Class number 2 criteria for real quadratic fields of Richaud-Degert type

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In this paper we continue the method of our previous paper [2] and get various class number 2 criteria for real quadratic fields of Richaud-Degert type. In the appendix, as an application of our method, we construct real quadratic fields of class number divisible by n, where n is any positive integer.

1 Preliminaries

In this section we introduce the result which will be necessary in our work without proof. Let k be a real quadratic field and $\zeta_k(s)$ denote the Dedekind zeta function of k. There are two ways of computing special values of $\zeta_k(s)$, due to C.L.Siegel and H.Lang. We first state Siegel's formula.

Theorem 1.1 *Let k be a real quadratic field with discriminant D. Then*

$$\zeta_k(-1) = \frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D \pmod 4}} \sigma_1\left(\frac{D-b^2}{4}\right),$$

where $\sigma_1(r)$ denote the sum of divisors of r.

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Proof: See [6] [12]. □

However there is another method of computing special values of $\zeta_k(s)$ if k is a real quadratic field due to H. Lang.

Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant D and A an ideal class of k. Let ϵ be the fundamental unit of k and $\boldsymbol{a}$ be any integral ideal belonging to A^{-1}. Let r_1, r_2 be an integral basis of $\boldsymbol{a}$ and r_1', r_2' be their conjugates. We put

$$\delta(\boldsymbol{a}) = r_1 r_2' - r_1' r_2.$$

Since $\epsilon r_1, \epsilon r_2$ are also an integral basis of $\boldsymbol{a}$, we can find an integral matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying

$$\epsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \cdot \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

Now we can state Lang's formula.

Theorem 1.2 *By keeping the above notation, we have*

$$\begin{aligned} \zeta_k(-1, A) &= \frac{sgn\,\delta(\boldsymbol{a}) r_2 r_2'}{360 N(\boldsymbol{a}) c^3} \{(a+d)^3 - 6(a+d)N(\epsilon) \\ &\quad -240c^3\,(sgn\,c) S^3(a,c) + 180ac^3\,(sgn\,c) S^2(a,c) \\ &\quad -240c^3\,(sgn\,c) S^3(d,c) + 180dc^3\,(sgn\,c) S^2(d,c)\}, \end{aligned}$$

where $S^i(a,c) = S_4^i(a,c)$ denote the generalized Dedekind sum.

Proof: This is a main theorem of [7]. □

To use Lang's formula, we need to compute a, b, c, d and generalized Dedekind sums.

Lemma 1.3 *Put $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then*

$$a = tr(\frac{r_1 r_2' \epsilon}{\delta(\boldsymbol{a})}), \qquad b = tr(\frac{r_1 r_1' \epsilon'}{\delta(\boldsymbol{a})})$$

$$c = tr(\frac{r_2 r_2' \epsilon}{\delta(\boldsymbol{a})}), \qquad d = tr(\frac{r_1 r_2' \epsilon'}{\delta(\boldsymbol{a})}).$$

Furthermore, $\det M = N(\epsilon)$ *and* $bc \neq 0$.

Proof: See [6]. □

Applying reciprocity law for generalized Dedekind sums (see, for example, [1, 3]), we have the following results.

Lemma 1.4 *Let m be a positive integer. Then we have*

(i) $S^3(\pm 1, m) = \pm \dfrac{-m^4 + 5m^2 - 4}{120m^3}$,

(ii) $S^2(\pm 1, m) = \dfrac{m^4 + 10m^2 - 6}{180m^3}$.

Proof: See [6]. □

Lemma 1.5 *Let m be a positive even integer. Then we have*

(i) $S^3(m+1, 2m) = S^1(m+1, 2m) = \dfrac{-m^4 + 50m^2 - 4}{120(2m)^3}$,

(ii) $S^3(m-1, 2m) = -S^1(m+1, 2m) = \dfrac{m^4 - 50m^2 + 4}{120(2m)^3}$,

(iii) $S^2(m-1, 2m) = S^2(m+1, 2m) = \dfrac{m^4 + 100m^2 - 6}{180(2m)^3}$.

