Class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$

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Abstract. Let t be a square free integer. We shall show that there exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both not divisible by 3.

1 Introduction

Let r and s be the 3-rank of the ideal class group of a real quadratic field $\mathbb{Q}(\sqrt{D})$ and an imaginary quadratic field $\mathbb{Q}(\sqrt{-3D})$. Scholz [7] showed that

$$r \leq s \leq r + 1.$$

This is a classical case of Leopoldt's reflection theorem. On the other hand, the Davenport-Heilbronn theorem [3] and a subsequent refinement by Nakagawa and Horie [6] say that there exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-3D})$ are not divisible by 3. Thus we can make the following observation:

> *There exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{-3D})$ are both not divisible by 3.*

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Recently, combining this observation and the Gross-Zagier theorem [1] [2] on the Heegner points and derivatives of L-series, Vatsal [8] obtained a positive proportion of rank-one quadratic twists of the modular elliptic curve $X_0(19)$. The aim of this note is extend the above observation to the pair of fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ with any square free integer t.

Theorem 1.1 *Let t be a square free integer. Then there exist infinitely many positive fundamental discriminants $D > 0$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both not divisible by 3.*

From this theorem and the class number product formula of bicyclic biquadratic fields, due to Kubota [5], we can easily obtain

Corollary 1.2 *Let t be a square free integer such that the class number of the quadratic field $\mathbb{Q}(\sqrt{t})$ is not divisible by 3. Then there exist infinitely many bicyclic biquadratic fields $\mathbb{Q}(\sqrt{t}, \sqrt{D})$ whose class number is not divisible by 3.*

Finally, as an application, we shall use Theorem 1.1 to get another positive proportion of rank-one twists of the modular elliptic curve $X_0(19)$.

Remark For the complementary question, Komatsu [4] explicitly constructed a family of infinite pairs of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ whose class numbers are both divisible by 3, for any square free integer t.

2 Preliminaries

We recall the result of Nakagawa and Horie in [6], which is a refinement of the result of Davenport-Heilbronn in [3]. Let m and N be two positive integers satisfying the following condition:

> $(*)$ If an odd prime number p is a common divisor of m and N, then p^2 divides N but not m. Further if N is even, then (i) 4 divides N and $m \equiv 1 \pmod 4$, or (ii) 16 divides N and $m \equiv 8$ or $12 \pmod{16}$.

For any positive real number $X > 0$, we denote by $S_+(X)$ the set of positive fundamental discriminants $D < X$ and by $S_-(X)$ the set of negative fundamental discriminants $D > -X$, and put

$$S_+(X, m, N) := \{D \in S_+(X) \mid D \equiv m \pmod{N}\},$$

$$S_-(X, m, N) := \{D \in S_-(X) \mid D \equiv m \pmod{N}\}.$$

Theorem 2.1 *(Nakagawa and Horie) Let D be a fundamental discriminant and $r_3(D)$ be the 3-rank of the quadratic field $\mathbb{Q}(\sqrt{D})$. Then for any two positive integers m, N satisfying $(*)$,*

$$\lim_{X\to\infty} \sum_{D\in S_+(X,m,N)} 3^{r_3(D)} / \sum_{D\in S_+(X,m,N)} 1 = \frac{4}{3}$$

and

$$\lim_{X\to\infty} \sum_{D\in S_-(X,m,N)} 3^{r_3(D)} / \sum_{D\in S_-(X,m,N)} 1 = 2.$$

From Theorem 2.1 and the following fact

$$\begin{aligned} &\sum_{\substack{D\in S_\pm(X,m,N)\\ r_3(D)=0}} 3^{r_3(D)} + 3\Big(\sum_{D\in S_\pm(X,m,N)} 1 - \sum_{\substack{D\in S_\pm(X,m,N)\\ r_3(D)=0}} 3^{r_3(D)}\Big) \\ \leq\ &\sum_{D\in S_\pm(X,m,N)} 3^{r_3(D)}, \end{aligned}$$

we can easily obtain the following lemma.

Lemma 2.2 *Let D be a fundamental discriminant and $h(D)$ the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$. Then for any two positive integers m, N satisfying $(*)$,*

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X, m, N) \mid h(D) \not\equiv 0 \ (\mathit{mod}\ 3)\}}{\sharp S_+(X, m, N)} \geq \frac{5}{6}$$

and

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S_-(X, m, N) \mid h(D) \not\equiv 0 \ (\mathit{mod}\ 3)\}}{\sharp S_-(X, m, N)} \geq \frac{1}{2}.$$

3 Proof of Theorem 1.1

Theorem 1.1 follows from the following proposition.

