REAL QUADRATIC FUNCTION FIELDS OF MINIMAL TYPE

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Abstract. In this paper, we will introduce the notion of the real quadratic function fields of minimal type, which is a function field analogue to Kawamoto and Tomita's notion of real quadratic fields of minimal type. As number field cases, we will show that there are exactly 6 real quadratic function fields of class number one that are not of minimal type.

1. Introduction

Let q be an odd prime and $k = \mathbb{F}_q$ the finite field of order q. Let $D \in k[x]$ be a monic square-free polynomial of even degree and $K = k(x)(\sqrt{D})$ the *real* quadratic function field over k. Let $O_K = k[x] + k[x]\sqrt{D}$ be the integral closure of $k[x]$ in K, $h(O_K)$ the ideal class number of O_K, and $\epsilon_K = t_D + u_D\sqrt{D}$ the fundamental unit of K.

Let $D \in k[x]$ be a monic square-free polynomial of even degree. Then $\sqrt{D}$ has a continued fraction expansion

$$\sqrt{D} = [a_0, \overline{a_1, a_2, \ldots, a_{l-1}, 2a_0}],$$

where the sequence of non-constant polynomials $a_1, \ldots, a_{l-1}$ is *palindromic*, that is, $a_{l-i} = a_i$ for $1 \le i \le l-1$. Here l is the *period* of D.

For a given positive integer $l > 1$ and a palindromic sequence of non-constant polynomials a_1, …, a_{l-1} in $k[x]$, let $S_k(l; a_1, \ldots, a_{l-1})$ be the set of all monic square-free polynomials D of even degree in $k[x]$ such that $\sqrt{D} = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0}]$ for some non-constant polynomial a_0 in $k[x]$.

In this paper, we will show the following theorem, which is a function field analogue to [B-L] and [Ha].

Theorem 1.1. *Let q be an odd prime and $k = \mathbb{F}_q$. For a given positive integer $l > 1$ and a palindromic sequence of non-constant polynomials a_1, …, a_{l-1} in $k[x]$, $\deg D > \deg u_D$ for all $D \in S_k(l; a_1, \ldots, a_{l-1})$ with one* *possible exception. If the exception exists, then it has the least degree in* $S_k(l; a_1, \ldots, a_{l-1})$.

The following definition is a function field analogue to Kawamoto and Tomita's notion of real quadratic fields of minimal type in [K-T].

Definition 1.2. *Let* q *be an odd prime and* $k = \mathbb{F}_q$. *For a given positive integer* $l > 1$ *and a palindromic sequence of non-constant polynomials* a_1, $\ldots$, a_{l-1} *in* $k[x]$, *if* $S_k(l; a_1, \ldots, a_{l-1})$ *has the exception* D *in Theorem 1.1, then we call* $K = k(x)(\sqrt{D})$ *a real quadratic function field of* <u>*minimal type*</u> *over* k.

In [K-T], Kawamoto and Tomita proved that there exist exactly 51 real quadratic fields of ideal class number one that are not of minimal type with one more possible exception. In this paper, we will show the following similar theorem.

Theorem 1.3. *Let* q *be an odd prime and* $k = \mathbb{F}_q$. *If* $q = 3$ *or 5, there are exactly 6 real quadratic function fields over* k *with ideal class number one that are not of minimal type. If* $q \geq 7$, *all real quadratic function fields with ideal class number one is of minimal type.*

2. Proof of Theorem 1.1

Let q be an odd prime and $k = \mathbb{F}_q$. For a continued fraction expansion in $k((\frac{1}{x}))$

$$[a_0, a_1, a_2, \ldots],$$

we define for $i \in \mathbb{N}_0$,

$$p_{-2} := 0, \quad p_{-1} := 1, \quad p_i := a_i p_{i-1} + p_{i-2},$$

$$q_{-2} := 1, \quad q_{-1} := 0, \quad q_i := a_i q_{i-1} + q_{i-2}.$$

Then we have for $i \in \mathbb{N}_0$,

$$\frac{p_i}{q_i} = [a_0, a_1, a_2, \ldots, a_i],$$

$$p_i q_{i-1} - p_{i-1} q_i = (-1)^{i-1},$$

$$p_{i+1} = a_{i+1} p_i + p_{i-1},$$

$$q_{i+1} = a_{i+1} q_i + q_{i-1}.$$

We call $\frac{p_i}{q_i}$ *the i-th approximant of the continued fraction* $[a_0, a_1, a_2, \ldots]$.

