Class number 2 problem for certain real quadratic fields of Richaud-Degert type

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Abstract. In this paper we will apply Biró's method in [1] [2] to class number 2 problem of real quadratic fields of Richaud-Degert type and will show that there are exactly 4 real quadratic fields of the form $K = \mathbb{Q}(\sqrt{n^2+1})$ with class number 2, where n^2+1 is a even square free integer.

1 Introduction

Let $K = \mathbb{Q}(\sqrt{d})$, where d is a positive square free integer and let $h(d)$ be the class number of this field.

Definition 1.1 *Let $d = n^2 + r$, $d \neq 5$, be a positive square free integer satisfying $r|4n$ and $-n < r \leq n$. Then we call $K = \mathbb{Q}(\sqrt{d})$ a real quadratic field of Richaud-Degert type. Specially if $|r| \in \{1, 4\}$, then $K = \mathbb{Q}(\sqrt{d})$ is called a narrow-Richaud-Degert type.*

As an analogue of Gauss class number 1 problem for imaginary quadratic fields, many authors studied to determine all the real quadratic fields of Richaud-Degert type with class number 1. For example, see [6] [13] [17]. In

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[13], Mollin and Williams classified all the real quadratic fields of Richaud-Degert type of class number 1 with one more possible exception and under the assumption of the Generalized Riemann Hypothesis (GRH) they showed that this classification is true without any exception.

Recently in [1] [2], Biró proved Chowla's conjecture; $h(4n^2+1) > 1$ *if* $n > 13$ and Yokoi's conjecture; $h(n^2+4) > 1$ *if* $n > 17$ without GRH. Following his idea, Byeon, Kim and Lee [5] proved Mollin's conjecture; $h(n^2-4) > 1$ *if* $n > 21$ without GRH and completed the class number 1 problem of the real quadratic fields of narrow-Richaud-Degert type without any exception.

On the other hand, many authors also studied to determine all the real quadratic fields of Richaud-Degert type with class number 2. For example, see [11] [14]. In [14], Mollin and Williams classified all the real quadratic fields of Richaud-Degert type of class number 2 with one more possible exception and under the assumption of GRH they showed that this classification is true without any exception. But in [3], Biró mentioned that his method in [1] [2] seems not to be applied to class number 2 problem (at least in Yokoi's and Chowla's case).

The aim of this paper is, on the contrary to Biró, to show that his method also can be applied to class number 2 problem of the real quadratic fields of Richaud-Degert type $K = \mathbb{Q}(\sqrt{d})$, $d \not\equiv 5 \pmod 8$ whose class number 2 criteria are well known as in [4] [11] [12] and as an example, to show the following conjecture in [11] [14] is true without GRH.

Conjecture *Let n be an odd integer and $d = n^2+1$ be a even positive square free integer. Then $h(d) = 2$ if and only if*

$$d = 10, 26, 122, 362$$

.

Theorem 1.2 *If $d = n^2+1$ is a even positive square free integer with $n > 3045$, then $h(d) > 2$. Specially the conjecture is true.*

Remark We believe that we can also determine all the real quadratic fields of Richaud-Degert type $K = \mathbb{Q}(\sqrt{d})$, $d \not\equiv 5 \pmod 8$ with class number 2 by the similar method without any exception. The results on more cases will be found in [10].

2 Preliminaries

To prove Theorem 1.2, we need the following propositions.

Proposition 2.1 *([8]) Let $K = \mathbb{Q}(\sqrt{d})$, where $d = n^2 + 1$ be a positive square-free integer. Then the fundamental unit ϵ and its norm $N_K(\epsilon)$ are given as follows :*

$$\epsilon = n + \sqrt{n^2 + 1}, \quad N_K(\epsilon) = -1.$$

Proposition 2.2 *Let $K = \mathbb{Q}(\sqrt{d})$, where $d = n^2+1$ be a positive square-free integer. Then*

(i) 2 splits in K if $d \equiv 1 \pmod 8$ i.e. $(2) = (2, \frac{1+\sqrt{d}}{2})(2, \frac{1-\sqrt{d}}{2})$,

(ii) 2 ramifies in K if $d \equiv 2 \pmod 4$ i.e. $(2) = (2, \sqrt{d})^2$,

(iii) 2 remains prime in k if $d \equiv 5 \pmod 8$.

Proof: See [7]. □

Proposition 2.3 *Let $K = \mathbb{Q}(\sqrt{d})$, where $d = n^2+1$ be a positive square-free integer, and $\mathbf{b} = (2, \frac{1\pm\sqrt{d}}{2})$ or $\mathbf{b} = (2, \sqrt{d})$ the ideals of K in Proposition 2.2 (i), (ii). Then if $d \equiv 2 \pmod 4$, $\mathbf{b}$ is not a principal ideal except $d = 2$ and if $d \equiv 1 \pmod 8$, $\mathbf{b}$ is not a principal ideal except $d = 17$.*

Proof: See the proof of Theorem 2.6 in [4]. □

Proposition 2.4 *([Corollary 3.2, 4]) Let $(2k+1)^2 + 1$ be a positive square-free integer with $k > 1$. Then $h((2k+1)^2 + 1) = 2$ if and only if $2k^2 + 2k + 1 - 2t^2 (0 \leq t \leq k)$ are primes.*

3 Computation of special values of zeta functions

Let n be an odd integer, $d = n^2 + 1 > 3$ be a even positive square free integer and $K = \mathbb{Q}(\sqrt{d})$. Let $\mathbf{b} = (2, \sqrt{n^2+1}) = (2, n+1+\sqrt{n^2+1})$ be the ideal of K. Then by Proposition 2.1, we know that $\epsilon = n + \sqrt{n^2+1}$ is the fundamental unit of K and by Proposition 2.2 (ii) and Proposition 2.3, we know that the ideal $\mathbf{b} = (2, \sqrt{n^2+1})$ is not a principal ideal in K.

Let $N(\mathbf{a})$ be the number of the elements of $O(K)/\mathbf{a}$ for an integral ideal $\mathbf{a}$, $N_K(\alpha) = \alpha \cdot \bar{\alpha}$ and $Tr_K(\alpha) = \alpha + \bar{\alpha}$ for $\alpha \in K$, where $\bar{\alpha}$ is the conjugation of α in K. Let χ be an odd primitive character with conductor q and $I(K)$ be the set of nonzero fractional ideals in K. Let K^+ be the set of totally positive elements in K and $i(K^+)$ be the set of principal fractional ideals generated by elements in K^+. Then we easily have the following lemma.

