ON THE CANCELLATION PROBLEM OF ZARISKI

Dongho Byeon and Hyun Kwang Kim

Department of Mathematics
Pohang University of Science and Technology, Pohang, Korea
E-mail: faust@ posmath. postech. ac. kr

Abstract

Let K_1 and K_2 be extension fields over a field K with $charK = p > 0$. Assume $L = K_1(x_1) = K_2(x_2) \supset K$ where x_i is transcendental over K_i, for $i = 1, 2$. In this paper we prove that if K_1 is a perfect field, then $K_1 = K_2$.

Let K_1 and K_2 be finitely generated extensions of a field K and let x_i be the transcendental over K_i, $i = 1, 2$. The cancellation problem of Zariski [5] asks if $K_1(x_1) = K_2(x_2)$ must K_1 and K_2 be K-isomorphic? In general the answer is no [1]. However there are some special cases in which the answer is yes [2,3,4,5]. For example, it is known that the problem holds true if $charK = 0$ and $x_1 = x_2$ [2,5]. But for the case of a finite base field, very little is known. In this paper we shall prove the problem holds true for an important case of a finite base field i.e. if $charK = p > 0$ and K_1 is a perfect field then $K_1 \cong K_2$. In this case we have more strong result, say $K_1 = K_2$.

THEOREM. *Let K_1 and K_2 be extension fields over a field K with $charK = p > 0$. Assume $L = K_1(x_1) = K_2(x_2) \supset K$ where x_i is transcendental over K_i, for $i = 1, 2$. If K_1 is a perfect field, then $K_1 = K_2$.*

REMARK. *In [2,3,4,5], we assume that K_1 and K_2 are finitely generated extensions of K. However, in our THEOREM we don't need to assume that K_1 and K_2 are finitely generated extensions of K.*

We start with a lemma.

LEMMA. *Let* K_1 *and* K_2 *be fields as in the THEOREM. If* K_1 *is a perfect field, then so is* K_2.

PROOF. Let φ be the Frobenius automorphism of L so that $\varphi(a) = a^p$ for all $a \in L$ where $p = char K > 0$. Then $\varphi(L) = L^p = K_1^p(x_1^p) = K_2^p(x_2^p)$. Since $K_1^p = K_1$, $K_1(x_1^p) = K_2^p(x_2^p)$. Thus $[K_2(x_1) : K_2^p(x_2^p)] = [K_1(x_1) : K_1(x_1^p)] = p$. However $p = [K_2(x_2) : K_2^p(x_2^p)] = [K_2(x_2) : K_2(x_2^p)] \times [K_2(x_2^p) : K_2^p(x_2^p)] = p \times [K_2(x_2^p) : K_2^p(x_2^p)]$. So $[K_2(x_2^p) : K_2^p(x_2^p)] = 1$, i.e. $K_2^p(x_2^p) = K_2(x_2^p)$. This implies that $K_2^p = K_2$. $\square$

PROOF OF THEOREM. Let K_1K_2 be the compositum of K_1 and K_2 in L. Then $L = K_1K_2(x_1, x_2)$ since $K_1K_2(x_1, x_2) \subset L$ and $L \subset K_1K_2(x_1, x_2)$ by definition of compositum. First we show that L is a trascendental extension over K_1K_2. By LEMMA K_2 is also a perfect field. So $L^{p^n} = K_1^{p^n}(x_1^{p^n}) = K_2^{p^n}(x_2^{p^n}) = K_1(x_1^{p^n}) = K_2(x_2^{p^n})$ for every positive integer n. Thus $K_1K_2 \subset L^{p^n}$ for every positive integer n.

$$
\begin{array}{ccccc}
 & & L & = & K_1(x_1) = K_2(x_2) = K_1K_2(x_1, x_2) \\
 & & | & & \\
 & & L^p & = & K_1(x_1^p) = K_2(x_2^p) \\
 & & \vdots & & \\
 & & L^{p^n} & = & K_1(x_1^{p^n}) = K_2(x_2^{p^n}) \\
 & & \vdots & & \\
 & & K_1K_2 & & \\
 & / & & \backslash & \\
K_1 & & & & K_2 \\
 & \backslash & & / & \\
 & & K & &
\end{array}
$$

But $[L : L^{p^n}] = p^n$ for every positive integer n. So L is an infinite dimensional extension over K_1K_2. Since $L = K_1K_2(x_1, x_2)$, L should be a transcendental extension over K_1K_2 . Now we claim that K_1K_2 must be algebraic over K_i, for $i = 1, 2$. Otherwise $1 = tr.d_{K_i} K_i(x_i) = tr.d_{K_i} K_1K_2 +$

$tr.d_{K_1K_2}L \geq 2$, for $i = 1, 2$. Since K_i is algebraically closed in L, for $i = 1, 2$, we conclude that $K_1K_2 \subset K_i$, for $i = 1, 2$ or $K_1K_2 = K_1 = K_2$. □

Reference

[1] A. Beauville, J.L. Colliot-Thélène, J.J. Sansuc and Sir P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationelles,* Ann. of Math. 121 (1986) 283-315.

[2] J. Deveney, *Automorphism groups of ruled function fields and a problem of Zariski,* Proc. Amer. Math. Soc. 90 (1984) 178-180.

[3] J. Deveney, *The cancellation problem for function fields,* Proc. Amer. Math. Soc. 103 (1988) 363-364.

[4]M. Kang, *A note on the birational cancellation problem,* J. of Pure and Appl. Algebra 77 (1992) 141-154.

[5] M. Nagata, *A Theorem on valuation rings and its applications,* Nagoya Math. J. 29 (1967) 85-91.