Proof: See [6]. □

Lemma 1.6 *Let m be a positive even integer. Then we have*

(i) $S^3(m+1, 4m) = \dfrac{-m^4 - 180m^3 + 410m^2 - 4}{120(4m)^3}$,

(ii) $S^3(m-1, 4m) = \dfrac{m^4 - 180m^3 - 410m^2 + 4}{120(4m)^3}$,

(iii) $S^2(m-1, 4m) = S^2(m+1, 4m) = \dfrac{m^4 + 820m^2 - 6}{180(4m)^3}$.

Proof: See [6]. □

2 Main theorem

In this section, we compare special values of zeta function and derive our main theorem. We start from a definition.

Definition 2.1 Let $d = n^2 + r$, $d \neq 5$, be a positive square free integer satisfying the conditions

$$r|4n \quad \text{and} \quad -n < r \leq n.$$

In this situation, the real quadratic field $k = \mathbb{Q}(\sqrt{d})$ is called a real quadratic field of Richaud-Degert (R-D) type.

Proposition 2.2 *Let $k = \mathbb{Q}(\sqrt{d})$, $d > 0$, be a real quadratic field of R-D type. Then the fundamental unit ϵ and its norm $N(\epsilon)$ are given as follows :*

$$\epsilon = n + \sqrt{n^2 + r}, \quad N(\epsilon) = -sgn\, r \quad if |r| = 1,$$

$$\epsilon = \frac{n + \sqrt{n^2 + r}}{2}, \quad N(\epsilon) = -sgn\, r \quad if |r| = 4,$$

and

$$\epsilon = \frac{2n^2 + r}{|r|} + \frac{2n}{|r|}\sqrt{n^2 + r}, \quad N(\epsilon) = 1 \quad if |r| \neq 1, 4.$$

Proof: See Degert [5]. □

Proposition 2.3 *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with square-free integer d. Then*

(i) 2 splits in k if $d \equiv 1$ (mod 8) i.e. $(2) = (2, \frac{1+\sqrt{d}}{2})(2, \frac{1-\sqrt{d}}{2})$.

(ii) 2 ramifies in k if $d \equiv 2, 3$ (mod 4) i.e. $(2) = (2, \alpha + \sqrt{d})^2$ where $\alpha = 0$ if $d \equiv 2$ (mod 4)and $\alpha = 1$ if $d \equiv 3$ (mod 4).

(iii) 2 remains prime in k if $d \equiv 5$ (mod 8).

Proof: See [4]. □

Let A be the ideal class of principal ideals and B the ideal class containing $(2, \frac{1\pm\sqrt{d}}{2})$ or $(2, \alpha + \sqrt{d})$ as in Proposition 2.3 i),ii).

Now we compute $\zeta_k(-1, A)$ and $\zeta_k(-1, B)$ and compare these values. Finally we conclude that the ideal B is not principal with some exceptions.

Theorem 2.4 *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of R-D type and let A denote the ideal class of principal ideals of k. Then,*

I. $d = n^2 + r \equiv 2, 3$ *(mod 4)*

(i) $|r| \neq 1, 4$

$$\zeta_k(-1, A) = \frac{4n^3(r^2+1) + 2nr(3r^2+5r+3)}{180r^2}$$

(ii) $|r| = 1$

$$\zeta_k(-1, A) = \frac{4n^3 + 5n \pm 6n}{180}$$

II. $d = n^2 + r \equiv 1$ *(mod 4)*

(i) $|r| \neq 1, 4$

$$\zeta_k(-1, A) = \frac{2n^3(r^2+1) + n(3r^3+50r^2+3r)}{720r^2} \quad \text{if } n \text{ even}$$

$$\zeta_k(-1, A) = \frac{2n^3(r^2+16) + n(3r^3+20r^2+48r)}{720r^2} \quad \text{if } n \text{ odd}$$

(ii) $|r| = 4$ *(hence n odd)*

$$\zeta_k(-1, A) = \frac{n^3 + 5n \pm 6n}{360}$$

(iii) $|r| = 1$ *(hence $r = 1$ and n even)*

$$\zeta_k(-1, A) = \frac{n^3 + 14n}{360}.$$

Proof: This is one of main theorems of [2]. Basic idea of proof is the same as that of Theorem 2.5 below. □

Theorem 2.5 *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of R-D type and let B be the ideal class containing $(2, \frac{1+\sqrt{d}}{2})$ or $(2, \alpha + \sqrt{d})$ as in Proposition 2.3 i),ii). Then,*