Lemma 3.1 *If $h(d) = 2$, then*

$$I(K)/i(K^+) = (q) \cdot i(K^+) \cup (q)\mathbf{b} \cdot i(K^+).$$

Thus if $h(d) = 2$, then we have

$$\begin{aligned} \zeta_K(s,\chi) &:= \sum_{\substack{\mathbf{a} \in I(K) \\ integral}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} \\ &= \sum_{\substack{\mathbf{a} \in (q) \cdot i(K^+) \\ integral}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} + \sum_{\substack{\mathbf{a} \in (q)\mathbf{b} \cdot i(K^+) \\ integral}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}. \end{aligned}$$

In this section we shall compute the special values of $\zeta_K(s,\chi)$ at $s = 0$ under the assumption $h(d) = 2$. Let $O(K)$ be the ring of integers of K, $O(K)^*$ the group of units in $O(K)$ and $O(K)^*_+$ the group of totally positive units in $O(K)$. For an integral ideal $\mathbf{a}$ of K, let

$$R(\mathbf{a}) := \{a+b\epsilon^2 \mid a, b \in \mathbb{Q} \text{ with } 0 < a \leq 1, 0 \leq b < 1 \text{ and } \mathbf{a}\cdot(a+b\epsilon^2) \subset O(K)\}.$$

Lemma 3.2 *Let $\mathbf{d}$ be an integral ideal in K. The integral ideal $\mathbf{a}$ of K is in $\mathbf{d} \cdot i(K^+) := \{\mathbf{d} \cdot \mathbf{c} \,|\, \mathbf{c} \in i(K^+)\}$ if and only if*

$$\mathbf{a} = \mathbf{d} \cdot (a + b\epsilon^2 + n_1 + n_2\epsilon^2)$$

for $a + b\epsilon^2 \in R(\mathbf{d})$ and nonnegative integers n_1,n_2.

Proof: Let $(\alpha) \in i(K^+)$. Suppose $(\alpha) = (\beta)$ and $\beta > 0$, $\bar{\beta} > 0$. Then $\frac{\alpha}{\beta} \in O(K)^*_+$. Since $O(K)^*_+$ is an infinite cyclic group generated by ϵ^2, $\alpha = \epsilon^{2j}\beta$ for some integer j. Thus we can find that there exists a unique $\beta \in K$ such that $(\alpha) = (\beta)$, $1 \leq \frac{\beta}{\bar{\beta}} < \epsilon^4$ and $\beta > 0$, $\bar{\beta} > 0$. Since $K = \mathbb{Q}(\epsilon^2)$, $\beta = X + Y\epsilon^2$ for some rational X and Y. So $X + Y\bar{\epsilon^2} \leq X + Y\epsilon^2 < \epsilon^4(X + Y\bar{\epsilon^2})$. And we have $Y(\epsilon^2 - \bar{\epsilon^2}) \geq 0$ and $X(\epsilon^4 - 1) > 0$. This implies that $X > 0$ and $Y \geq 0$. Also if $X > 0$ and $Y \geq 0$ then $X + Y\epsilon^2 \in K^+$. Thus

$$\begin{aligned} & i(K^+) \\ = \; & \{(X + Y\epsilon^2) \mid X, Y \in \mathbb{Q} \text{ with } X > 0 \text{ and } Y \geq 0\} \\ = \; & \{(a + b\epsilon^2 + n_1 + n_2\epsilon^2) \mid a, b \in \mathbb{Q} \text{ with } 0 < a \leq 1, 0 \leq b < 1 \\ & \quad \text{and } n_1, n_2 \text{ nonnegative integers}\} \end{aligned}$$

And the integral ideal $\mathbf{d}$ is generated by τ_1 and τ_2, for $\tau_i \in O(K)$. So

$$\begin{aligned} & \mathbf{d} \cdot (a + b\epsilon^2 + n_1 + n_2\epsilon^2) \text{ is an integral ideal.} \\ \Longleftrightarrow \; & \tau_i \cdot (a + b\epsilon^2 + n_1 + n_2\epsilon^2) \in O(K). \\ \Longleftrightarrow \; & \tau_i \cdot (a + b\epsilon^2) \in O(K). \\ \Longleftrightarrow \; & \mathbf{d} \cdot (a + b\epsilon^2) \text{ is an integral ideal.} \end{aligned}$$

This proves the lemma. □

Lemma 3.3 *If* $(q)\mathbf{b} \cdot (x + y\epsilon^2 + n_1 + n_2\epsilon^2)$ *is an integral ideal, then*

$$N((q)\mathbf{b} \cdot (x + y\epsilon^2 + n_1 + n_2\epsilon^2)) \equiv N((q)\mathbf{b} \cdot (x + y\epsilon^2)) \pmod{q},$$

for $0 < x \leq 1$, $0 \leq y < 1$ *and nonnegative integers* n_1, n_2.

Proof: We note that

$$\begin{aligned} & N((q)\mathbf{b} \cdot (x + y\epsilon^2 + n_1 + n_2\epsilon^2)) \\ = \; & 2 \cdot N_K(q \cdot (x + y\epsilon^2 + n_1 + n_2\epsilon^2)) \\ = \; & 2q^2 \cdot (x + y\epsilon^2 + n_1 + n_2\epsilon^2) \cdot (x + y\bar{\epsilon}^2 + n_1 + n_2\bar{\epsilon}^2) \\ = \; & 2q^2(x + y\epsilon^2) \cdot (x + y\bar{\epsilon}^2) + 2q^2 \cdot (x + y\epsilon^2) \cdot (n_1 + n_2\bar{\epsilon}^2) \\ + \; & 2q^2 \cdot (x + y\bar{\epsilon}^2) \cdot (n_1 + n_2\epsilon^2) + 2q^2 \cdot (n_1 + n_2\epsilon^2) \cdot (n_1 + n_2\bar{\epsilon}^2). \end{aligned}$$

Since $(q)\mathbf{b}\cdot(x+y\epsilon^2+n_1+n_2\epsilon^2)$ is an integral ideal, $2q(x+y\epsilon^2)$ is in $O(K)$. So $2q\cdot(x+y\epsilon^2)\cdot(n_1+n_2\bar{\epsilon}^2)$ is in $O(K)$. Thus

$$\begin{aligned} & Tr_K(2q\cdot(x+y\epsilon^2)\cdot(n_1+n_2\bar{\epsilon}^2)) \\ = \; & 2q\cdot(x+y\epsilon^2)\cdot(n_1+n_2\bar{\epsilon}^2)+2q\cdot(x+y\bar{\epsilon}^2)\cdot(n_1+n_2\epsilon^2)\in\mathbb{Z}. \end{aligned}$$

Hence

$$\begin{aligned} & 2q^2(x+y\epsilon^2)\cdot(x+y\bar{\epsilon}^2)+2q^2\cdot(x+y\epsilon^2)\cdot(n_1+n_2\bar{\epsilon}^2) \\ + \; & 2q^2\cdot(x+y\bar{\epsilon}^2)\cdot(n_1+n_2\epsilon^2)+2q^2\cdot(n_1+n_2\epsilon^2)\cdot(n_1+n_2\bar{\epsilon}^2) \\ = \; & 2q^2(x+y\epsilon^2)\cdot(x+y\bar{\epsilon}^2)\ (\mathrm{mod}\ q). \end{aligned}$$

This proves the lemma. □

Lemma 3.4 *([15][16]) For $x,y\in\mathbb{Q}$ with $x>0$, $y\geq 0$ and γ, its conjugate $\bar{\gamma}\in K$ with $\gamma,\bar{\gamma}>0$ and $\gamma\cdot\bar{\gamma}=1$,*

$$\sum_{n_1,n_2=0}^{\infty} N_K(x+y\gamma+n_1+n_2\gamma)^{-s}|_{s=0}=B_1(x)B_1(y)+\frac{1}{4}(\gamma+\overline{\gamma})(B_2(x)+B_2(y)),$$

where B_1 and B_2 are the 1st and 2nd Bernoulli polynomials.

