Class number one problem for pure cubic fields of Rudman-Stender type

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1 Preliminaries

In [1], Louboutin obtained a lower bound for class numbers of pure cubic number fields and applied this bound to classify all pure cubic fields of the form $\mathbb{Q}(\sqrt[3]{m^3 \pm 1})$ whose class numbers are smaller than three. In this paper, using Louboutin's bound we classify all pure cubic fields of Rudman- Stender type of class number one.

Definition 1.1 *Let $d = m^3 + r$, where $d, m, r \in \mathbb{Z}$, with $d, m > 0, |r| > 1$ and d cube-free. If $r|3m^2$ then the field $k = \mathbb{Q}(\sqrt[3]{d})$ is called a pure cubic field of Rudman- Stender type.*

Rudman and Stender proved

Theorem 1.2 *Let $k = \mathbb{Q}(\sqrt[3]{m^3 + r})$ be a pure cubic field of Rudman- Stender type. Let η be the fundamental unit of k and $\epsilon = r/(\omega - m)^3$, where $\omega = \sqrt[3]{m^3 + r}$. Then*

$$\epsilon = \eta,$$

with the following exceptions:

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$$\epsilon = \begin{cases} \eta^2 & \text{if (m,r)=(2,-6), (1,3), (2,2), (3,1), and (5,-25),} \\ \eta^3 & \text{if (m,r)=(2,-4).} \end{cases}$$

Proof: See [3]. □

Theorem 1.3 *Let k be a pure cubic field. Then*

$$h_k R_k \geq \frac{1}{9}\sqrt{\frac{d_k}{log d_k}}, \quad d_k \geq 3 \cdot 10^4,$$

where h_k,d_k and R_k are the class number, the absolute value of discriminant and the regulator of k, respectively.

Proof: See [1]. □

2 Main theorems

In this section, we obtain a lower bound for class numbers of pure cubic fields of Rudman-Stender type. We apply this bound to determine all pure cubic fields of Rudman- Stender type of class number one.

Theorem 2.1 *Let k be a pure cubic field of Rudman-Stender type. Then*

$$h_k \geq \frac{1}{9}\frac{1}{log(12d_k^2)}\sqrt{\frac{d_k}{log d_k}}, \quad d_k \geq 3 \cdot 10^4,$$

where h_k, and d_k are the class number and the absolute value of discriminant of k respectively.

Proof: Set $d = m^3 + r$. Let $k = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field of Rudman-Stender type and $\epsilon = r/(\omega - m)^3$. Define a and b by means of $(a, b) = 1$ and $d = ab^2$. Then $d_k = 3(ab)^2$ or $d_k = 27(ab)^2$ according as $d \equiv \pm 1 \pmod 9$ or not. Thus $d_k \geq 3d$. Since $\epsilon = (\omega^2 + m\omega + m^2)^3/r^2$ and $\sqrt[3]{2}\omega > m$, we easily see that

$$\epsilon \leq 12d_k^2.$$

By Theorem 1.2 we have

$$R_k \leq log\epsilon \leq log(12d_k^2).$$

where R_k is the regulator of k. From Theorem 1.3 we get the desired lower bound for class number of k. □

Theorem 2.2 *There are exactly five pure cubic fields of Rudman-Stender type of class number one, i.e.,* $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(\sqrt[3]{5})$, $\mathbb{Q}(\sqrt[3]{6})$, $\mathbb{Q}(\sqrt[3]{10})$, $\mathbb{Q}(\sqrt[3]{12})$.

Proof: Set $d = m^3 + r$. Let $k = \mathbb{Q}(\sqrt[3]{d})$ be a pure cubic field of Rudman-Stender type. By Theorem 2.1 we have $h_k > 1$ if $d_k \geq 1.05 \cdot 10^6$. Note that if $m \geq 72$ then $d_k \geq 1.05 \cdot 10^6$. If $d \leq 1000$, then we find exactly five d which $h_k = 1$,i.e., $d = 2$ if (m,r)=(2,-6), (1,4), or (2,-4), $d = 5$ if (m,r)=(2,-3), $d = 6$ if (m,r)=(2,-2), $d = 10$ if (m,r)=(2,2) or (5,-25), $d = 12$ if (m,r)=(2,4) or (3,-9) from the table in [2]. Thus to prove the theorem, it is enough to show that if $d > 1000$, $m \leq 71$ and $d_k < 1.05 \cdot 10^6$, then $h_k > 1$. Using MATHEMATICA we know that there are only 26 pairs of (m, r), i.e., (m, r)=(10,25), (10,100), $\cdots$, (30,225), satisfying the above conditions. For each case, computing the actual value of the regulator we have more sharper lower bound than Theorem 2.1 and easily show that its class number is greater than one. For example, we consider the case (m, r)=(10,25). In this case, $d = 1025$, $d_k = 126075$ and $R_k \approx 10.7$. Applying these values to Theorem 1.3 we have $h_k > 1$. The other cases can be treated similarly. This complets the proof of the theorem. □

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