I. $d = n^2 + r \equiv 2, 3$ *(mod 4)* *(i)* $|r| \neq 1, 4$

$$\zeta_k(-1, B) = \frac{2n^3(r^2+1) + nr(3r^2+50r+3)}{360r^2}$$

if $d \equiv 2 \pmod 4$ *and* n *odd or if* $d \equiv 3 \pmod 4$ *and* n *even.*

$$\zeta_k(-1, B) = \frac{2n^3(r^2+16) + nr(3r^2+20r+48)}{360r^2}$$

if $d \equiv 2 \pmod 4$ *and* n *even or if* $d \equiv 3 \pmod 4$ *and* n *odd.*

(ii) $|r| = 1$

$$\zeta_k(-1, B) = \frac{2n^3 + 25n \pm 3n}{360}$$

II. $d = n^2 + r \equiv 1 \pmod 8$

(i) $|r| \neq 1, 4$

$$\zeta_k(-1, B) = \frac{2n^3(r^2+1) + n(3r^3+410r^2+3r)}{2880r^2}$$

(ii) $|r| = 1$ *(hence* $r = 1$ *and* n *even)*

$$\zeta_k(-1, B) = \frac{n^3 + 104n}{1440}.$$

Proof: We know that $\{\frac{1\pm\sqrt{d}}{2}, 2\} and \{\alpha + \sqrt{d}, 2\}$ are integral bases for $(\frac{1\pm\sqrt{d}}{2}, 2)$ and $(\alpha + \sqrt{d}, 2)$ in Proposition 2.3 i),ii), respectively. Hence we can take $\boldsymbol{a} = [\frac{1\mp\sqrt{d}}{2}, 2]$ or $[\alpha + \sqrt{d}, 2]$ in Theorem 1.2.

We give detailed computation only for the case I(i) $d \equiv 2 \pmod 4$ and n odd, since the other cases are similar to this case.

Now assume that $d = n^2 + r \equiv 2 \pmod 4$, where n is odd and $|r| \neq 1, 4$. In this case, $D = 4d$ and $r_1 = \sqrt{n^2+r}$, $r_2 = 2$ form an integral basis for $\boldsymbol{a}$. By Proposition 2.2,

$$\epsilon = \frac{2n^2+r}{|r|} + \frac{2n}{|r|}\sqrt{n^2+r}$$

is the fundamental unit of k and $N(\epsilon) = 1$. By Lemma 1.3, we have

$$\begin{aligned}
\epsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} &= \left(\frac{2n^2+r}{|r|} + \frac{2n}{|r|}\sqrt{n^2+r}\right) \begin{bmatrix} \sqrt{n^2+r} \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2n^2+r}{|r|} & \frac{n(n^2+r)}{|r|} \\ \frac{4n}{|r|} & \frac{2n^2+r}{|r|} \end{bmatrix} \begin{bmatrix} \sqrt{n^2+r} \\ 2 \end{bmatrix}.
\end{aligned}$$

Note that

$$\frac{2n^2+r}{|r|} = \frac{n-1}{2}\frac{4n}{|r|} + \frac{2n}{|r|} + \operatorname{sgn} r \equiv \frac{2n}{|r|} + \operatorname{sgn} r \pmod{\frac{4n}{|r|}}.$$

Now put $\eta = \operatorname{sgn} r$. Then, by Lemma 1.5,

$$\begin{aligned} 240c^3 \operatorname{sgn} c S^3(a,c) &= 240c^3 S^3(\frac{2n}{|r|}+\eta, \frac{4n}{|r|}) = -\frac{8\eta}{r^4}(4n^4 - 50n^2r^2 + r^4), \\ 180ac^3 \operatorname{sgn} c S^2(a,c) &= 180ac^3 S^2(\frac{2n}{|r|}+\eta, \frac{4n}{|r|}) = \frac{2\eta}{r^5}(2n^2+r)(8n^4+200n^2r^2-3r^4), \end{aligned}$$

and

$$(a+d)^3 - 6(a+d)N(\epsilon) = 8\eta\frac{(2n^2+r)^3}{r^3} - 12\eta\frac{2n^2+r}{r}.$$

By substitution these results to Theorem 1.2, we get

$$\zeta_k(-1,B) = \frac{2n^3(r^2+1)+nr(3r^2+50r+3)}{360r^2}.$$

□

Theorem 2.6 *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of R-D type and h_d be the class number of k. Then,*

I. $d = n^2 + r \equiv 2, 3$ *(mod 4)*

- *(i)* $|r| \neq 1, 4$
  $h_d > 1$ *except* $r = \pm 2$
- *(ii)* $|r| = 1$
  $h_d > 1$ *except* $d = 2, 3$

II. $d = n^2 + r \equiv 1$ *(mod 8)*

- *(i)* $|r| \neq 1, 4$
  $h_d > 1$ *except* $d = 33$
- *(ii)* $|r| = 1$ *(hence* $r = 1$ *and* n *even)*
  $h_d > 1$ *except* $d = 17$.