From the above lemmas we have the following proposition.

Proposition 3.5 *If $h(d)=2$, then*

$$\begin{aligned} & \zeta_K(0,\chi) \\ = & \sum_{x+y\epsilon^2\in R((q))}\chi(N((q)\cdot(x+y\epsilon^2)))\cdot(B_1(x)B_1(y)+\frac{1}{4}(\epsilon^2+\overline{\epsilon^2})(B_2(x)+B_2(y))) \\ + & \sum_{x+y\epsilon^2\in R((q)\cdot\mathbf{b})}\chi(N((q)\mathbf{b}\cdot(x+y\epsilon^2)))\cdot(B_1(x)B_1(y)+\frac{1}{4}(\epsilon^2+\overline{\epsilon^2})(B_2(x)+B_2(y))). \end{aligned}$$

Proof: If $h(d)=2$, then from Lemma 3.1, 3.2 and 3.3 we have

$$\begin{aligned} & \zeta_K(s,\chi) \\ = & \sum_{\substack{\mathbf{a}\in(q)\cdot i(K^+)\\ integral}}\frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}+\sum_{\substack{\mathbf{a}\in(q)\mathbf{b}\cdot i(K^+)\\ integral}}\frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} \end{aligned}$$

$$= \sum_{x+y\epsilon^2 \in R((q))} \sum_{n_1,n_2=0}^{\infty} \frac{\chi(N((q)\cdot(x+y\epsilon^2+n_1+n_2\epsilon^2)))}{N((q)\cdot(x+y\epsilon^2+n_1+n_2\epsilon^2))^s}$$
$$+ \sum_{x+y\epsilon^2 \in R((q)\cdot\mathbf{b})} \sum_{n_1,n_2=0}^{\infty} \frac{\chi(N((q)\mathbf{b}\cdot(x+y\epsilon^2+n_1+n_2\epsilon^2)))}{N((q)\mathbf{b}\cdot(x+y\epsilon^2+n_1+n_2\epsilon^2))^s}$$
$$= \sum_{x+y\epsilon^2 \in R((q))} \chi(N((q)\cdot(x+y\epsilon^2)) \sum_{n_1,n_2=0}^{\infty} N((q)\cdot(x+y\epsilon^2+n_1+n_2\epsilon^2))^{-s}$$
$$+ \sum_{x+y\epsilon^2 \in R((q)\cdot\mathbf{b})} \chi(N((q)\mathbf{b}\cdot(x+y\epsilon^2))) \sum_{n_1,n_2=0}^{\infty} N((q)\mathbf{b}\cdot(x+y\epsilon^2+n_1+n_2\epsilon^2))^{-s}.$$

So

$$\zeta_K(0,\chi)$$
$$= \sum_{x+y\epsilon^2 \in R((q))} \chi(N((q)\cdot(x+y\epsilon^2))) \sum_{n_1,n_2=0}^{\infty} q^{-2s}\cdot N_K(x+y\epsilon^2+n_1+n_2\epsilon^2)^{-s}|_{s=0}$$
$$+ \sum_{x+y\epsilon^2 \in R((q)\cdot\mathbf{b})} \chi(N((q)\mathbf{b}\cdot(x+y\epsilon^2))) \sum_{n_1,n_2=0}^{\infty} 2^{-s}q^{-2s}\cdot N_K(x+y\epsilon^2+n_1+n_2\epsilon^2)^{-s}|_{s=0}.$$

Now the proposition follows from Lemma 3.4. □

To get the explicit values of $\zeta_K(0,\chi)$, we need to find explicit (x,y) such that $x+y\epsilon^2 \in R((q))$ and $R((q)\cdot\mathbf{b})$ in Proposition 3.5.

Lemma 3.6

$$\begin{aligned} & \{(x,y)|x+y\epsilon^2 \in R((q))\} \\ = \; & \{(x,y)|x = \frac{C}{q} - \frac{D+qj}{2nq} + \sigma_1(j) \text{ and } y = \frac{D+qj}{2nq} \\ & \text{for } j = 0,1,2,\cdots,(2n-1) \text{ and } 0 \le C,D \le q-1\}, \end{aligned}$$

where

$$\sigma_1(j) = \begin{cases} 0 \text{ if } 0 \le j \le \lceil \frac{2nC-D}{q} \rceil - 1, \\ 1 \text{ if } \lceil \frac{2nC-D}{q} \rceil \le j \le 2n-1. \end{cases}$$

Proof: We note that the set $\{C+D\epsilon \text{ for } 0 \le C,D \le q-1\}$ represents every elements in $O(K)/qO(K)$ because $\{1,\epsilon\}$ is an integral basis for $O(K)$.

So

$$\begin{aligned} & \{(x,y)|x+y\epsilon^2 \in R((q))\} \\ = \ & \{(x,y)|x+y\epsilon^2 \in q^{-1}O(K) \text{ and } 0<x\leq 1 \ , \ 0\leq y<1\} \\ = \ & \{(x,y)|q(x+y\epsilon^2)=C+D\epsilon+q(i+j\epsilon) \text{ for some } 0\leq C,D\leq q-1, i,j\in\mathbb{Z} \\ & \text{ and } 0<x\leq 1 \ , \ 0\leq y<1\}. \end{aligned}$$

From the equation

$$\epsilon = \frac{\epsilon^2-1}{2n}$$

we obtain the following equivalent conditions

$$\begin{aligned} & q(x+y\epsilon^2)=C+D\epsilon+q(i+j\epsilon) \\ \iff \ & q(x+y\epsilon^2)=C+qi+(D+qj)(\frac{\epsilon^2-1}{2n}). \end{aligned}$$

So

$$\begin{aligned} & y=\frac{D+qj}{2nq} \text{ and } 0\leq y<1 \text{ for inegers } j \\ \iff \ & y=\frac{D+qj}{2nq} \text{ for } j=0,1,2,\cdots,(2n-1) \end{aligned}$$

and

$$\begin{aligned} & x=\frac{C}{q}-\frac{D+qj}{2nq}+i \text{ and } 0<x\leq 1 \text{ for inegers } i \text{ and } j=0,1,2,\cdots,(2n-1) \\ \iff \ & x=\frac{C}{q}-\frac{D+qj}{2nq}+i \\ & \text{for } i=\Big[1+\frac{D+qj}{2nq}-\frac{C}{q}\Big]=\begin{cases} 0 \text{ if } 0\leq j\leq \lceil\frac{2nC-D}{q}\rceil-1, \\ 1 \text{ if } \lceil\frac{2nC-D}{q}\rceil\leq j\leq 2n-1. \end{cases} \end{aligned}$$

