Rank-one quadratic twists of an infinite family of elliptic curves

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Abstract. A conjecture of Goldfeld implies that a positive proportion of quadratic twists of an elliptic curve $E/\mathbb{Q}$ has (analytic) rank 1. This assertion has been confirmed by Vatsal [V1] and the first author [By] for only two elliptic curves. Here we confirm this assertion for infinitely many elliptic curves $E/\mathbb{Q}$ using the Heegner divisors, the 3-part of the class groups of quadratic fields, and a variant of the binary Goldbach problem for polynomials.

1 Introduction

Let $E/\mathbb{Q} : y^2 = x^3 + ax + b$ be an elliptic curve over $\mathbb{Q}$ and let $L(s, E) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be its Hasse-Weil $L$-function defined for $\Re(s) > \frac{3}{2}$. The work of Breuil, Conrad, Diamond, Taylor and Wiles [B-C-D-T] [T-W] [Wi] implies that $L(s, E)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation relating the values at $s$ and $2 - s$. Let $D$ be the fundamental discriminant of the quadratic field $\mathbb{Q}(\sqrt{D})$, and let $\chi_D = (\frac{D}{\cdot})$ denote the usual Kronecker character. For $D$ coprime to the conductor of $E$, the Hasse-Weil $L$-function of the quadratic twist $E_D : Dy^2 = x^3 + ax + b$ of $E$ is the twisted $L$-function $L(s, E_D) = \sum_{n=1}^{\infty} \chi_D(n)a(n)n^{-s}$ which also has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation relating the values at $s$ and $2 - s$. Goldfeld [Go] conjectured that

$$\sum_{|D| < X} \text{Ord}_{s=1}L(s, E_D) \sim \frac{1}{2} \sum_{|D| < X} 1.$$
A weaker version of this conjecture is that for \( r = 0 \) or \( 1 \),

\[
\sharp \{|D| < X \mid \text{Ord}_{s=1} L(s, E_D) = r\} \gg X.
\]  

(1)

For the case \( r = 0 \), there is remarkable progress \([J]\) \([O]\) \([O-S]\) \([V2]\). In particular, it is known that there are infinitely many \( E \) such that (1) holds with \( r = 0 \) \([V2]\).

But for the case \( r = 1 \), we know only two elliptic curves satisfying (1) \([By]\) \([V1]\). For more results on Goldfeld’s conjecture, see Chapter 9 of \([O1]\). In this direction, we shall show the following theorem.

**Theorem 1.1** There are infinitely many elliptic curves \( E/\mathbb{Q} \) such that \( \text{Ord}_{s=1} L(s, E_D) = 1 \) for a positive proportion of fundamental discriminants \( D \).

**Remark 1.** By “infinitely many elliptic curves \( E/\mathbb{Q} \)”, we mean infinitely many \( E/\mathbb{Q} \) with distinct \( j \)-invariants.

**Remark 2.** Theorem 1.1 answers Problem 9.33 in \([O1]\).

In Section 3, as in \([By]\) \([V1]\), using a theorem of Davenport and Heilbronn \([D-H]\) on the 3-parts of the class groups of quadratic fields, a theorem of Gross \([Gr]\) on the non-triviality of Heegner points, and Gross and Zagier’s theorem \([G-Z]\) on Heegner points and derivatives of \( L \)-series, we shall prove the following Theorem 1.2. A new ingredient in this theorem is the relation between Dedekind eta-products and cuspidal divisors, which will be stated in Section 2 and used to show the non-triviality of Heegner points.