Proof: Basic idea is as follows. We compare $\zeta_k(-1, A)$ in Theorem 2.4 and $\zeta_k(-1, B)$ in Theorem 2.5. We have $h_d > 1$ if $\zeta_k(-1, A) \neq \zeta_k(-1, B)$. We give detailed computation only for the case II (ii), since the other cases are similar to this case.

Now assume that $d = n^2 + 1 \equiv 1 \pmod 8$. Then by Theorem 2.4

$$\zeta_k(-1, A) = \frac{n^3 + 14n}{360},$$

and by Theorem 2.5

$$\zeta_k(-1, B) = \frac{n^3 + 104n}{1440}.$$

If $\frac{n^3+14n}{360} = \frac{n^3+104n}{1440}$ then $3n(n^2 - 16) = 0$. Thus $d = 17$. Hence $h_d > 1$ except $d = 17$. □

Combining Theorem 1.1, Theorem 2.4,Theorem 2.5 and Theorem 2.6 we obtain

Theorem 2.7 *Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of R-D type and D be the discriminant of k. Then, for each case, the following equality is equivalent to the condition that $h_d = 2$.*

I. $d = n^2 + r \equiv 2, 3$ *(mod 4)*

(i) $|r| \neq 1, 4$ *except* $r = \pm 2$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1(\frac{D - b^2}{4}) = \frac{2n^3(r^2 + 1) + n(3r^3 + 14r^2 + 3r)}{72r^2}$$

if $d \equiv 2$ *(mod 4) and* n *odd or if* $n \equiv 3$ *(mod 4) and* n *even,*

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1(\frac{D - b^2}{4}) = \frac{2n^3(r^2 + 4) + n(3r^3 + 8r^2 + 12r)}{72r^2}$$

if $d \equiv 2$ *(mod 4) and* n *even or if* $n \equiv 3$ *(mod 4) and* n *odd.*

(ii) $|r| = 1$ *except* $d = 2, 3$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1(\frac{D - b^2}{4}) = \frac{10n^3 + 35n \pm 15n}{360}$$

II. $d = n^2 + r \equiv 1$ *(mod 8)*

(i) $|r| \neq 1, 4$ *except* $d = 33$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1(\frac{D-b^2}{4}) = \frac{2n^3(r^2+1) + n(3r^3 + 122r^2 + 3r)}{576r^2} \quad \text{if } n \text{ even}$$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1(\frac{D-b^2}{4}) = \frac{2n^3(r^2+13) + n(3r^3 + 98r^2 + 39r)}{576r^2} \quad \text{if } n \text{ odd}$$

(ii) $|r| = 1$ *(hence* $r = 1$ *and* n *even) except* $d = 17$

$$\frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D(4)}} \sigma_1(\frac{D-b^2}{4}) = \frac{n^3 + 32n}{288}$$

Proof: Let A and B be the ideal class in Theorem 2.4 and Theorem 2.5 respectively. By Theorem 2.6, B is not equal to A in each case. Hence

$$\zeta_k(-1) = \zeta_k(-1, A) + \zeta_k(-1, B)$$

if and only if $h_d = 2$. By Theorem 1.1, 2.4, 2.5 and easy computation we have the result. □

3 Class number 2 criteria for real quadratic fields of Richaud-Degert type

In this section we shall apply Theorem 2.7 to obtain class number 2 criteria for some real quadratic fields of R-D type. Recall that $k = \mathbb{Q}(\sqrt{d})$ is a real quadratic field of R-D type if $d(\neq 5)$ is a square free integer of the form $n^2 + r$ such that $r|4n$, $-n < r \leq n$. We devide the situation into two cases.