This proves the lemma. □

Lemma 3.7 *Let* $\delta=n+1+\sqrt{n^2+1}$. *Then*

$$\begin{aligned} & \{(x,y)|x+y\epsilon^2\in R((q)\cdot\mathbf{b})\} \\ = \ & \{(x,y)|x=\frac{A}{2q}-\frac{B+2ql}{4nq}+\sigma_2(l) \textit{ and } y=\frac{B+2ql}{4nq} \textit{ for } l=0,1,2,\cdots,(2n-1) \\ & \textit{and } 0\leq A,B\leq 2q-1 \ , \ A=B \textit{ (mod 2)}\}, \end{aligned}$$

where

$$\sigma_2(l) = \{ \begin{array}{l} 0 \ if \ \ 0 \le l \le \lceil \frac{2nA-B}{2q} \rceil - 1 \\ 1 \ if \ \ \lceil \frac{2nA-B}{2q} \rceil \le l \le 2n-1. \end{array}$$

Proof: We note that

$$\begin{array}{rl} & \{(x,y)|x+y\epsilon^2 \in R((q)\cdot \mathbf{b})\} \\ = & \{(x,y)|x+y\epsilon^2 \in q^{-1}\mathbf{b}^{-1}O(K) \text{ and } 0 < x \le 1 \ , \ 0 \le y < 1\} \\ = & \{(x,y)|2q(x+y\epsilon^2) \ , \ \delta q(x+y\epsilon^2) \in O(K), \text{ and } 0 < x \le 1 \ , \ 0 \le y < 1\} \end{array}$$

and the set $\{A+B\epsilon | 0 \le A, B \le 2q-1\}$ represents all elements in $O(K)/2qO(K)$ because $\{1, \epsilon\}$ is integral basis for $O(K)$. Thus we have

$$\begin{array}{rl} & 2q(x+y\epsilon^2) \in O(K) \\ \iff & 2q(x+y\epsilon^2) = A + B\epsilon + 2q(k+l\epsilon), \\ & \text{for } 0 \le A, B \le 2q-1 \text{ and integers } k, l \\ \iff & 2q(x+y\epsilon^2) = A + 2qk + (B+2ql)(\frac{\epsilon^2-1}{2n}), \\ & \text{for } 0 \le A, B \le 2q-1 \text{ and integers } k, l. \end{array}$$

So

$$\begin{array}{rl} & y = \dfrac{B+2ql}{4nq} \text{ and } 0 \le y < 1 \text{ for inegers } l \\ \iff & y = \dfrac{B+2ql}{4nq} \text{ for } l = 0, 1, 2, ..., (2n-1) \end{array}$$

and

$$\begin{array}{rl} & x = \dfrac{A}{2q} - \dfrac{B+2ql}{4nq} + k \text{ and } 0 < x \le 1 \\ & \text{for inegers } k \text{ and } l = 0, 1, 2, ..., (2n-1) \\ \iff & x = \dfrac{A}{2q} - \dfrac{B+2ql}{4nq} + k \\ & \text{for } k = \Big[1 + \dfrac{B+2ql}{4nq} - \dfrac{A}{2q}\Big] = \{ \begin{array}{l} 0 \text{ if } \ 0 \le l \le \lceil \frac{2nA-B}{2q} \rceil - 1, \\ 1 \ \text{ if } \ \ \lceil \frac{2nA-B}{2q} \rceil \le l \le 2n-1. \end{array} \end{array}$$

Let

$$\sigma_2(l) = \{ \begin{array}{l} 0 \text{ if } \ 0 \le l \le \lceil \frac{2nA-B}{2q} \rceil - 1 \\ 1 \text{ if } \ \lceil \frac{2nA-B}{2q} \rceil \le l \le 2n-1. \end{array}$$

Then

$$\begin{aligned} & 2q(x+y\epsilon^2) \in O(K) \text{ and } 0 < x \leq 1 \ , \ 0 \leq y < 1 \\ \iff \ & x = \frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l) \text{ and } y = \frac{B+2ql}{4nq} \\ & \text{for } l = 0, 1, 2, ..., (2n-1) \text{ and } 0 \leq A, B \leq 2q-1. \end{aligned}$$

So

$$\begin{aligned} & 2q(x+y\epsilon^2), \delta q(x+y\epsilon^2) \in O(K) \text{ and } 0 < x \leq 1 \ , \ 0 \leq y < 1 \\ \iff \ & x = \frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l) \text{ and } y = \frac{B+2ql}{4nq} \\ & \text{for } l = 0, 1, 2, ..., (2n-1) \text{ and } 0 \leq A, B \leq 2q-1 \\ & \text{and } \delta q\Big(\frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l) + \frac{B+2ql}{4nq}\epsilon^2\Big) \in O(K). \end{aligned}$$

Since

$$\begin{aligned} & \delta q\Big(\frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l) + \frac{B+2ql}{4nq}\epsilon^2\Big) \\ &= \frac{A}{2}\delta + q\delta\sigma_2(l) + \frac{B+2ql}{4nq}(\epsilon^2-1)q\delta \\ &= \frac{A}{2}\delta + q\delta\sigma_2(l) + \frac{B+2ql}{2}\epsilon\delta \\ &= \frac{A}{2}\delta + \frac{B}{2}\epsilon\delta + q\delta\sigma_2(l) + q\delta\epsilon l \\ &= \frac{A}{2}\delta + \frac{B}{2}(\delta^2-\delta) + q\delta\sigma_2(l) + q\delta\epsilon l \\ &= \frac{A-B}{2}\delta + \frac{B}{2}\delta^2 + q\delta\sigma_2(l) + q\delta\epsilon l \end{aligned}$$

and

$$\frac{\delta^2}{2} = -n + (n+1)\delta \in O(K),$$

we have

$$\delta q\Big(\frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l) + \frac{B+2ql}{4nq}\epsilon^2\Big) \in O(K) \iff \ A = B(\text{mod } 2).$$

Finally we find the following equivalent conditions

$$\begin{aligned} & 2q(x+y\epsilon^2), \delta q(x+y\epsilon^2) \in O(K) \text{ and } 0 < x \leq 1 \ , \ 0 \leq y < 1 \\ \iff \ & x = \frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l) \text{ and } y = \frac{B+2ql}{4nq} \\ & \text{for } l = 0, 1, 2, ..., (2n-1) \text{ and } 0 \leq A, B \leq 2q-1 \ , \ A = B \ (\mod 2 \ ). \end{aligned}$$

This proves the lemma. □

Since for $(x+y\epsilon^2) \in R((q))$,

$$\begin{aligned} N((q)\cdot(x+y\epsilon^2)) &= N_K(q(x+y\epsilon^2)) \\ &= N_K(C+D\epsilon+q(i+j\epsilon)) \\ &\equiv N_K(C+D\epsilon) \pmod q \\ &= C^2 - D^2 + 2nCD, \end{aligned}$$

and for $(x+y\epsilon^2) \in R((q)\cdot\mathbf{b})$,

$$\begin{aligned} N((q)\mathbf{b}\cdot(x+y\epsilon^2)) &= 2\cdot N_K(q(x+y\epsilon^2)) \\ &= 2\cdot N_K(\frac{A}{2} + \frac{B\epsilon}{2} + q(k+l\epsilon)) \\ &= 2\cdot N_K(\frac{A}{2} + \frac{B\epsilon}{2}) \pmod q \\ &= \frac{A^2}{2} - \frac{B^2}{2} + nAB, \end{aligned}$$

from Lemma 3.6 and Lemma 3.7 we immediately have the following lemma.