Before stating Theorem 1.2, we shall briefly explain some notions and facts. Let \( E/\mathbb{Q} \) be an elliptic curve of conductor \( N \) and \( X_0(N) \) the modular curve of level \( N \) with Jacobian \( J_0(N) \). The work of Breuil, Conrad, Diamond, Taylor and Wiles \([B-C-D-T]\) \([T-W]\) \([Wi]\) shows that there is a surjective morphism \( \phi : X_0(N) \to E \) defined over \( \mathbb{Q} \), which uniquely factors in \( J_0(N) \) through a homomorphism \( \pi : J_0(N) \to E \). An elliptic curve \( E/\mathbb{Q} \) is said be optimal if \( \ker(\pi) \) is connected. There is a unique optimal elliptic curve \( E \) in any isogeny class of elliptic curves defined over \( \mathbb{Q} \) of conductor \( N \). Let \( \delta \) denote a positive divisor of \( N \) and let \( r = (r_{\delta}) \) a family of rational integers \( r_{\delta} \in \mathbb{Z} \). Let \( \eta(z) \) be the Dedekind eta-function and \( \eta_{\delta}(z) := \eta(\delta z) \). It is known \([Li]\) that if \( D_0 \) is a \( \mathbb{Q} \)-rational cuspidal divisor of order \( l \) in \( J_0(N) \), then there is a Dedekind eta-product \( g_r = \prod_{\delta \mid N} \eta_{\delta}^{r_{\delta}} \) which is a modular function on \( X_0(N) \) defined over \( \mathbb{Q} \) and satisfies \( \text{div} \ g_r = lD_0 \). The Dedekind eta-product \( g_r \) is said to be \( l \)-power like if \( \prod_{\delta \mid N} \delta^{t_{\delta}} \) is the \( l \)th-power of a rational number.

For more details on Dedekind eta-products, see Section 2.
Theorem 1.2 Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$. Let $X_0(N)$ be the modular curve of level $N$ with Jacobian $J_0(N)$, $\phi : X_0(N) \to E$ a surjective morphism, which factors in $J_0(N)$ through $\pi : J_0(N) \to E$, and $\pi^* : E \to J_0(N)$ its dual map. Suppose that

(i) the sign $\epsilon$ of the functional equation of $L(s, E)$ is equal to $+1$,
(ii) $E$ has a $\mathbb{Q}$-rational $3$-torsion point $P$,
(iii) $\pi^*(P)$ is a $\mathbb{Q}$-rational cuspidal divisor of order $3$ in $J_0(N)$,
(iv) the Dedekind eta-product $g_r$ such that $\text{div} \ g_r = 3 \pi^*(P)$ is not $3$-power like.

Then $\text{Ord}_{s=1} L(s, E_D) = 1$, for a positive proportion of fundamental discriminants $D$.

In Section 4, we shall find a family of elliptic curves which satisfy the conditions in Theorem 1.2. The most expected family would come from the following result [Theorem 1.2, Du] [V3]:

Let $l$ be a prime number. Let $E'/\mathbb{Q}$ be an elliptic curve of conductor $N$ such that $l^2 \nmid N$, and let $E$ be the optimal elliptic curve in the isogeny class of $E'$. If $E'$ has a $\mathbb{Q}$-rational $l$-torsion, then $E$ has a $\mathbb{Q}$-rational $l$-torsion point $P$ such that $\pi^*(P)$ is a $\mathbb{Q}$-rational cuspidal divisor of order $l$ in $J_0(N)$.

But we do not know whether these elliptic curves also satisfy the condition (iv) in Theorem 1.2 or not. In fact, some of these elliptic curves have the corresponding $g_r$ which is $l$-power like. So, instead of using this result, we will use the more explicit result of Dummitian [Du], which will be stated in Proposition 4.1. And we shall show the following theorem.

Theorem 1.3 Let $E/\mathbb{Q}$ be an optimal elliptic curve of square-free conductor $N$. Let $F$ be the associated newform, and for $d|N$ let $\omega_d = \pm 1$ be such that $W_d F = \omega_d F$, where $W_d$ is the Atkin-Lehner involution. Suppose that

(i) $N = pq$, where $p, q$ are different primes such that $\omega_p = -1$, $\omega_q = 1$ and $p \neq 3$, $q \equiv -1 \pmod{9}$,
(ii) there is an elliptic curve $E'/\mathbb{Q}$ which is isogenous over $\mathbb{Q}$ to $E$ and has a $\mathbb{Q}$-rational $3$-torsion point.

Then $\text{Ord}_{s=1} L(s, E_D) = 1$, for a positive proportion of fundamental discriminants $D$.

