On the finiteness of certain Rabinowitsch polynomials II

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Let m be a positive integer and $f_m(x)$ be a polynomial of the form $f_m(x) = x^2 + x - m$. We call a polynomial $f_m(x)$ a Rabinowitsch polynomial if for $t = [\sqrt{m}]$ and consecutive integers $x = x_0, x_0 + 1, \cdots, x_0 + t - 1$, $|f(x)|$ is either 1 or prime. In [1], we showed that there are only finitely many Rabinowitsch polynomials $f_m(x)$ such that $1 + 4m$ is square free. In this note we shall remove the condition that $1 + 4m$ is square free.

1 Introduction

Let m be a positive integer and $f_m(x)$ be a polynomial of the form $f_m(x) = x^2 + x - m$. We call a polynomial $f_m(x)$ a Rabinowitsch polynomial if for

$t = [\sqrt{m}]$ and consecutive integers $x = x_0,\ x_0+1,\ \cdots,\ x_0+t-1$, $|f(x)|$ is either 1 or prime. In [1], we proved that every Rabonowitsch polynomial of the form $f_m(x) = x^2 + x - m$ is one of the following types.
(i) $x^2 + x - 2$,
(ii) $x^2 + x - t^2$, where t is 1 or a prime,
(iii) $x^2 + x - (t^2 + t + n)$, where $-t < n \le t$ and where $|n|$ is 1 or $|n| = \frac{2t+1}{3}$ is an odd prime.

Using this and the Siegel-Brauer theorem, we showed that there are only finitely many Rabinowitch polynomials $f_m(x) = x^2 + x - m$ such that $1 + 4m$ is square free. In this short note, we shall remove the condition that $1 + 4m$ is square free by proving the following proposition.

Proposition. *If $f_m(x) = x^2 + x - m$ is a Rabinowitsch polynomial, then $1 + 4m$ is square free except $m = 2$.*

Thus we immediately have the following theorem.

Theorem. *There are only finitely many Rabinowitsch polynomials $f_m(x) = x^2 + x - m$.*

Finally we shall give the table of all 14 Rabinowitsch polynomials $f_m(x) = x^2 + x - m$ with at most one possible exception. If we assume the generalized Riemann hypothesis is true, then there is no exception.

2 Proof of Proposition

Let $D = 1 + 4m$. Suppose $p^2|D$ for some prime p (which must be odd). We see that for any $x \equiv \frac{p-1}{2} \pmod{p}$, we have

$$p^2|f_m(x) = \frac{1}{4}[(2x+1)^2 - D].$$

Let $D = 1 + 4t^2$ in the type (ii). If $1 + 4t^2 = r^2$ for some positive integer r, then $1 = (r - 2t)(r + 2t)$. But it is impossible. So D can not be a square and $D = D_0 p^2$, where $D_0 \ne 1, 4$. If $D = 2p^2$ or $3p^2$, then $D \equiv 2$ or $\equiv 3$ (mod 4). It is a contradiction to $D \equiv 1$ (mod 4). Thus we have that $D = D_0 p^2$, where $D_0 \ge 5$. Now $D = 1 + 4t^2 \ge 5p^2$ implies $t \ge p$.

Let $D = (2t+1)^2 + 4n$ in the type (iii). If $(2t+1)^2 + 4n = r^2$ for some positive integer r, then $4n = (r - 2t - 1)(r + 2t + 1)$. We easily see that it is impossible for $|n| = 1$. Let n be an odd prime such that $|n| = \frac{2t+1}{3}$. Then n divides r and n^2 divides $4n$. But it is also impossible. So D can not be a square. By a similar argument to the type (ii), we have that $D = D_0 p^2$, where $D_0 \geq 5$. Now $D = (2t+1)^2 + 4n \geq 5p^2$ also implies $t \geq p$.

Therefore for the type (ii) and (iii), there is an $x \equiv \frac{p-1}{2} \pmod{p}$ in the range of $\{x = x_0, x_1, \cdots, x_0 + t - 1\}$ and $p^2 | f_m(x)$. Thus $f_m(x)$ is not a Rabinowitsch polynomial and we proved the proposition.

3 Table

From Proposition and results in [1] [2] [4], we have the following table of all 14 Rabinowitsch polynomials $f_m(x) = x^2 + x - m$ with at most one possible exception.

m	t	n	x_0	D=1+4m	type
1(1)	1(1)	(-1)	0	5	ii(iii)
2	1		0		i
3	1	1	0	13	iii
4	2		1	17	ii
7	2	1	0	29	iii
9	3		1	37	ii
13	3	1	0	53	iii
25	5		1	101	ii
43	6	1	0	173	iii
49	7		1	197	ii
73	8	1	0	293	iii
103	10	-7	4	413*	iii
169	13		1	677	ii
283	16	11	6	1133*	iii

Remark 1. While making the table, we know that Mollin and Williams [3] obtained a similar result to our previous work in [1], by a different method. They treated the case of $x_0 = 1$, square-free $1 + 4m$ and produced narrow Richaud-Degert type of real quadratic fields associated to Rabinowitsch polynomials. But we considered the case of arbitrary x_0, $1 + 4m$ and obtained

not only narrow Richaud-Degert type but also wide Richaud-Degert type of real quadratic fields (* in table).

Remark 2. Example (iii)-4 in [1] is not correct, because $1+4m = 245 = 5\cdot 7^2$ is not square free and $10^2 + 10 - 61 = 49 = 7^2$ is not prime.

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References

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