Let $a_1, \cdots, a_{l-1}$ be a palindromic sequence of non-constant polynomials in $k[x]$. Suppose that there exists a monic square-free polynomial D of even degree and $a_0 \in k[x]$ such that

$$\sqrt{D} = [a_0, \overline{a_1, a_2, \cdots, a_{l-1}, 2a_0}].$$

Then we have

$$\sqrt{D} = a_0 + \beta,$$

where $\beta = [0, \overline{a_1, a_2, \cdots, a_{l-1}, 2a_0}]$. Let $\frac{p_i}{q_i}$ be the i-th approximant of the continued fraction $[0, \overline{a_1, a_2, \cdots, a_{l-1}, 2a_0}]$. Then we have

$$p_{l-1} = q_{l-2}$$

and

$$\sqrt{D} = \sqrt{a_0^2 + \frac{2a_0p_{l-1} + p_{l-2}}{q_{l-1}}}$$

(cf. [p.338, Mc]). We note that $\deg a_0^2 > \deg \frac{2a_0p_{l-1}+p_{l-2}}{q_{l-1}}$. If we let $v := \frac{2a_0p_{l-1}+p_{l-2}}{q_{l-1}}$, then we have $v \in k[x]$ and

$$2a_0p_{l-1} + p_{l-2} = vq_{l-1}.$$

So we have

$$2a_0p_{l-1} - vq_{l-1} = -p_{l-2} = (-1)^{l+1}p_{l-2}(p_{l-1}q_{l-2} - p_{l-2}q_{l-1})$$

and

$$p_{l-1}(2a_0 + (-1)^l p_{l-2}q_{l-2}) = q_{l-1}(v + (-1)^l p_{l-2}^2).$$

Since p_{l-1} and q_{l-1} are relatively prime, there exists $w \in k[x]$ such that

$$2a_0 + (-1)^l p_{l-2}q_{l-2} = wq_{l-1},$$

and

$$v + (-1)^l p_{l-2}^2 = wp_{l-1}.$$

Thus we have

$$D = (\frac{wq_{l-1} + (-1)^{l-1}p_{l-2}q_{l-2}}{2})^2 + wp_{l-1} + (-1)^{l-1}p_{l-2}^2$$

and obtain the following lemma.

Lemma 2.1. *Let q be an odd prime, $k = \mathbb{F}_q$, and D a monic square-free polynomial of even degree in $k[x]$. Let $l > 1$ be a positive integer, $a_1, \cdots, a_{l-1}$ a palindromic sequence of non-constant polynomials in $k[x]$, and $\frac{p_i}{q_i}$ the i-th approximant of the continued fraction $[0, a_1, a_2, \ldots, a_{l-1}]$. Suppose that there exists $a_0 \in k[x]$ such that*

$$\sqrt{D} = [a_0, \overline{a_1, a_2, \cdots, a_{l-1}, 2a_0}].$$

Then there exists $w \in k[x]$ such that

$$D = (\frac{wq_{l-1} + (-1)^{l-1}p_{l-2}q_{l-2}}{2})^2 + wp_{l-1} + (-1)^{l-1}p_{l-2}^2.$$

The following lemma is well known.

Lemma 2.2. *Let q be an odd prime and $k = \mathbb{F}_q$. Let D be a monic square-free polynomial of even degree in $k[x]$ such that $\sqrt{D} = [a_0, \overline{a_1, a_2, \cdots, a_{l-1}, 2a_0}]$ and $\frac{p_i}{q_i}$ the i-th approximant of the continued fraction. Let $K = k(x)(\sqrt{D})$ be the real quadratic function field over k and ϵ_K the fundamental unit of K. Then*

$$\epsilon_K = \begin{cases} p_{l-1} + q_{l-1}\sqrt{D} & \text{if } l \text{ is eqaul to the quasi-period of } D, \\ p_{\frac{l}{2}-1} + q_{\frac{l}{2}-1}\sqrt{D} & \text{if } l \text{ is eqaul to two times of the quasi-period of } D. \end{cases}$$

Remark. It is well known that the period of D is equal to the quasi-period or two times of the quasi-period of D. For detail and definition of quasi-period, see [St].

Proof of Theorem 1.1. Let q be an odd prime and $k = \mathbb{F}_q$. For a given positive integer $l > 1$ and a palindromic sequence of polynomials $a_1, \ldots, a_{l-1}$ in $k[x]$, suppose that $D \in S_k(l; a_1, \ldots, a_{l-1})$. Then by Lemma 2.1, we have

$$D = (\frac{wq_{l-1} + (-1)^{l-1}p_{l-2}q_{l-2}}{2})^2 + wp_{l-1} + (-1)^{l+1}p_{l-2}^2$$

for some $w \in k[x]$, where $\frac{p_i}{q_i}$ the i-th approximant of the continued fraction $[0, a_1, a_2, \dots, a_{l-1}]$. So if $\deg wq_{l-1} \neq \deg p_{l-2}q_{l-2}$, then we have

$$\begin{aligned} \deg D &\geq \deg a_0^2 \\ &= \deg \left(\frac{wq_{l-1} + (-1)^{l-1}p_{l-2}q_{l-2}}{2}\right)^2 \\ &> 2\deg q_{l-1} \\ &> 2\deg q_{\frac{l}{2}-1} \end{aligned}$$