Lemma 3.8

$$\begin{aligned} (i) \qquad & \sum_{x+y\epsilon^2 \in R((q))} \chi(N((q)\cdot(x+y\epsilon^2)))\cdot S(x,y) \\ = \ & \sum_{0\leq C,D\leq q-1} \chi(C^2-D^2+2nCD) \sum_{j=0}^{2n-1} S(\frac{C}{q} - \frac{D+qj}{2nq} + \sigma_1(j), \frac{D+qj}{2nq}), \end{aligned}$$

$$\begin{aligned} (ii) \qquad & \sum_{x+y\epsilon^2 \in R((q)\cdot\mathbf{b})} \chi(N((q)\mathbf{b}\cdot(x+y\epsilon^2)))\cdot S(x,y) \\ = \ & \sum_{\substack{0 \leq A, B \leq 2q-1 \\ A = B \pmod 2}} \chi(\frac{A^2}{2} - \frac{B^2}{2} + nAB) \sum_{l=0}^{2n-1} S(\frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l), \frac{B+2ql}{4nq}), \end{aligned}$$

where

$$S(x,y) = (B_1(x)B_1(y) + \frac{1}{4}(\epsilon^2 + \overline{\epsilon^2})(B_2(x) + B_2(y))).$$

By direct computation using MATHEMATICA, we can easily check the following lemma.

Lemma 3.9

$$\begin{aligned}
(i) \qquad & \sum_{j=0}^{2n-1} S(\frac{C}{q} - \frac{D+qj}{2nq} + \sigma_1(j), \frac{D+qj}{2nq}) \\
= \; & \frac{C^2 n}{q^2} + \frac{D^2 n}{q^2} - \frac{2CDn^2}{q^2} + \frac{2C^2 n^3}{q^2} - \frac{Dn}{q} + \frac{Cn^2}{q} + \frac{s_{C,D}(n)}{2} - \frac{n s_{C,D}(n)}{2} \\
& - \frac{C s_{C,D}(n)}{q} + \frac{Dn s_{C,D}(n)}{q} - \frac{2Cn^2 s_{C,D}(n)}{q} + \frac{n s_{C,D}(n)^2}{2} \\
= \; & \frac{n t_{C,D}(n)^2}{2q^2} - \frac{n t_{C,D}(n)}{2q} + \frac{C^2 n}{q^2} + \frac{D^2 n}{2q^2} - \frac{Dn}{2q} + \frac{s_{C,D}(n)}{2} - \frac{C s_{C,D}(n)}{q},
\end{aligned}$$

$$\begin{aligned}
(ii) \qquad & \sum_{l=0}^{2n-1} S(\frac{A}{2q} - \frac{B+2ql}{4nq} + \sigma_2(l), \frac{B+2ql}{4nq}) \\
= \; & \frac{A^2 n}{4q^2} + \frac{B^2 n}{4q^2} - \frac{ABn^2}{2q^2} + \frac{A^2 n^3}{2q^2} - \frac{Bn}{2q} + \frac{An^2}{2q} + \frac{v_{A,B}(n)}{2} - \frac{n v_{A,B}(n)}{2} \\
& - \frac{A v_{A,B}(n)}{2q} + \frac{Bn v_{A,B}(n)}{2q} - \frac{An^2 v_{A,B}(n)}{q} + \frac{n v_{A,B}(n)^2}{2} \\
= \; & \frac{n u_{A,B}^2(n)}{8q^2} - \frac{A u_{A,B}(n)}{4q^2} - \frac{n u_{A,B}(n)}{4q} + \frac{AB}{4q^2} - \frac{A^2 n}{4q^2} + \frac{B^2 n}{8q^2} - \frac{Bn}{4q} + \frac{v_{A,B}(n)}{2},
\end{aligned}$$

where

$$\begin{aligned}
s_{C,D}(n) &= \lceil \frac{2nC - D}{q} \rceil \\
t_{C,D}(n) &= D - 2nC + q s_{C,D}(n) \\
v_{A,B}(n) &= \lceil \frac{2nA - B}{2q} \rceil \\
u_{A,B}(n) &= B - 2nA + 2q v_{A,B}(n).
\end{aligned}$$

Finally combining Proposition 3.6, Lemma 3.8 and Lemma 3.9, we immediately have the following main theorem in this section.

Theorem 3.10 *If $h(d) = 2$, then*

$$\begin{aligned}
&\zeta_K(0,\chi)\\
&= \sum_{0\le C,D\le q-1} \chi(C^2 - D^2 + 2nCD)\cdot\Big(\frac{nt_{C,D}(n)^2}{2q^2} - \frac{nt_{C,D}(n)}{2q} + \frac{C^2n}{q^2} + \frac{D^2n}{2q^2}\\
&-\frac{Dn}{2q} + \frac{s_{C,D}(n)}{2} - \frac{Cs_{C,D}(n)}{q}\Big)\\
&+ \sum_{\substack{0\le A,B\le 2q-1\\ A=B \pmod 2}} \chi(\frac{A^2}{2} - \frac{B^2}{2} + nAB)\cdot\Big(\frac{nu_{A,B}^2(n)}{8q^2} - \frac{Au_{A,B}(n)}{4q^2} - \frac{nu_{A,B}(n)}{4q}\\
&+\frac{AB}{4q^2} - \frac{A^2n}{4q^2} + \frac{B^2n}{8q^2} - \frac{Bn}{4q} + \frac{v_{A,B}(n)}{2}\Big),
\end{aligned}$$

where $s_{C,D}$, $t_{C,D}$, $v_{A,B}$, $u_{A,B}$ are in Lemma 3.9.

Corollary 3.11 *If $h(d) = 2$ and $n = qk + r$ for $0 \le r < q$ then*

$$\zeta_K(0,\chi) = \frac{1}{8q^2}(B_\chi(r)k + A_\chi(r)),$$

where

$$\begin{aligned}
&A_\chi(r)\\
&= \sum_{0\le C,D\le q-1} \chi(C^2 - D^2 + 2rCD)\cdot\Big(4rt_{C,D}(r)^2 - 4qrt_{C,D}(r) + 8C^2r + 4D^2r\\
&-4qDr + 4q^2s_{C,D}(r) - 8qCs_{C,D}(r)\Big)\\
&+ \sum_{\substack{0\le A,B\le 2q-1\\ A=B \pmod 2}} \chi(\frac{A^2}{2} - \frac{B^2}{2} + rAB)\cdot\Big(ru_{A,B}^2(r) - 2Au_{A,B}(r) - 2qru_{A,B}(r)\\
&+2AB - 2A^2r + B^2r - 2qBr + 4q^2v_{A,B}(r)\Big)
\end{aligned}$$

and

$$B_\chi(r)$$

$$= \sum_{0\le C,D\le q-1} \chi(C^2-D^2+2rCD)\cdot\Big(-8C^2q+4D^2q+8Cq^2-4Dq^2$$
$$-4q^2t_{C,D}(r)+4qt_{C,D}(r)^2\Big)$$
$$+ \sum_{\substack{0\le A,B\le 2q-1\\ A=B \pmod 2}} \chi(\frac{A^2}{2}-\frac{B^2}{2}+rAB)\cdot\Big(-2A^2q+B^2q+4Aq^2-2Bq^2$$
$$-2q^2u_{A,B}(r)+qu_{A,B}(r)^2\Big),$$

where $s_{C,D}$, $t_{C,D}$, $v_{A,B}$, $u_{A,B}$ are in Lemma 3.9.

