A note on basic Iwasawa λ-invariants of imaginary quadratic fields and the congruence of modular forms

by
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1 Introduction and statement of results

For a number field k and a prime number l, we denote by $h(k)$ the class number of k and by $\lambda_l(k)$ the Iwasawa λ- invariant of the cyclotomic $\mathbb{Z}_l$-extension of k, where $\mathbb{Z}_l$ is the ring of l-adic integers.

Let l be an odd prime number. Using the Kronecker class number relation for quadratic forms, Hartung [3] proved that there exist infinitely many imaginary quadratic fields k whose class numbers are not divisible by l. For the case $l = 2$, this is an immediate consquence of Gauss genus theory. For the case $l = 3$, Davenport and Heilbronn [2] proved the stronger result that a positive proportion of imaginary quadratic fields has class number coprime to 3. Recently, using Sturm's work [11] on the congruence of modular forms, Kohnen and Ono [7] obtained a lower bound for the number of D_k, $-X < D_k < 0$, where D_k is the discriminant of an imaginary quadratic field k such that $h(k) \not\equiv 0 \pmod{l}$ and X is a sufficiently large positive real number. Using the same method, subject to a mild condition on l, Ono [9] obtained similar results for real quadratic fields.

On the other hand, using the idea of Hartung and Eichler trace formula combined with the l-adic Galois representation attached to the Jacobian variety $J = J_0(l)$ of the modular curve $X = X_0(l)$, Horie [4] proved that

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there exist infinitely many imaginary quadratic fields k such that l does not split in k and l does not divide $h(k)$. Later Horie and Onishi [5] obtained more refined results. By a theorem of Iwasawa [6], these results imply that there exist infinitely many imaginary quadratic fields k with $\lambda_l(k) = 0$. For the case $l = 2$, this is also an immediate consquence of Gauss genus theory. For the case $l = 3$, by refining Davenport and Heilbronn's result [2], Nakagawa and Horie [8] gave a positive lower bound on the density of imaginary quadratic fields k and real quadratic fields k with $\lambda_l(k) = 0$. Recently, Taya [12] improved the result of Nakagawa and Horie on real quadratic fields for the case $l = 3$ and Ono [9] obtained a lower bound on the number of real quadratic fields k with $\lambda_l(k) = 0$ for the case $3 < l < 5000$.

In this note, refining Kohnen and Ono's method [7, 9], we obtain a lower bound for the number of D_k, $-X < D_k < 0$, where D_k is the discriminant of an imaginary quadratic field k such that $h(k) \not\equiv 0 \pmod{l}$ and l does not split in k and X is a sufficiently large positive real number. Similary, by a theorem of Iwasawa [6], this is also a lower bound for the number of imaginary quadratic fields k with $\lambda_l(k) = 0$.

Theorem 1.1 *Let $l > 3$ be an odd prime and p be an odd prime such that $p \equiv 1 \pmod{8}$, $p \equiv -2 \pmod{l}$ and $(\frac{t}{p}) = 1$ for all prime t, $2 < t < l$. Then there exist an integer d_{lp}, $1 \le d_{lp} \le \frac{3}{4}(l+1)(p+1)$ such that $d_{lp}lp \neq nlp^2$ for any n, $1 \le n \le l$ and let $k = \mathbb{Q}(\sqrt{-d_{lp}lp})$ be the imaginary quadratic field, then $h(k) \not\equiv 0 \pmod{l}$ and l does not split in k.*

Corollary 1.2 *Let $l > 3$ be an odd prime and $\epsilon > 0$. Let D_k be the discriminant of imaginary quadratic field k with $\lambda_l(k) = 0$. Then for all sufficiently large $X > 0$,*

$$\sharp\{D_k| - X < D_k < 0\} >>_l \frac{\sqrt{X}}{logX}.$$

2 Proof of results

Proof of Theorem 1.1. Let l and p be odd primes. Let $\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2}$ be the classical theta function, where $q = e^{2\pi iz}$, $z \in \mathbb{C}$. Define $r(n)$ by

$$\sum_{n=0}^{\infty} r(n)q^n := \theta^3(z) = 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + \cdots.$$