Case I. $d = n^2 + r \equiv 2, 3 \pmod 4$

Corollary 3.1 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 - 1$, $n > 1$. *Then*

$$h_d = 2 \quad \Leftrightarrow \quad 2n^2 - 2t^2 - 2t - 1\,(0 \le t \le n) \quad \textit{are primes.}$$

Corollary 3.2 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 + 1$, $n > 1$. *Then*

$$h_d = 2 \quad \Leftrightarrow \quad 2n^2 + 2n + 1 - 2t^2\,(0 \le t \le n) \quad \textit{are primes.}$$

Corollary 3.3 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 + r$, $r \equiv 1(4)$, $r|2n+1$, $r > 1$. *Write* $2n+1 = rm$. *Then*

$$\begin{aligned} h_d = 2 \quad \Leftrightarrow \quad & r, rm \pm \frac{r-1}{2}, \frac{rm^2+1}{2}, \\ & r^2m^2 + r - t^2\,(1 \le t \le rm, 2\not|t, r\not|t, \textit{and } t \ne \frac{r+1}{2}), \\ & \frac{r^2m^2 + r - 4s^2}{2}\,(1 \le s \le \frac{rm-1}{2}, r\not|s), \\ & rm^2 + 1 - ru^2\,(1 \le u \le m-1, 2\not|u), \\ & \frac{rm^2 + 1 - 4rv^2}{2}\,(1 \le v \le \frac{m-1}{2}) \quad \textit{are primes.} \end{aligned}$$

Corollary 3.4 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 - r$, $r \equiv 1(4)$, $r|2n+1$, $r > 1$. *Write* $2n+1 = rm$. *Then*

$$\begin{aligned} h_d = 2 \quad \Leftrightarrow \quad & r, rm \pm \frac{r+1}{2}, \frac{rm^2-1}{2}, \\ & r^2m^2 - r - t^2\,(1 \le t \le rm-1, 2\not|t, r\not|t, \textit{and } t \ne \frac{r-1}{2}), \\ & \frac{r^2m^2 - r - 4s^2}{2}\,(1 \le s \le \frac{rm-1}{2}, r\not|s), \\ & rm^2 - 1 - ru^2\,(1 \le u \le m-1, 2\not|u), \\ & \frac{rm^2 + 1 - 4rv^2}{2}\,(1 \le v \le \frac{m-1}{2}) \quad \textit{are primes} \end{aligned}$$

Corollary 3.5 *Let* $k = \mathbb{Q}(\sqrt{d})$, $d = (2n+1)^2 + 2r$, $r \equiv 1, 3(4)$, $r|2n+1$, $r > 1$. *Write* $2n+1 = rm$. *Then*

$$h_d = 2 \quad \Leftrightarrow \quad r, rm^2 + 2,$$

$$r^2m^2-2r-t^2\,(1\le t\le rm, 2\not|t+1, r\not|t),$$
$$\frac{r^2m^2+2r-(2s-1)^2}{2}\,(1\le s\le \frac{rm+1}{2}, r\not|2s-1),$$
$$rm^2-2-ru^2\,(1\le u\le m-1, 2\not|u+1),$$
$$\frac{rm^2+2-r(2v-1)^2}{2}\,(1\le v\le \frac{m-1}{2})\quad \textit{are primes}$$

Corollary 3.6 *Let* $k=\mathbb{Q}(\sqrt{d})$, $d=(2n+1)^2-2r$, $r\equiv 1,3(4)$, $r|2n+1$, $r>1$. *Write* $2n+1=rm$. *Then*

$$h_d=2 \iff r, rm^2-2,$$
$$r^2m^2-2r-t^2\,(1\le t\le rm-1, 2\not|t+1, r\not|t),$$
$$\frac{r^2m^2-2r-(2s-1)^2}{2}\,(1\le s\le \frac{rm-1}{2}, r\not|2s-1),$$
$$rm^2-2-ru^2\,(1\le u\le m-1, 2\not|u+1),$$
$$\frac{rm^2-2-r(2v-1)^2}{2}\,(1\le v\le \frac{m-1}{2})\quad \textit{are primes}$$