Finally, in Section 5, using some results [B-K-W] [Pe] on the binary Goldbach problem for polynomials, we shall show that there are infinitely many elliptic curves satisfying the conditions in Theorem 1.3 and complete the proof of Theorem 1.1.
2 Dedekind eta-products and cuspidal divisors

Let $N$ be a positive integer and let $\delta$ denote a positive divisor of $N$. Let $r = (r_\delta)$ be a family of rational integers $r_\delta \in \mathbb{Z}$ indexed by all the positive divisors $\delta$ of $N$. Let

$$g_r = \prod_{\delta \mid N} \eta_{\delta}^{r_\delta},$$

where $\eta(z)$ is the Dedekind eta-function and $\eta_{\delta}(z) := \eta(\delta z)$. Then we have the following Proposition.

**Proposition 2.1** ([Proposition 3.2.1, Li]) *The Dedekind eta-product $g_r$ is a modular function on $X_0(N)$ defined over $\mathbb{Q}$, i.e., $g_r \in \mathbb{Q}(X_0(N))$ if and only if the following conditions are satisfied:*

1. $\sum_{\delta \mid N} r_\delta = 0$,
2. $\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24},$
3. $\sum_{\delta \mid N} \frac{N}{d} r_\delta \equiv 0 \pmod{24},$
4. $\prod_{\delta \mid N} \delta^{r_\delta} \in \mathbb{Q}^2.$

To state Theorem 1.2, we need the following definition.

**Definition 2.2** For an odd prime $l$, the Dedekind eta-product $g_r = \prod_{\delta \mid N} \eta_{\delta}^{r_\delta}$ is said to be $l$-power like if $\prod_{\delta \mid N} \delta^{r_\delta}$ is the $l$th-power of a rational number.

As representatives of the cusps of $X_0(N)$, we use the rational numbers $\frac{x}{d}$ where $d \mid N$, $d > 0$ and $(x, d) = 1$ with $x$ taken modulo $(d, N/d)$. We say that such a cusp $\frac{x}{d}$ is of level $d$ and it is defined over $\mathbb{Q}(\zeta_m)$, where $m = (d, N/d)$. Let $(P_d)$ denote the divisor on $X_0(N)$ defined as the sum of all the cusps of level $d$ (each with multiplicity one). Then $(P_d)$ is invariant under $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and the $\mathbb{Q}$-rational cuspidal subgroup $C(N)$ of $J_0(N)$ is generated by divisor classes of divisors of the kind

$$\phi((d, N/d))P_1 - (P_d),$$
as \( d \) runs through the positive divisors of \( N \). And we have the following relation between \( \mathbb{Q} \)-rational cuspidal divisors of degree 0 and Dedekind eta-products.

**Proposition 2.3** ([Proposition 3.2.10, Li]) Let \( D_0 = \sum_{d|N} m_d(P_d) \) be a \( \mathbb{Q} \)-rational cuspidal divisor of degree 0 of \( X_0(N) \), i.e.,
\[
\sum_{d|N} \phi((d, N/d))m_d = 0.
\]
Then there exists a Dedekind eta-product \( g_r \in \mathbb{Q}(X_0(N)) \) such that \( \text{div } g_r = lD_0 \) and \( l \) is the order of \( D_0 \).

The following proposition is needed to state and prove Theorem 1.2.

**Proposition 2.4** Let \( E/\mathbb{Q} \) be an elliptic curve of conductor \( N \). Let \( X_0(N) \) be the modular curve of level \( N \) with Jacobian \( J_0(N) \), \( \phi : X_0(N) \to E \) a surjective morphism, which factors in \( J_0(N) \) through \( \pi : J_0(N) \to E \), and \( \pi^* : E \to J_0(N) \) its dual map. Suppose that \( E \) has a \( \mathbb{Q} \)-rational \( l \)-torsion divisor point \( P = [A] \), where \( A \) is a \( \mathbb{Q} \)-rational divisor of degree 0 of \( E \), and \( \pi^*(A) = [B] \) where \( B \) is a \( \mathbb{Q} \)-rational cuspidal divisor of degree 0 with order \( l \) of \( X_0(N) \). Let \( f \in \mathbb{Q}(E) \) such that \( \text{div } f = lA \). Then \( f \circ \phi = \alpha g_r \cdot g^l \in \mathbb{Q}(X_0(N)) \) for some constant \( \alpha \in \mathbb{Q} \), Dedekind eta-product \( g_r \) and \( g \) in \( \mathbb{Q}(X_0(N)) \).