and by Lemma 2.2, we have

$$\deg D > \deg u_D.$$

Suppose that $\deg wq_{l-1} = \deg p_{l-2}q_{l-2}$. Then we can write

$$(-1)^{l-1}p_{l-2}q_{l-2} = \alpha q_{l-1} + \gamma$$

for some α, $\gamma \in k[x]$ such that $\deg \gamma < \deg q_{l-1}$ or $\gamma = 0$ and we have

$$wq_{l-1} + (-1)^{l-1}p_{l-2}q_{l-2} = (w + \alpha)q_{l-1} + \gamma.$$

So if $w \neq -\alpha$, then by Lemma 2.1 we also have $\deg D > 2\deg q_{l-1}$ and by Lemma 2.2 we have

$$\deg D > \deg u_D.$$

Thus we proved that $\deg D > \deg u_D$ for all $D \in S_k(l; a_1, \dots, a_{l-1})$ with one possible exception, which is the case $w = -\alpha$ and in this case D has the least degree in $S_k(l; a_1, \dots, a_{l-1})$. □

3. Proof of Theorem 1.3

Let q be an odd prime and $k = \mathbb{F}_q$. Let $D \in k[x]$ be a monic square-free polynomial of even degree and $K = k(x)(\sqrt{D})$ the real quadratic function field over k. Let $h(O_K)$ be the ideal class number of O_K and h_K the divisor class number of K. Then we have

$$h_K = R_K h(O_K),$$

where R_K is the regulator of K. If $\epsilon_K = t_D + u_D\sqrt{D}$, then $R_K = \deg t_D$. We note that if $\deg D = 2N$, then the genus g_K of K is equal to $N - 1$.

Let $l > 1$ be a positive integer and $a_1, \cdots, a_{l-1}$ a palindromic sequence of non-constant polynomials in $k[x]$. Suppose that $D \in S_k(l : a_1, \cdots, a_{l-1})$

and $\sqrt{D} = [a_0, \overline{a_1, a_2, \cdots, a_{l-1}, 2a_0}]$ for some $a_0 \in k[x]$. Let $\frac{p_i}{q_i}$ be the i-th approximant of the continued fraction $[a_0, \overline{a_1, a_2, \cdots, a_{l-1}, 2a_0}]$. If D does not have the least degree in $S_k(l : a_1, \cdots, a_{l-1})$, then $\deg D > 2\deg q_{l-1}$. So we have

$$\deg D = 2N > 2\deg q_{l-1} = 2(\deg a_1 + \cdots + \deg a_{l-1})$$

and

$$R_K \leq \deg p_{l-1} = \deg a_0 + \deg a_1 + \cdots + \deg a_{n-1} = N + \deg q_{l-1} < 2N.$$

So we have the following lower bound of the ideal class number of $K = k(x)(\sqrt{D})$ from the work of Madan and Queen [M-Q] and the work of Feng and Hu [F-H].

$$\begin{aligned} h(O_K) &\geq \frac{(q-1)(q^{2g_K-1} + 1 - 2g_K q^{\frac{2g_K-1}{2}})}{R_K(2g_K-1)(q^{g_K}-1)} \\ &> \frac{(q-1)(q^{2N-3} + 1 - 2(N-1)q^{\frac{2N-3}{2}})}{2N(2N-3)(q^{N-1}-1)}. \end{aligned}$$

This lower bound implies the following proposition.

Proposition 3.1. *Let q be an odd prime and $k = \mathbb{F}_q$. Let $D \in k[x]$ be a monic square-free polynomial of even degree and $K = k(x)(\sqrt{D})$ the real quadratic function field over k. If K is not minimal type and the ideal class number of K is equal to one, then*

$$\deg D = \begin{cases} 4, 6, 8, 10, & \text{if } q = 3 \\ 4, 6, & \text{if } q = 5 \\ 4, & \text{if } q = 7 \end{cases}$$

In particular, if $q \geq 11$, then all real quadratic function fields with ideal class number one is of minimal type.

Remark. If $\deg D = 2$, D is not contained in $S_k(l : a_1, \cdots, a_{l-1})$ for $l > 1$. In this case, $K = k(x)(\sqrt{D})$ always has ideal class number one.

Proof of Theorem 1.3. Theorem 1.3 follows from Proposition 3.1 and the table of class numbers of quadratic function field in [F-H]. The complete list of 6 real quadratic function fields over k with ideal class number one that are not of minimal type is following.

q, deg$D$	D	R	$h(O_K)$
$q=3$, deg$D=4$	x^4+2x^2+2	2	1
$q=3$, deg$D=6$	$x^6+x^4+x^3+x^2+2x+2$	3	1
	$x^6+x^4+2x^3+x^2+x+2$	3	1
	$x^6+x^5+2x^3+2x^2+2$	5	1
	$x^6+2x^5+x^3+2x^2+2$	5	1
$q=5$, deg$D=4$	x^4+2	2	1

□

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