Proposition 3.1 *Let t be a square free integer and m, N be two positive integers satisfying $(*)$ and $(m,t) = 1$. Then there exist infinitely many positive fundamental discriminants $D \equiv m \pmod N$ with a positive density such that the class numbers of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{tD})$ are both not divisible by 3.*

Proof: We shall give the details of the case $t \equiv 1 \pmod{16}$ and $(mN, t) = 1$, because the other cases are routine modifications of this case. Let

$$S'_+(X, m, tN) := \{tD \mid D \in S_+(X, m, tN)\}.$$

Since t is relatively prime to any $D \in S_+(X, m, tN)$, we have $\sharp S'_+(X, m, tN) = \sharp S_+(X, m, tN)$ and

$$\begin{aligned} S'_+(X, m, tN) &= S_+(tX, tm, t^2N) \quad \text{if } t \text{ is positive,} \\ S'_+(X, m, tN) &= S_-(-tX, tm, t^2N) \text{ if } t \text{ is negative.} \end{aligned}$$

Note that two positive integers m, tN satisfy the condition $(*)$. Then from Lemma 2.2, we have

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X, m, tN) \mid h(D) \not\equiv 0 \,(\mathrm{mod}\ 3)\}}{\sharp S_+(X, m, tN)} \geq \frac{5}{6}. \tag{1}$$

Assume t is positive. Since tm, t^2N also satisfy $(*)$, we know

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S'_+(X, m, tN) \mid h(D) \not\equiv 0 \,(\mathrm{mod}\ 3)\}}{\sharp S'_+(X, m, tN)} \tag{2}$$

$$= \liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X, tm, t^2N) \mid h(D) \not\equiv 0 \,(\mathrm{mod}\ 3)\}}{\sharp S_+(X, tm, t^2N)} \geq \frac{5}{6}. \tag{3}$$

Suppose that

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X, m, tN) \mid h(D) \not\equiv 0 \text{ and } h(tD) \not\equiv 0 \,(\mathrm{mod}\ 3)\}}{\sharp S_+(X, m, tN)} < \frac{2}{3}. \tag{4}$$

Then (4) contradicts (3) and we get

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X,m,tN) \mid h(D) \not\equiv 0 \text{ and } h(tD) \not\equiv 0 \pmod 3\}}{\sharp S_+(X,m,tN)} \geq \frac{2}{3}.$$

If we assume that n is negative then we have

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S'_+(X,m,tN) \mid h(D) \not\equiv 0 \pmod 3\}}{\sharp S'_+(X,m,tN)} \geq \frac{1}{2}.$$

By the similar argument in the case t is positive, we can obtain

$$\liminf_{X\to\infty} \frac{\sharp\{D \in S_+(X,m,tN) \mid h(D) \not\equiv 0 \text{ and } h(tD) \not\equiv 0 \pmod 3\}}{\sharp S_+(X,m,tN)} \geq \frac{1}{3}.$$

The proof is completed since $S_+(X,m,tN)$ has a positive density in $S_+(X,m,N)$.
□

4 Application to rank-one twists

We shall follow Vatsal's paper [8]. Let $t < 0$ be a negative square free integer such that 19 splits in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{t})$ and $t \equiv 1 \pmod 4$. Let $c > 0$ be a positive square free integer satisfying the following conditions:

(i) $c \equiv 1 \pmod 4$ and $(c, t \cdot 19) = 1$,

(ii) 19 is inert in $K = \mathbb{Q}(\sqrt{c})$,

(iii) the class numbers of quadratic fields $\mathbb{Q}(\sqrt{c})$ and $\mathbb{Q}(\sqrt{t \cdot c})$ are both not divisible by 3.

Let E be the modular elliptic curve $X_0(19)$ and $L(s,E)$ be the corresponding L-series. We denote ψ and ψ' the quadratic Galois characters associated to $k = \mathbb{Q}(\sqrt{c})$ and $k' = \mathbb{Q}(\sqrt{t \cdot c})$. Then Gross-Zagier theorem [1] [2] on the Heegner points and derivatives of L-series implies that if c satisfies the above condition (i)(ii)(iii) then $L(s, E \otimes \psi)$ has a simple zero and $L(s, E \otimes \psi')$ is non-zero at $s = 1$.

On the other hand, we can find two positive integers m, N, which depend on $\{4, 19, t\}$ and satisfy $(*)$, such that if c satisfies the congruence $c \equiv m$ (mod N) then c satisfies the above two conditions (i)(ii). From Proposition 3.1, we know that a positive proportion of $c \equiv m$ (mod N) satisfies the condition (iii). Finally if we add the condition $3|c$, which is different from Vatsal's $(3, c) = 1$, then we have another positive proportion of rank-one twists of $X_0(19)$.

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