**Proof:** Let \( \phi^* : \text{Div}^0(E) \to \text{Div}^0(X_0(N)) \) be the homomorphism corresponding to \( \phi : X_0(N) \to E \). Since \( \pi^*(P) = [B] \), we can write \( \phi^*(A) = B + \text{div } g \), for some \( g \in \mathbb{Q}(X_0(N)) \). By Proposition 2.3, there exists a Dedekind eta-product \( g_r \in \mathbb{Q}(X_0(N)) \) such that \( \text{div } g_r = lB \). But
\[
\text{div}(f \circ \phi) = \phi^*(\text{div } f) = \phi^*(lA) = l\phi^*(A) = lB + l\text{div } g.
\]
(Cf. [p. 33 Proposition 3.6, Si].) Thus \( \text{div } (f \circ \phi) = \text{div } (g_r \cdot g^l) \) and we have \( f \circ \phi = \alpha g_r \cdot g^l \) for some constant \( \alpha \in \mathbb{Q} \). (Cf. [p. 32 Proposition 3.1, Si].)

### 3 Proof of Theorem 1.2

To prove Theorem 1.2, we need the following proposition.
Proposition 3.1 Let \( E/\mathbb{Q} \) be an elliptic curve of conductor \( N \). Let \( X_0(N) \) be the modular curve of level \( N \) with Jacobian \( J_0(N), \phi : X_0(N) \to E \) a surjective morphism, which factors in \( J_0(N) \) through \( \pi : J_0(N) \to E \), and \( \pi^* : E \to J_0(N) \) its dual map. Suppose that

(i) the sign \( \epsilon \) of the functional equation of \( L(s, E) \) is equal to \(+1\),
(ii) \( E \) has a \( \mathbb{Q} \)-rational \( l \)-torsion point \( P \),
(iii) \( \pi^*(P) \) is a \( \mathbb{Q} \)-rational cuspidal divisor of order \( l \) in \( J_0(N) \),
(iv) the corresponding \( g_\tau \) to \( \pi^*(P) \) in Proposition 2.4 is not \( l \)-power like.

Let \( K \) be an imaginary quadratic field with the discriminant \( D_K(\neq -3) \). Suppose that

(v) every prime factor of \( N \) splits completely in \( K \),
(vi) \( E_{D_K} \) has no \( \mathbb{Q} \)-rational \( l \)-torsion point,
(vii) \( l \) does not divide the class number \( h(D_K) \) of \( K \).

Then we have

\[ \text{Ord}_{s=1} L(s, E_{D_K}) = 1. \]

Proof: Let \( O_K \) be the ring of integers of \( K \) and \( a \) an ideal of \( O_K \). By the condition (v), we can define the Heegner point on \( X_0(N) \) with coordinates \( j(a), j(n^{-1}a) \), where \( (N) = n \cdot n^\tau \) in \( K \) and \( \tau \) is the complex conjugation. We denote it by

\[ (O_K, n, [a]), \]

where \([a]\) denotes the ideal class of \( K \) containing \( a \). Following Birch, Stephens [B-S] and Gross [Gr], let

\[ P_E^*(D_K, 1, 1) := \sum_{a \in \text{Pic}(O_K)} \phi((O_K, n, [a])) - \sum_{a \in \text{Pic}(O_K)} \phi((O_K, n, [a])^\tau). \]

Then by the condition (i),

\[ P_E^*(D_K, 1, 1) \in E_{D_K}(\mathbb{Q}). \]

By the condition (ii), there is \( f \in \mathbb{Q}(E) \) such that \( \text{div} f = lA \), where \( P = [A] \) and \( A \in \text{Div}^0(E) \). Then by Weil’s reciprocity law, \( f \) induces a homomorphism

\[ \delta : E(K)/IE(K) \to K^*/(K^*)^l, \]
in particular, which gives

$$\delta(P^*_E(D_K, 1, 1)) = \prod_{a \in \text{Pic}(O_K)} \frac{f(\phi((O_K, n, [a])))}{f(\phi((O_K, n, [a])^d))}. $$