Proof: We observe the following equations

$$s_{C,D}(qk+r) = 2kC + s_{C,D}(r)$$
$$t_{C,D}(qk+r) = t_{C,D}(r)$$
$$v_{A,B}(qk+r) = kA + v_{A,B}(r)$$
$$u_{A,B}(qk+r) = u_{A,B}(r).$$

And since the character χ has a conductor q,

$$\chi(C^2-D^2+2(qk+r)CD) = \chi(C^2-D^2+2rCD)$$
$$\chi(\frac{A^2}{2}-\frac{B^2}{2}+(qk+r)AB) = \chi(\frac{A^2}{2}-\frac{B^2}{2}+rAB).$$

By the above equations and some simple computations, we can easily obtain the corollary. □

4 Proof of Theorem 1.2

Let n be an odd integer, $d=n^2+1>3$ be a even positive square free integer and $K=\mathbb{Q}(\sqrt{d})$. Let $q>2$ be an integer with $(q,d)=1$, χ an odd primitive character with conductor q, $\chi_D(\cdot)=(\frac{D}{\cdot})$ the usual Kronecker character and L_χ the field generated over $\mathbb{Q}$ by the values $\chi(a)$ $(1\le a\le q)$. Since K is an real quadratic field, its quadratic character $\chi_{4(n^2+1)}$ is even. Thus from the same argument on the formula (2.2) in [1], we have

$$\zeta_K(0,\chi) = \frac{1}{4q^2(n^2+1)}\sum_{a=1}^{q} a\chi(a)\sum_{b=1}^{4q(n^2+1)} b\chi(b)\chi_{4(n^2+1)}(b).$$

From now on we assume that $h(d) = 2$. Then by Corollary 3.11, we have

$$\frac{1}{8q^2}(B_\chi(r)k + A_\chi(r)) = \frac{1}{4q^2(n^2+1)} \sum_{a=1}^{q} a\chi(a) \sum_{b=1}^{4q(n^2+1)} b\chi(b)\chi_{4(n^2+1)}(b),$$

where $n = qk + r$, $0 \leq r < q$. Let

$$m_\chi = \sum_{a=1}^{q} a\chi(a).$$

Then we have

$$(B_\chi(r)k + A_\chi(r)) = 8q \cdot m_\chi \cdot \Big(\frac{1}{4q(n^2+1)} \sum_{b=1}^{4q(n^2+1)} b\chi(b)\chi_{4(n^2+1)}(b)\Big).$$

Since $\frac{1}{4q(n^2+1)} \sum_{b=1}^{4q(n^2+1)} b\chi(b)\chi_{4(n^2+1)}(b)$ is an algebraic integer in L_χ from the same argument on the Fact A in [1], we have

$$B_\chi(r)k + A_\chi(r) \equiv 0 \pmod{I},$$

where I is a prime ideal of L_χ for which $m_\chi \in I$.

Condition(*): *The integer q is odd, p is an odd prime, and there is an odd prime character χ with conductor q and a prime ideal I of L_χ lying over p such that $m_\chi \in I$ and the residue field of I is a prime field.*

If the integers q and p satisfy the Condition(*), then for r such that $B_\chi(r) \notin I$ we have

$$n \equiv -q\frac{A_\chi(r)}{B_\chi(r)} + r \pmod{I},$$

and there exists a unique $T(r) \in \{0, 1, 2, ...p-1\}$ such that

$$-q\frac{A_\chi(r)}{B_\chi(r)} + r + I = T(r) + I.$$

Moreover

$$n \equiv T(r) \pmod{p}. \tag{1}$$

We will denote by $q \to p$ that q, p satisfy Condition(*). From the Section 4 in [1], we can find that

$$175 \to 61, \quad 61 \to 1861, \quad 175 \to 1861.$$

Let χ_i and I_i be the characters and ideals defined in Example1, Example3 and Example2 of Section 4 in [1] respectively for $i = 1, 2, 3$. For r with $B_\chi(r) \notin I$, the functions $T_1(r)$, $T_2(r)$ and $T_3(r)$ are defined as follows

$$\begin{aligned}
-175\frac{A_{\chi_1}(r)}{B_{\chi_1}(r)} + r + I_1 &= T_1(r) + I_1, \\
-61\frac{A_{\chi_2}(r)}{B_{\chi_2}(r)} + r + I_2 &= T_2(r) + I_2, \\
-175\frac{A_{\chi_3}(r)}{B_{\chi_3}(r)} + r + I_3 &= T_3(r) + I_3.
\end{aligned}$$

Let

$$U_m = \{a \in \mathbb{Z} | (\frac{a^2+1}{p}) = -1, \text{ for any prime } p \text{ dividing } m\}.$$

Let a_{175} be an residue modulo 175 with $B_{\chi_1}(a_{175}) \notin I_1$ and b_{61} be the residue modulo 61 for which

$$b_{61} = T_1(a_{175}).$$

For b_{61} such that $B_{\chi_2}(b_{61}) \notin I_2$, let c_{1861} be the residue modulo 1861 for which

$$c_{1861} = T_2(b_{61}).$$

And let d_{1861} be the residue modulo 1861 for which

$$d_{1861} = T_3(a_{175})$$

for a_{175} with $B_{\chi_3}(a_{175}) \notin I_3$. Then by computer work (the program is in Appendix), we can check that

$$a_{175} \in U_{175} \to B_{\chi_1}(a_{175}) \notin I_1, B_{\chi_2}(b_{61}) \notin I_2 \text{ and } B_{\chi_3}(a_{175}) \notin I_3.$$

So we can calculate $T_1(a_{175})$, $T_2(b_{61})$ and $T_3(a_{175})$ for $a_{175} \in U_{175}$ and we obtain the following table.

$a_{175} \in U_{175}$	b_{61}	c_{1861}	d_{1861}
± 4	± 4	± 4	± 4
± 9	± 18	± 623	± 121
± 11	± 11	± 11	± 11
± 16	± 47	± 1095	± 540
± 19	± 19	± 19	± 19
± 24	± 34	± 1394	± 1720
± 26	± 40	± 1737	± 894
± 31	0	0	± 695
± 39	± 15	± 566	± 668
± 44	± 45	± 469	± 1073
± 46	± 19	± 19	± 1679
± 51	± 51	± 558	± 1579
± 54	± 49	± 701	± 1491
± 59	± 10	± 1303	± 1601
± 61	± 12	± 1160	± 620
± 66	± 60	± 1860	± 1648
± 74	± 55	± 405	± 62
± 79	± 32	± 1312	± 597
± 81	± 33	± 352	± 934
± 86	± 14	± 766	± 601

First we will find the upper bound of odd $n \notin U_{175}$ with $h(d) = 2$.