Corollary 3.7 *Let* $k=\mathbb{Q}(\sqrt{d})$, $d=4n^2+r$, $r\equiv 3(4)$, $r|n$, $r>1$. *Write* $n=rm$. *Then*

$$h_d=2 \iff r, 2rm\pm\frac{r-1}{2}, 4rm+1,$$
$$4r^2m^2+r-t^2\,(1\le t\le 2rm, 2\not|t+1, r\not|t, \textit{ and } t\ne\frac{r+1}{2}),$$
$$\frac{4r^2m^2+r-(2s-1)^2}{2}\,(1\le s\le rm, r\not|2s-1),$$
$$4rm^2+1-ru^2\,(1\le u\le 2m-1, 2\not|u+1),$$
$$\frac{4rm^2+1-r(2v-1)^2}{2}\,(1\le v\le m)\quad \textit{are primes}$$

Corollary 3.8 *Let* $k=\mathbb{Q}(\sqrt{d})$, $d=4n^2-r$, $r\equiv 1(4)$, $r|n$, $r>1$. *Write* $n=rm$. *Then*

$$h_d=2 \iff r, 2rm\pm\frac{r+1}{2}, 4rm-1,$$
$$4r^2m^2-r-t^2\,(1\le t\le 2rm-1, 2\not|t+1, r\not|t, \textit{ and } t\ne\frac{r-1}{2}),$$

$$\frac{4r^2m^2 - r - (2s-1)^2}{2} \, (1 \le s \le rm, r \not| 2s-1),$$
$$4rm^2 - 1 - ru^2 \, (1 \le u \le 2m-1, 2 \not| u+1),$$
$$\frac{4rm^2 - 1 - r(2v-1)^2}{2} \, (1 \le v \le m) \quad \textit{are primes}$$

Corollary 3.9 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + 2r$, $r \equiv 1, 3(4)$, $r|n$, $r > 1$. Write $n = rm$. Then*

$$\begin{aligned} h_d = 2 \quad \Leftrightarrow \quad & r, 2rm^2 + 1, \\ & 4r^2m^2 + 2r - t^2 \, (1 \le t \le 2rm, 2 \not| t, r \not| t), \\ & 2r^2m^2 + r - 2s^2 \, (1 \le s \le rm, r \ne s), \\ & 4rm^2 + 2 - ru^2 \, (1 \le u \le 2m, 2 \not| u), \\ & 2rm^2 + 1 - 2rv^2 \, (1 \le v \le m) \quad \textit{are primes} \end{aligned}$$

Corollary 3.10 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 - 2r$, $r \equiv 1, 3(4)$, $r|n$, $r > 1$. Write $n = rm$. Then*

$$\begin{aligned} h_d = 2 \quad \Leftrightarrow \quad & r, 2rm^2 - 1, \\ & 4r^2m^2 - 2r - t^2 \, (1 \le t \le 2rm-1, 2 \not| t, r \not| t), \\ & 2r^2m^2 - r - 2s^2 \, (1 \le s \le rm-1, r \ne s), \\ & 4rm^2 - 2 - ru^2 \, (1 \le u \le 2m-1, 2 \not| u), \\ & 2rm^2 - 1 - 2rv^2 \, (1 \le v \le m-1) \quad \textit{are primes} \end{aligned}$$

Case II. $d = n^2 + r \equiv 1 \pmod 8$

Corollary 3.11 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = 4n^2 + 1$. Then*

$$h_d = 2 \Leftrightarrow d = 65.$$

Corollary 3.12 *Let $k = \mathbb{Q}(\sqrt{d})$, $d = n^2 + r$, $r \ne 1$. Then*

$$h_d = 2 \Leftrightarrow d = 105.$$

We give the proof of Corollary 3.3, the other cases can be done similarly.
Proof of Corollary 3.3: We have $D = 4d$. By Siegel's computation,

$$\begin{aligned}
\zeta_k(-1) &= \frac{1}{60} \sum_{\substack{|b|<\sqrt{D} \\ b^2 \equiv D (\mathrm{mod}\, 4)}} \frac{1}{60} \sigma_1(\frac{D-b^2}{4}) \\
&= \frac{1}{60} \left\{ 2 \sum_{t=1}^{rm} \sigma_1(r^2m^2 + r - t^2) + \sigma_1(r^2m^2 + r) \right\}.
\end{aligned}$$