By the condition (iii) and Proposition 2.4, $f \circ \phi = \alpha g_r \cdot g$ for some constant $\alpha \in \mathbb{Q}$, Dedekind eta-product $g_r$ and $g \in \mathbb{Q}(X_0(N))$. Thus

$$\delta(P^*_E(D_K, 1, 1)) = \prod_{a \in \text{Pic}(O_K)} \frac{\alpha g_r((O_K, n, [a]))}{\alpha g_r((O_K, n, [a])^d)} \cdot \left( \prod_{a \in \text{Pic}(O_K)} \frac{g((O_K, n, [a]))}{g((O_K, n, [a])^{d})} \right)^{\beta} = \beta \cdot \prod_{a \in \text{Pic}(O_K)} \frac{g_r((O_K, n, [a]))}{g_r((O_K, n, [a])^d)},$$

for some $\beta \in K$.

For each positive divisor $d$ of $N$, we denote by $n_d$ the unique $O_K$-ideal of norm $d$ with $n_d | n$. From the definition of $g_r$ and the condition (i) in Proposition 2.1, we have that

$$g_r((O_K, n, [a])^{24} = \prod_{d|N} \Delta(n_d a)^{rd} = \prod_{d|N} \left( \frac{\Delta(n_d a)}{\Delta(a)} \right)^{rd}.$$  

And we know that $\Delta(a)/\Delta(n_d a)$ is an integer in the Hilbert class field $H$ of $K$ which generates the ideal $n_d^{12}$ and from the condition (iv) in Proposition 2.1, we have

$$\prod_{d|N} n_d^{rd} = m^{-2},$$

for some fractional $O_K$-ideal $m$. Thus $\delta(P^*_E(D_K, 1, 1))$ is an element in $K^*$ which generates the ideal $\langle \beta \rangle \cdot (m/m^r)^{h(D_K)}$ and

$$\delta(P^*_E(D_K, 1, 1)) = \zeta \cdot \beta \cdot \gamma^{h(D_K)/O(m)},$$

where $\zeta$ is a root of unity in $K^*$, $\gamma$ is a generator of the principal ideal $(m/m^r)^{O(m)}$ and $O(m)$ is the order of $m$ in Pic$(O_K)$. Hence by the conditions (iv),(vii), $\delta(P^*_E(D_K, 1, 1))$ is not an $l$th-power and by the condition (vi), $\delta(P^*_E(D_K, 1, 1))$ has infinite order in $E_{D_K}(\mathbb{Q})$.

Finally Gross and Zagier’s theorem [G-Z] on Heegner points and derivatives of $L$-series implies that $\text{Ord}_{s=1} L(s, E_{D_K}) = 1$ and we completed the proof.  

Now we can prove Theorem 1.2.
Proof of Theorem 1.2: For any elliptic curve $E/\mathbb{Q}$, there are only finitely many fundamental discriminants $D$ such that $E_D$ has a $\mathbb{Q}$-rational 3-torsion point. A theorem of Davenport and Heilbronn [D-H] (as refined by Nakagawa and Horie [N-H]) on the 3-parts of the class groups of quadratic fields implies that for a positive proportion of negative fundamental discriminants $D$, every prime factor of $N$ splits in the imaginary quadratic fields $\mathbb{Q}(\sqrt{D})$ and 3 does not divide the class number of $\mathbb{Q}(\sqrt{D})$. Thus Theorem 1.2 follows from Proposition 3.1 for the case of $l = 3$.

4 Proof of Theorem 1.3

Let $E/\mathbb{Q}$ be an optimal elliptic curve of square-free conductor $N$. Let $l$ be an odd prime such that $l \not| N$. Under some conditions, Dummigan [Theorem 4.1, Du] shows that if an elliptic curve $E'/\mathbb{Q}$ in the isogeny class of $E$ has a $\mathbb{Q}$-rational point of order $l$ then so has $E$. To prove it, he [Proposition 3.2 and Corollary 3.3, Du] use Dedekind eta-products and explicitly construct a $\mathbb{Q}$-rational cuspidal divisor of degree 0 in $J_0(N)$ whose order is divisible by $l$. In order to apply Dummigan’s result to prove Theorem 1.3, we combine Proposition 3.2, Corollary 3.3, Theorem 4.1 in [Du] and obtain the following proposition.