It is well known that

$$r(n) = \begin{cases} 12H(4n) & \text{if } n \equiv 1, 2 \pmod 4 \\ 24H(n) & \text{if } n \equiv 3 \pmod 4 \\ r(n/4) & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{if } n \equiv 7 \pmod 4, \end{cases} \tag{1}$$

where $H(N)$ is the Hurwitz-Kronecker class number for a natural number $N \equiv 0, 3 \pmod 4$. If $-N = D_k f^2$ where D_k is the discriminant of an imaginary quadratic field k, then $H(N)$ is related to class number of k by the formula [1]:

$$H(N) = \frac{h(k)}{\omega(k)} \sum_{d|f} \mu(d)(\frac{D_k}{d})\sigma_1(f/d), \tag{2}$$

where $\omega(k)$ is half the number of units in $k = \mathbb{Q}(\sqrt{D_k})$, $\sigma_1(n)$ denotes the sum of the positive divisors of n and $\mu(d)$ is Möbius function defined by $\mu(d) = (-1)^k$ if d is equal to a product of k distinct primes (including $k = 0$) and $\mu(d) = 0$ otherwise.

Define $(U_{lp}\theta^3)(z)$, $(V_{lp}\theta^3)(z)$ and $(U_l V_p \theta^3)(z)$ in the usual way, i.e,

$$\begin{aligned} (U_{lp}\theta^3)(z) &:= \textstyle\sum_{n\geq 0} r(lpn)q^n = 1 + \sum_{n\geq 1} r(lpn)q^n, \\ (V_{lp}\theta^3)(z) &:= \textstyle\sum_{n\geq 0} r(n)q^{lpn} = 1 + \sum_{n\geq 1} r(n)q^{lpn} \\ (U_l V_p\theta^3)(z) &:= \textstyle\sum_{n\geq 0} r(nl)q^{np} = 1 + \sum_{n\geq 1} r(nl)q^{np}. \end{aligned} \tag{3}$$

Then $U_{lp}\theta^3$, $V_{lp}\theta^3$, and $U_l V_p \theta^3$ are modular forms of weight $\frac{3}{2}$ on $\Gamma_0(4lp)$ with character $(\frac{4lp}{\cdot})$ [10].

To prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 *Let l and p be odd primes. If $(\frac{-np}{l}) = 1$ for some n, $1 \leq n \leq p$, then $r(npl^2) \equiv 0 \pmod l$.*

Proof: Suppose that $(\frac{-np}{l}) = 1$ for some n, $1 \leq n \leq p$. Then from (2), we have

$$r(npl^2) = r(np)(l + 1 - (\frac{-np}{l})) = r(np)l \equiv 0 (\text{mod } l)$$

and prove the lemma. □

Lemma 2.2 *Let l be an odd prime such that $l \equiv 5$ or 7 (mod 8). Let p be an odd prime such that $p \equiv 1$ (mod 8), $p \equiv -2$ (mod l) and $(\frac{t}{p}) = 1$ for all prime t, $2 < t < l$. Then $r(nlp^2) \equiv 0$ (mod l) for all n, $1 \leq n < l$.*

Proof: From the assumption on l and p in the above, we easily see that $(\frac{-nl}{p}) = -1$ for all n, $1 \leq n < l$. Thus from (2), we have

$$r(nlp^2) = r(nl)(p + 1 - (\frac{-nl}{p})) \equiv 0 \pmod{l},$$

for all n, $1 \leq n < l$ and prove the lemma. □

Similary we have

Lemma 2.3 *Let l be an odd prime such that $l \equiv 1$ or 3 (mod 8). Let p be an odd prime such that $p \equiv 1$ (mod 8), $p \equiv -2$ (mod l) and $(\frac{t}{p}) = 1$ for all prime t, $2 < t < l$. Then $r(nlp^2) \equiv -2r(nl)$ (mod l) for all n, $1 \leq n < l$.*

If $g = \sum_{n=0}^{\infty} a(n)q^n$ has integer coefficients then define $\mathrm{ord}_l(g)$ by

$$\mathrm{ord}_l(g) := \min\{n | a(n) \not\equiv 0 (\mathrm{mod}\ l)\}.$$

Let $M_k(\Gamma_0(N), \chi)$ be the space of modular forms of weight k on $\Gamma_0(N)$ with character χ. Sturm [11] proved that if $g \in M_k(\Gamma_0(N), \chi)$ has integer coefficients and

$$\mathrm{ord}_l(g) > \frac{k}{12}[\Gamma_0(1) : \Gamma_0(N)],$$

then $g \equiv 0$ (mod l). He proved this for integral k and trivial χ but Kohnen and Ono[7] noted that this is also true for the general case.