Since $r^2m^2+r-t^2 = r(rm^2+1)-t^2$ and r,m are odd integers, $r^2m^2+r-t^2$ has the following trivial divisors in each case:

$$\begin{aligned}
t &= 2s, 1 \le s \le \frac{rm-1}{2} \text{ and } r \not| s \;; 1, r^2m^2 + r - t^2, 2, \frac{r^2m^2 + r - 4s^2}{2} \\
t &= ru, 1 \le u \le m-1 \text{ and } 2 \not| u \;; 1, r^2m^2 + r - t^2, r, rm^2 + 1 - ru^2 \\
t &= 2rv, 1 \le v \le \frac{m-1}{2} \;; 1, r^2m^2 + r - t^2, 2, \frac{r^2m^2 + r - 4r^2v^2}{2}, \\
&\quad\; r, rm^2 + 1 - 4rv^2, 2r, \frac{rm^2 + 1 - 4rv^2}{2} \\
t &= \frac{r+1}{2} \;; 1, r^2m^2 + r - t^2, rm - \frac{r-1}{2}, rm + \frac{r-1}{2}.
\end{aligned}$$

Similary $r^2m^2 + r = r(rm^2 + 1)$ has the following trivial divisors;

$$1, r^2m^2 + r, 2, \tfrac{r^2m^2+r}{2}, r, rm^2 + 1, 2r, \tfrac{rm^2+1}{2}.$$

Hence we have

$$\begin{aligned}
\zeta_k(-1) \;\ge\; & \frac{1}{30} \sum_{t=1}^{rm} (1 + r^2m^2 + r - t^2) \\
& + \frac{1}{30} \sum_{s=1}^{\frac{rm-1}{2}} (2 + \frac{r^2m^2 + r - 4s^2}{2}) \\
& + \frac{1}{30} \sum_{u=1}^{m-1} (r + rm^2 + 1 - ru^2)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{30}\sum_{v=1}^{\frac{m-1}{2}}(2r+\frac{rm^2+1-4rv^2}{2}) \\
& +\frac{1}{30}(rm-\frac{r-1}{2}+rm+\frac{r-1}{2}) \\
& +\frac{1}{60}(1+r+rm^2+1+2+\frac{r^2m^2+r}{2}+2r+\frac{rm^2+1}{2}+r^2m^2+r) \\
= & \frac{10r^3m^3+10rm^3+15r^2m+70rm+15m}{360} \\
= & \zeta_k(-1,A)+\zeta_k(-1,B).
\end{aligned}
$$

Note that equality holds if and only if $r, rm\pm\frac{r-1}{2}, \frac{rm^2+1}{2}, r^2m^2+r-t^2$ $(1\leq t\leq rm, 2\not| t, r\not| t, \text{and } t\neq\frac{r+1}{2})$, $\frac{r^2m^2+r-4s^2}{2}$ $(1\leq s\leq\frac{rm-1}{2}, r\not| s)$, rm^2+1-ru^2 $(1\leq u\leq m-1, 2\not| u)$, $\frac{rm^2+1-4rv^2}{2}$ $(1\leq v\leq\frac{m-1}{2})$ are primes.

□

Remark 1. In [9], R. A. Mollin obtained results similar to the above corollaries in different ways. He used theory of continued fractions and algebraic arguements to obtain his results.

Remark 2. There is an interesting result concerning the number of real quadratic fields of Richaud-Degert type of class number two. Let d be a square-free rational integer of the form $d=n^2+4$ or n^2+1 where n is a natural number. In [8], M. G. Leu showed that there are exactly 16 values of d, namely

$$d=10,26,65,85,122,362,365,485,533,629,965,1157,1685,1853,2117,2813,$$

such that $k=\mathbb{Q}(\sqrt{d})$ has class number two with one more possible exception, and under the assumption of the generalized Riemman Hypothesis this is true without any exception.

4 Appendix. Construction of real quadratic fields of class number divisible by n

In this appendix, as an application of our method, we construct real quadratic fields whose class number is divisible by n, where n is any positive integer.