Proposition 4.1 ([Du]) Let $E/\mathbb{Q}$ be an optimal elliptic curve of square-free conductor $N$. Let $X_0(N)$ be the modular curve of level $N$ with Jacobian $J_0(N)$, $\phi : X_0(N) \rightarrow E$ a surjective morphism, which factors in $J_0(N)$ through $\pi : J_0(N) \rightarrow E$, and $\pi^* : E \rightarrow J_0(N)$ its injective dual map. Let $F$ be the associated newform, and for $d | N$ let $\omega_d = \pm 1$ be such that $W_d F = \omega_d F$, where $W_d$ is the Atkin-Lehner involution. Let $G$ be the product of those primes such that $\omega_p = 1$. Define a divisor $Q$ supported on the cusps of $X_0(N)$ and the Dedekind eta-product $g_r$:

$$Q := \sum_{d | (N/G)} \omega_d(P_d G) \quad \text{and} \quad g_r := (\prod_{g | G} \prod_{d | (N/G)} \eta_{\omega_d g} \mu(g) g)^{24/h},$$

where $h := (r, 24)$, $r := \prod_{p | G} (p^2 - 1) \prod_{p | (N/G)} (p - 1)$, and $\mu$ is the M"obius function.

(i) $Q$ is a $\mathbb{Q}$-rational cuspidal divisor of degree 0,
(ii) $g_r^2 \in \mathbb{Q}(X_0(N))$ and $\text{div}(g_r^2) = (-1)^t \omega_N(2n)Q$, where $n := r/h$ and $t$ is the number of prime divisors of $N$,
(iii) the exact order of the rational point $[Q]$ in $J_0(N)$ is either $n$ or $2n$. 

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(iv) if there is an elliptic curve $E'/\mathbb{Q}$ which is isogenous over $\mathbb{Q}$ to $E$ and has a $\mathbb{Q}$-rational $l$-torsion point, where $l$ is an odd prime such that $l \nmid N$ and $l|n$, then $E$ has a $\mathbb{Q}$-rational $l$-torsion point $P$ such that $\pi^*(P) = R := \frac{2n}{3}[Q]$.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3: We will show that if $E$ is an optimal elliptic curve which satisfies the conditions in Theorem 1.3, $E$ satisfies the conditions in Theorem 1.2.

By the condition (i) in Theorem 1.3, we have $\epsilon = -\omega_N = -\omega_p \cdot \omega_q = +1$; the condition (i) in Theorem 1.2. And by Proposition 4.1, we can construct $\mathbb{Q}$-rational cuspidal divisor $Q$ of degree 0;

$$Q = \sum_{\delta \mid p} \omega_\delta (P_{\delta q}) = (P_q) - (P_{pq}),$$

and the Dedekind eta-product $g_r \in \mathbb{Q}(X_0(N))$;

$$g_r = \left( \prod_{g \mid a} \prod_{d \mid p} \eta_{d g^{(a)}} \right)^{24/h} = \left( \frac{\eta_{pq} \eta_p \eta_q}{\eta_{pq} \eta_p \eta_q} \right)^{24/h},$$

where $h = ((q^2 - 1)(p - 1), 24)$. And

$$\text{div}(g_r^2) = -(2n)Q,$$

where $n = (q^2 - 1)(p - 1)/((q^2 - 1)(p - 1), 24)$. We note that $3 \nmid N = pq$ and $3|n$. Thus by the condition (ii) in Theorem 1.3 and Proposition 4.1 (iv), we easily see that $E$ satisfies the conditions (ii),(iii) in Theorem 1.2.

Finally the corresponding eta-product $g_r^{(q^{-1}) \cdot \frac{24}{(q^2 - 1)(p - 1), 24}}$ is not a cubic because $p^{(q^{-1}) \cdot \frac{24}{(q^2 - 1)(p - 1), 24}}$ is not a cubic. Thus $E$ satisfies the condition (iv) in Theorem 1.2 and we completed the proof.