Now we can prove Theorem 1.1. From now on we assume that $l > 3$ be an odd prime and p be an odd prime such that $p \equiv 1$ (mod 8), $p \equiv -2$ (mod l) and $(\frac{t}{p}) = 1$ for all prime t, $2 < t < l$.

Case I $l \equiv 5$ or 7 (mod 8). First we claim that $(U_{lp}\theta^3)(z) \not\equiv (V_{lp}\theta^3)(z)$ (mod l). To see this, by (3), it is enough to show that the coefficients of q^{lp} in $(U_{lp}\theta^3)(z)$ and $(V_{lp}\theta^3)(z)$ are not congruent modulo l, i.e., $r(l^2p^2) \not\equiv 6$ (mod l). From (1) and (2), we see that

$$r(l^2p^2) = 12H(4l^2p^2) = 6(l + 1 - (\frac{-4}{l}))(p + 1 - (\frac{-4}{p})).$$

Thus from the choice of l and p, we have

$$r(l^2p^2) \equiv \begin{cases} 0 & \pmod{l} \quad \text{if } l \equiv 5 \pmod{8} \\ -24 & \pmod{l} \quad \text{if } l \equiv 7 \pmod{8} \end{cases}$$

and prove the claim .

Now we note that the relevant Sturm bound for the modular forms in $M_{\frac{3}{2}}(\Gamma_0(4lp), (\frac{4lp}{\cdot}))$ is $\frac{3}{4}(l+1)(p+1)$. Then by applying Sturm's theorem [11] to the modular form $g(z) = (U_{lp}\theta^3)(z) - (V_{lp}\theta^3)(z)$ in $M_{\frac{3}{2}}(\Gamma_0(4lp), (\frac{4lp}{\cdot}))$, we have that there exist an integer d_{lp}, $1 \leq d_{lp} \leq \frac{3}{4}(l+1)(p+1) < lp$ (when $l, p \geq 7$ or $l = 5$, $p > 9$) such that $r(d_{lp}lp) \not\equiv 0 \pmod{l}$. From Lemma 2.2, we know that for such d_{lp}, $d_{lp}lp \neq nlp^2$ for any n, $1 \leq n < l$. Furthermore from Lemma 2.1, we see that if $k = \mathbb{Q}(\sqrt{-d_{lp}lp})$ be the imaginary quadratic field and D_k be the discriminant of k then $(\frac{D_k}{l}) = 0$ or $(\frac{D_k}{l}) = -1$, i.e., l does not split in k. Thus we prove Theorem 1.1 for the case $l \equiv 5$ or $7 \pmod{8}$.

Case II $l \equiv 1$ or $3 \pmod{8}$. Let $f(z) = (U_{lp}\theta^3)(z) + 2(U_lV_p\theta^3)(z)$ and $g(z) = 3(V_{lp}\theta^3)(z)$ be modular forms in $M_{\frac{3}{2}}(\Gamma_0(4lp), (\frac{4lp}{\cdot}))$. Then we can also show that $f(z) \not\equiv g(z) \pmod{l}$. By the similar way in Case I, from Sturm's theorem, Lemma 2.1, and Lemma 2.3, we can prove Theorem 1.1 for the case $l \equiv 1$ or $3 \pmod{8}$.

Proof of Corollary 1.2. Let $l > 3$ be an odd prime. First we note that there exist a natural number r, $1 \leq r \leq 8l\prod t$, where the product runs over all prime t, $2 < t < l$, such that if a natural number $s \equiv r \pmod{8l\prod t}$, then $s \equiv 1 \pmod{8}$, $s \equiv -2 \pmod{l}$ and $s \equiv 1 \pmod{t}$ for all prime t, $2 < t < l$. Then we easily see that if a prime p be in the arithmetic progression such that $p \equiv r \pmod{8l\prod t}$ then p satisfies the conditions in Theorem 1.1.

Let $p_1 < p_2 < \cdots$ be the primes in such arithmetic progression in increasing order. Then in the notation from the proof of Theorem 1.1, if $i < j < k$ and D_i, D_j, D_k are the discriminants of imaginary quadratic fields associated to $d_{lp_i}lp_i$, $d_{lp_j}lp_j$, $d_{lp_k}lp_k$ by (1) and (2), then at least two of them are different by Theorem 1.1. Moreover, it is obvious that $D_i > -3lp_i(l+1)(p_i+1) > -4l^2p_i^2$ (when $l, p_i \geq 7$ or $l = 5$, $p_i > 9$). Thus from Dirichilet's theorem on primes in arithmetic progression, we have the corollary.

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