Let $d = (2p^n)^2 + 1$ be a square free integer where p and n are any positive rational integer, and $k = \mathbb{Q}(\sqrt{d})$ a real quadratic field. Then we have,

Theorem 4.1 *Let $k = \mathbb{Q}(\sqrt{(2p^n)^2 + 1})$ be a real quadratic field and C the ideal class of principal ideals. Then*

$$\zeta_k(-1, C) = \frac{2p^{3n} + 7p^n}{90}.$$

Proof: We can take $\boldsymbol{a} = \mathcal{O}_k = [r_1, r_2]$ where $r_1 = \frac{1+\sqrt{(2p^n)^2+1}}{2}$ and $r_2 = 1$ in Theorem 1.2. By Proposition 2.2,

$$\epsilon = 2p^n + \sqrt{(2p^n)^2 + 1}$$

is the fundamental unit of k and $N(\epsilon) = -1$. By Lemma 1.3, we have

$$\begin{aligned} \epsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} &= (2p^n + \sqrt{(2p^n)^2 + 1}) \begin{bmatrix} \frac{1+\sqrt{(2p^n)^2+1}}{2} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2p^n + 1 & 2p^{2n} \\ 2 & 2p^n - 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{(2p^n)^2+1}}{2} \\ 1 \end{bmatrix}. \end{aligned}$$

By lemma 1.4,

$$\begin{aligned} 240c^3 \operatorname{sgn} c S^3(a, c) &= 240c^3 S^3(1, c) = 0, \\ 240c^3 \operatorname{sgn} c S^3(d, c) &= 240c^3 S^3(-1, c) = 0, \\ 180ac^3 \operatorname{sgn} c S^2(a, c) &= 180ac^3 S^2(1, c) = 50(2p^n + 1), \\ 180dc^3 \operatorname{sgn} c S^2(d, c) &= 180ac^3 S^2(-1, c) = 50(2p^n - 1), \end{aligned}$$

and

$$(a + d)^3 - 6(a + d)N(\epsilon) = (4p^n)^3 + 6(4p^n).$$

By substitution these results to Theorem 1.2, we get

$$\zeta_k(-1, C) = \frac{2p^{3n} + 7p^n}{90}.$$

□

We know that the rational integer p factors in k such that

$$(p) = (\frac{1+\sqrt{(2p^n)^2+1}}{2}, p)(\frac{1-\sqrt{(2p^n)^2+1}}{2}, p).$$

And we easily see that for rational integer $1 \leq r \leq n$,

$$(\frac{1 \pm \sqrt{(2p^n)^2+1}}{2}, p)^r = (\frac{1 \pm \sqrt{(2p^n)^2+1}}{2}, p^r).$$

In fact, $\{\frac{1\pm\sqrt{(2p^n)^2+1}}{2}, p^r\}$ is an integral basis for $(\frac{1\pm\sqrt{(2p^n)^2+1}}{2}, p^r)$(See [4]).

Lemma 4.2 *The integral ideal* $(\frac{1\pm\sqrt{(2p^n)^2+1}}{2}, p^n)$ *is principal.*

Proof: We will prove that

$$(\frac{1 \pm \sqrt{(2p^n)^2+1}}{2}, p^n) = (\frac{1 \pm \sqrt{(2p^n)^2+1}}{2} + p^n).$$

To do this, it is enough to show that

$$p^n \in (\frac{1 \pm \sqrt{(2p^n)^2+1}}{2} + p^n).$$

But this is clear since

$$(\frac{1 \pm \sqrt{(2p^n)^2+1}}{2} + p^n)(\frac{1 \mp \sqrt{(2p^n)^2+1}}{2} + p^n) = p^n.$$

□

Theorem 4.3 *Let A^r be the ideal class of* $(\frac{1+\sqrt{(2p^n)^2+1}}{2}, p^r)$ *,for rational integer $1 \leq r \leq n$. Then*

$$\zeta_k(-1, A^r) = \frac{2p^{3n-2r} + 2p^{2r+n} + 5p^n}{90}.$$

In particular, if $r = n$, then $\zeta_k(-1, A^n) = \zeta_k(-1, C)$.

Proof: We can take $\boldsymbol{a}^r = [r_1, r_2]$ where $r_1 = \frac{1-\sqrt{(2p^n)^2+1}}{2}$ and $r_2 = p^r$. Then the result follows from the same way as in Theorem 4.1. $\square$

From Theorem 4.3 we have the following corollary.

Corollary 4.4 *The class number of the real quadratic* $\mathbb{Q}(\sqrt{(2p^n)^2+1})$ *is divisible by* n*.*

Proof: By Lemma 4.2, we only show that if the rational positive integer $r \neq n$, then

$$\zeta_k(-1, A^r) \neq \zeta_k(-1, C).$$

But it is easy. So we have the result. $\square$

Remark. The result of Corollary 4.4 is classical and well-known. For example, Y. Yamamoto [10] and P. J. Weinbger [11] obtained the same result in different manners.

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