5 Proof of Theorem 1.1

Let $G(x) \in \mathbb{Z}[x]$ be a polynomial of degree $k$ with positive leading coefficient. Perelli [Pe] and Brüdern, Kawada and Wooley [B-K-W] proved that almost all values of the polynomial $2G(m)$ are the sum of two primes. We slightly modify the result to show that there are infinitely many elliptic curves satisfying the conditions in Theorem 1.3.
Proposition 5.1 ([B-K-W]) Let \( G(x) \in \mathbb{Z}[x] \) be a polynomial of degree \( k \) with positive leading coefficient and \( A, B \) be positive integers such that \((A, B) = 1\). Let \( S_k(M, G) \) denote the number of natural numbers \( m \) with \( 1 \leq m \leq M \), for which the equation
\[
2G(m) = Ap_1 + Bp_2
\]
has no solution in primes \( p_1, p_2 \). Then there is an absolute constant \( c > 0 \) such that
\[
S_k(M, G) \ll_G M^{1-c/k}.
\]

**Proof:** We define \( S(A\alpha) := \sum_{p \leq X} (\log p) e(A\alpha p) \), where the summation is over prime numbers and define
\[
r(m) := \int_0^1 S(A\alpha)S(B\alpha)e(-\alpha m)d\alpha.
\]
Then \( r(2G(m)) \) counts the solutions of \( 2G(m) = Ap_1 + Bp_2 \) with weight \((\log p_1)(\log p_2)\).

If we directly follow the proof of Theorem 1 in [B-K-W], we obtain \( r(2G(m)) > 0 \) for each integer \( m \) with \( 1 \leq m \leq M \), with at most \( O(M^{1-c/k}) \) possible exceptions for a constant \( c > 0 \), which does not depend on \( A, B \) and \( G(x) \). \( \square \)

Now we can prove Theorem 1.1.

*Proof of Theorem 1.1:* Let \( E'/\mathbb{Q} : y^2 + a_1xy + a_3y = x^3, a_1, a_3 \in \mathbb{Z} \). Then the point \((0, 0) \in E'(\mathbb{Q})\) is a 3-torsion point. The discriminant \( \Delta \) of \( E' \) is
\[
\Delta = a_3^3(a_1^3 - 27a_3^2).
\]
Now we assume that \( 2, 3 \nmid \Delta \) and \((a_1, a_3) = 1\). Then since \( c_4 := a_1(a_1^3 - 24a_3) \), we easily see that for every prime factor \( t \) of \( \Delta \), \( E'/\mathbb{Q} \) has multiplicative reductions at \( t \). Thus the conductor \( N \) of \( E' \) is square-free. Furthermore, for every prime factors \( t \) of \( a_3 \), clearly \( E' \) has a split multiplicative reduction at \( t \). On the other hand, for every prime factor \( t \equiv -1 \pmod{3} \) of \((a_1^3 - 27a_3) \) has a non-split multiplicative reduction at \( t \) and for every prime factor \( t \equiv 1 \pmod{3} \) of \((a_1^3 - 27a_3) \) has a split multiplicative reduction at \( t \) because the slopes of the tangent lines at the node \((-a_1^2/9, a_1^3/27) \in E'(\mathbb{F}_t) \) are \((-3a_1 \pm a_1\sqrt{-3})/6\).

Let \( G(x) := (9(2x + 1) - 1)^3/2 \). Then by Proposition 5.1, we know that there are infinitely many \( m \) such that
\[
2G(m) = (9(2m + 1) - 1)^3 = 27p + q,
\]
for some primes $p, q$. For such $m, p, q$, let $a_1 := (9(2m + 1) - 1)$ and $a_3 := p \, (\neq 3)$. Then we have
\[
\Delta = p^3 q \quad \text{and} \quad N = pq,
\]
where $p \neq 3$ and $q \equiv -1 \pmod{9}$. So $E'$ has a split multiplicative reduction at $p$ and has a non-split multiplicative reduction at $q$. Since the signs of Atkin-Lehner involutions $\omega_t = -1$ or $+1$ according as the multiplicative reduction at primes $t$ is split or non-split, respectively, we have $\omega_p = -1$ and $\omega_q = +1$. Thus if we let $E'/\mathbb{Q}$ be the optimal elliptic curve of the isogeny class of $E'/\mathbb{Q}$, then $E$ satisfies all the conditions in Theorem 1.3.

Hence we proved that there are infinitely many elliptic curves $E$ satisfying the conditions in Theorem 1.3. And we easily see that these elliptic curves $E$ have different $j$-invariants by the form of the conductors of $E$. Finally Theorem 1.1 immediately follows from Theorem 1.3.

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