wedge \theta^1 + (h_{11}h_{22} - h_{12}^2)\theta^1 \wedge \theta^2 = \Phi^2_1.
\]

In matrix form, these equations are

\[
\text{adj}(H) \begin{bmatrix} \theta^2 \wedge \theta^3 \\ \theta^3 \wedge \theta^1 \\ \theta^1 \wedge \theta^2 \end{bmatrix} = \begin{bmatrix} \Phi^3_2 \\ -\Phi^3_1 \\ \Phi^2_1 \end{bmatrix}.
\]

To compute \( \text{adj}(H) = K \), evaluate on \( (e_k, e_l) \). Let \( \Phi^j_{i;k} = \Phi^j_i(e_k, e_l) \).

Then

\[
K = \begin{bmatrix} \Phi^3_{22} & \Phi^3_{23} & \Phi^3_{21} \\ -\Phi^3_{12} & -\Phi^3_{13} & -\Phi^3_{11} \\ \Phi^2_{12} & \Phi^2_{13} & \Phi^2_{11} \end{bmatrix}.
\]
Since \( K = \text{adj}(H) = (\det H)H^{-1} \),
\[
H = \frac{1}{\det H}K^{-1},
\]
\[
\det K = (\det H)^3(\det H)^{-1} = (\det H)^2.
\]

Thus \( \det H = \pm \sqrt{\det K} \). If \( \det K > 0 \), Gauss equation is solvable and the solution is unique up to sign and if \( \det K < 0 \), there is no solution.

(v) Check whether \( H \) satisfies Codazzi equation \( d(\text{t} \theta H) = -\text{(t} \theta H) \wedge \omega \).
If it holds, then \( H \) is a solution.

Here is a more general result of the codimension one case under some restrictions for \( M^n \) for \( n \geq 5 \). This result was shown by J. Vilms[6].

Let \( V \) be an \( n \)-dimensional real vector space with inner product. Let \( \Lambda^2 V \) denote the \( \binom{n}{2} \)-dimensional space of bivectors of \( V \). A linear map \( L : V \to V \) defines a linear map \( L \wedge L : \Lambda^2 V \to \Lambda^2 V \) by \( (L \wedge L)(x \wedge y) = Lx \wedge Ly \). When \( V \) is taken to be the tangent space at a point of \( M^n \), the curvature tensor \( R \) at that point can be thought of as a symmetric linear map \( R : \Lambda^2 V \to \Lambda^2 V \). Letting \( L \) denote the second fundamental form operator and denoting the covariant derivative by \( \nabla \), we can express the Gauss and Codazzi equations as \( R = L \wedge R \) and \( \nabla L \) is symmetric. On the above setting, the problem of locally isometrically embedding into \( \mathbb{R}^{n+1} \) a \( C^3 \) Riemannian manifold \( M^n \) with curvature of rank \( \geq 6 \) is reduced to the following algebraic question: Given a symmetric linear map \( R : \Lambda^2 V \to \Lambda^2 V \), find necessary and sufficient condition in order that there exists a symmetric linear map \( L : V \to V \) satisfying \( R = L \wedge L \).

**Theorem 2.6 (J. Vilms[6]).** Let \( M^n \), with \( n \geq 5 \), be a Riemannian manifold with nonsingular curvature tensor \( R \). Then \( M^n \) admits local isometric imbedding into \( \mathbb{R}^{n+1} \) if and only if
(1) \( R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4) + R(x_1 \wedge x_3) \wedge R(x_2 \wedge x_4) = 0, \) for all \( x_i \in V, \) and

(2) \( R_{ij}^{ij} R_{kl}^{kl} R_{pq}^{pq} + \frac{1}{4} R_{ij}^{ij} R_{pq}^{pq} R_{ij}^{ij} > 0. \)

Moreover, if \( n \equiv 3 \mod 4, \) then (1) can be replaced by \( \det R > 0. \)

References