Proposition 4.1 *If n is an odd integer such that $n \notin U_{175}$ with $n > 23$ and $d = n^2 + 1$ is a even positive square free integer, then $h(d) > 2$.*

Proof: We note that for an odd prime p

$$\begin{aligned} & 2k^2 + 2k + 1 - 2t_0^2 = 0 \pmod{p} \\ \iff \; & 4k^2 + 4k + 2 - 4t_0^2 = (2k+1)^2 + 1 - (2t_0)^2 = 0 \pmod{p}, \end{aligned}$$

and

$$\begin{aligned} & 2k + 1 \notin U_{175} \\ \iff \; & \Big(\frac{(2k+1)^2 + 1}{5}\Big) \neq -1 \;\text{ or }\; \Big(\frac{(2k+1)^2 + 1}{7}\Big) \neq -1 \\ \iff \; & k = 1, 2, 3 \pmod 5 \text{ or } \; k = 0, 3, 6 \pmod 7. \end{aligned}$$

Thus if $n = 2k+1 \notin U_{175}$ then there exists an integer $t_0 \geq 0$ such that $2k^2+2k+1-2t_0^2 = 0 \pmod 5$ or $2k^2+2k+1-2t_0^2 = 0 \pmod 7$. For example, we consider the case of $k = 7l+3$. If we take $t_0 = 11$, then $2k^2+2k+1-2t_0^2 = 7(14l^2+14l-31)$ can not be a prime for any integer l. Thus if $k = 3 \pmod 7$, then $h(d) > 2$ for $n > 23$ by Proposition 2.4. Applying the same method to the other cases, we can easily obtain the proposition. □

Now we will find the upper bound of odd $n \in U_{175}$ with $h(d) = 2$.

Proposition 4.2 *If n is an odd integer such that $n \in U_{175}$ with $n > 3045$ and $d = n^2+1$ is a even positive square free integer, then $h(d) > 2$.*

Proof: Suppose that $n \equiv a_{175} \pmod{175}$ for some $a_{175} \in U_{175}$ such that $a_{175} \neq \pm 4, \pm 11, \pm 19$. From the above table we know that if $h(d) = 2$ then $c_{1861} \neq d_{1861}$ for all c_{1861}, d_{1861}. It is contradiction to (1). So $h(d) > 2$ if $n \not\equiv \pm 4, \pm 11, \pm 19 \pmod{175}$.

Suppose $n = 2k+1 \in U_{175}$ and $n \equiv \pm 11 \pmod{175}$. If $h(d) = 2$ then from the above table we have $n = 2k+1 = \pm 11 \pmod{61}$. So $k = 61l+5$ or $61l+55$ for some integer l. But if we take $t_0 = 61$, then $2k^2+2k+1-2t_0^2 = 61(122l^2+22l-121)$ or $61(122l^2+222l-21)$. It can not be a prime. So from Proposition 2.4, we have $h(d) > 2$ if $n \equiv \pm 11 \pmod{175}$ and $n > 123$.

Suppose $n = 2k+1 \in U_{175}$ and $n \equiv \pm 19 \pmod{175}$. If $h(d) = 2$ then from the above table we have $n = 2k+1 = \pm 19 \pmod{61}$. So $k = 61l+9$ or $61l+51$ for some integer l. But if we take $t_0 = 50$, then $2k^2+2k+1-2t_0^2 = 61(122l^2+38l-79)$ or $61(122l^2+206l+5)$. It can not be a prime. So from Proposition 2.4, we have $h(d) > 2$ if $n \equiv \pm 19 \pmod{175}$ and $n > 101$.

Suppose $n = 2k+1 \in U_{175}$ and $n \equiv \pm 4 \pmod{175}$. If $h(d) = 2$ then from the above table we have $n = 2k+1 = \pm 4 \pmod{1861}$. So $k = 1861l+932$ or $1861l+928$ for some integer l. But if we take $t_0 = 1522$, then $2k^2+2k+1-2t_0^2 = 1861(3722l^2+3730l-1555)$ or $1861(3722l^2+3714l-1563)$. It can not be a prime. So from Proposition 2.4, we have $h(d) > 2$ if $n \equiv \pm 4 \pmod{175}$ and $n > 3045$. This proves the proposition. □

Proof of Theorem 1.2: In [Theorem 2', 11], Leu proved that if $d = n^2+1$ is a even square free integer with $n < 5000$ and $h(d) = 2$ then $d = 10, 26, 122, 362$. Thus Proposition 4.1 and Proposition 4.2 implies Theorem 1.2 and specially implies that the conjecture is true. □

5 Appendix

The following is the MATHEMATICA program to compute $T_i(r)$ in section 4.

(`f[x_,y_]` computes the logarithm of x with base 2 modulo y.)

```
f[x_, y_] := ( j = 0; m = Mod[x, y];
          If [Mod[x, y] == 0, Return[0]];
          While[ Mod[m, y] >1, m = Mod[m*2, y] ; j = j + 1];
          Return[y - 1 - j]);
```

(`g[x_,y_]` computes the logarithm of x with base 3 modulo y.)

```
g[x_, y_] := (j = 0; m = Mod[x, y];
      If[Mod[x, y] == 0, Return[0]];
      While[ Mod[m, y] >1, m = Mod[m*3, y]; j = j + 1];
      Return[y - 1 - j]);
g7[x_] := g[x, 7]; f25[x_] := (j = 0; m = Mod[x, 25];
        If[ Mod[m, 5] == 0, Return[0]];
        While[Mod[m, 25] >1, m = Mod[m*2, 25]; j = j + 1];
        Return[20 - j]);
f61[x_] := f[x, 61];
```

(`iv[x_,y]` computes the multiplicative inverse of x modulo y.)

```
iv[x_, y_] := (
      i = 1;
      While[Mod[ i*x, y] >1, i++];
      Return[i] );
```

(`chi[a_]` computes $\chi_i(a)$ modulo I_i.)

```
ch1[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];
      Return[Mod[PowerMod[8, f25[Mod[a, 25]], 61]*
      PowerMod[47, g7[Mod[a, 7] ], 61], 61]]);
ch2[a_] := (If[Mod[a, 61] == 0, Return[0]];
      Return[PowerMod[1833, f61[Mod[a, 61]], 1861]]);
ch3[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];
      Return[Mod[PowerMod[380, f25[Mod[a, 25]], 1861]*
      PowerMod[1406, g7[Mod[a, 7] ], 1861], 1861]]);
```

(The followings are needed to compute $A_{\chi_i}(r)$ and $B_{\chi_i}(r)$ modulo I_i.)

```
v[q_, n_, A_, B_] := -Floor[(B - 2 n A)/(2 q)];
u[q_, n_, A_, B_] := B - 2 n A + 2 q v[q, n, A, B];
s[q_, n_, C_, D_] := -Floor[(D - 2 n C)/q];
t[q_, n_, C_, D_] := D - 2 n C + q s[q, n, C, D];
SA1[q_, n_, A_, B_] := 2 A B - 2  A^2 n + B^2 n - 2 q B n -
      2 A u[q, n, A, B] - 2 q n u[q, n, A, B] + n u[q, n, A, B]^2 +
      4 q^2 v[q, n, A, B];
SA2[q_, n_, C_, D_] := 8 C^2 n + 4 D^2 n
      -4 q D n + 4 q^2  s[q, n, C, D] - 8 q C s[q, n,C, D]
      -4 n q t[q, n, C, D] + 4 n t[q, n, C, D]^2;
SB1[q_, n_, A_, B_] := -2 q A^2 + q B^2 - 2 q^2 B
       -2 q^2 u[q, n, A, B] + q u[q, n, A, B]^2 + 4 A q^2;
SB2[q_, n_, C_, D_] := -8 q C^2 + 4 q D^2 - 4 q^2 D + 8 q^2 C
      - 4 q^2 t[q, n, C, D] + 4 q t[q, n, C, D]^2;
```

(`RAi[q_,r_]` and `RBi[q_,r_]`) computes $A_{\chi_i}(r)$ and $B_{\chi_i}(r)$ modulo I_i respectively, where `q` is the conductor for the character χ_i for $i = 1, 2, 3$.)

```
RA1[q_, n_] :=
  Mod[Sum[Mod[Sum[Mod[ch1[2 a^2 - 2 b^2 + 4 n a b]
      *SA1[q,n, 2 a, 2 b], 61], {a, 0, q - 1}], 61], {b, 0, q - 1}], 61]
  + Mod[Sum[Mod[Sum[Mod[ch1[2 a^2 + 2 a - 2 b^2
      - 2 b + n (2 a + 1)(2 b + 1)]*SA1[q, n, 2 a + 1, 2 b + 1], 61],
          {a, 0, q - 1}], 61], {b, 0, q - 1}], 61]
  + Mod[Sum[Mod[Sum[Mod[ch1[c^2 - d^2 + 2 c d n]
       *SA2[q, n, c, d], 61], {c, 0, q - 1}], 61], {d, 0, q - 1}], 61];
RB1[q_, n_] :=
  Mod[Sum[Mod[Sum[Mod[ch1[2 a^2 - 2 b^2 + 4 n a b]
       *SB1[q, n, 2 a, 2 b], 61], {a, 0, q - 1}], 61], {b, 0, q - 1}], 61]
 + Mod[Sum[Mod[Sum[Mod[ch1[2 a^2 + 2 a - 2 b^2 -
         2 b + n (2 a + 1)(2 b + 1)]*SB1[q, n, 2 a + 1, 2 b + 1], 61],
          {a, 0, q - 1}], 61], {b, 0, q - 1}], 61]
 + Mod[Sum[Mod[Sum[Mod[ch1[c^2 - d^2 + 2 c d n]
          *SB2[q, n, c, d], 61], {c, 0, q - 1}], 61], {d, 0, q - 1}], 61];
RA2[q_, n_] :=
  Mod[Sum[Mod[Sum[Mod[ch2[2 a^2 - 2 b^2 + 4 n a b]
    *SA1[q,n, 2 a, 2 b], 1861], {a, 0, q - 1}], 1861], {b, 0, q - 1}],1861]
  + Mod[Sum[Mod[Sum[Mod[ch2[2 a^2 + 2 a - 2 b^2
```

```
      - 2 b + n (2 a + 1)(2 b + 1)]*SA1[q, n, 2 a + 1, 2 b + 1], 1861],
         {a, 0, q - 1}], 1861], {b, 0, q - 1}], 1861]
  + Mod[Sum[Mod[Sum[Mod[ch2[c^2 - d^2 + 2 c d n]
       *SA2[q, n, c, d], 1861], {c, 0, q - 1}], 1861], {d, 0, q - 1}], 1861];
RB2[q_, n_] :=
  Mod[Sum[Mod[Sum[Mod[ch2[2 a^2 - 2 b^2 + 4 n a b]
   *SB1[q, n, 2 a, 2 b], 1861], {a, 0, q - 1}], 1861], {b, 0, q - 1}], 1861]
 + Mod[Sum[Mod[Sum[Mod[ch2[2 a^2 + 2 a - 2 b^2 -
        2 b + n (2 a + 1)(2 b + 1)]*SB1[q, n, 2 a + 1, 2 b + 1], 1861],
         {a, 0, q - 1}],1861], {b, 0, q - 1}], 1861]
 + Mod[Sum[Mod[Sum[Mod[ch2[c^2 - d^2 + 2 c d n]
      *SB2[q, n, c, d], 1861], {c, 0, q - 1}], 1861], {d, 0, q - 1}], 1861];
RA3[q_, n_] :=
  Mod[Sum[Mod[Sum[Mod[ch3[2 a^2 - 2 b^2 + 4 n a b]
    *SA1[q,n, 2 a, 2 b], 1861], {a, 0, q - 1}], 1861], {b, 0, q - 1}],1861]
  +Mod[Sum[Mod[Sum[Mod[ch3[2 a^2 + 2 a - 2 b^2
      - 2 b + n (2 a + 1)(2 b + 1)]*SA1[q, n, 2 a + 1, 2 b + 1], 1861],
         {a, 0, q - 1}], 1861], {b, 0, q - 1}], 1861]
  + Mod[Sum[Mod[Sum[Mod[ch3[c^2 - d^2 + 2 c d n]
       *SA2[q, n, c, d], 1861], {c, 0, q - 1}], 1861], {d, 0, q - 1}],1861];
RB3[q_, n_] :=
  Mod[Sum[Mod[Sum[Mod[ch3[2 a^2 - 2 b^2 + 4 n a b]
     *SB1[q, n, 2 a, 2 b], 1861], {a, 0, q - 1}],1861], {b, 0, q - 1}],1861]
 + Mod[Sum[Mod[Sum[Mod[ch3[2 a^2 + 2 a - 2 b^2 -
        2 b + n (2 a + 1)(2 b + 1)]*SB1[q, n, 2 a + 1, 2 b + 1], 1861],
         {a, 0, q - 1}], 1861], {b, 0, q - 1}],1861]
 + Mod[Sum[Mod[Sum[Mod[ch3[c^2 - d^2 + 2 c d n]
       *SB2[q, n, c, d], 1861], {c, 0, q - 1}], 1861], {d, 0, q - 1}], 1861];
```

(`Ti[r_]`$(i = 1, 2, 3)$ are previously defined functions in section 4.)

```
T1[r_] := Mod[-RA1[175, r]*175*iv[RB1[175, r], 61] + r, 61];
T2[r_] := Mod[-RA2[61, r]*61*iv[RB2[61, r], 1861] + r, 1861];
T3[r_] := Mod[-RA3[175, r]*175*iv[RB3[175, r], 1861] + r, 1861];
```

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