ELLIPTIC CURVES WITH ALL CUBIC TWISTS OF THE SAME ROOT NUMBER

DONGHO BYEON AND NAYOUNG KIM

Abstract. Let E/K be an elliptic curve with j-invariant 0 defined over a number field K. In this paper, we give a simple condition on K which determines whether all cubic twists of E/K have the same root number or not. This is a cubic twist analogue to the work [D-D] of Dokchitser and Dokchitser on quadratic twists of elliptic curves.

1. Introduction and Main results

Let K be a number field of degree n over $\mathbb{Q}$. Let E/K be an elliptic curve over K with conductor $N(E/K)$, and let $L(E/K,s)$ be its Hasse-Weil L-function defined for $\Re(s) > \frac{3}{2}$. The Hasse-Weil conjecture asserts that $L(E/K,s)$ has an analytic continuation to the whole of $\mathbb{C}$ and the completed L-function

$$L^*(E/K,s) := \Gamma\big(\frac{s}{2}\big)^n\,\Gamma\big(\frac{s+1}{2}\big)^n\,\pi^{-ns}\,N(E/K)^{s/2}\,L(E/K,s)$$

satisfies a functional equation

$$L^*(E/K,2-s) = w(E/K)\,L^*(E/K,s)$$

with the sign given by the global root number $w(E/K) = \pm 1$. The global root number has played an important role, since from the functional equation this number determines the parity of $\mathrm{ord}_{s=1}L(E/K,s)$ and under the parity conjecture, the parity of the Mordell-Weil rank of E over K. However, such a functional equation is not yet known to exist in general. Therefore we adopt another representation-theoretic definition of the global root number which can be defined independent on any conjectures and is conjectured to be $w(E/K)$. The global root number $w(E/K)$ is the product of the local root numbers over all places of K,

$$w(E/K) = \prod_v w(E/K_v)$$

with $w(E/K_v)$ defined using local Galois actions on the Tate module.

In [D-D], Dokchitser and Dokchitser consider an elliptic curve E/K all of whose quadratic twists have the same global root number, and they gave the conditions on E/K which determine whether it is such a curve. The aim of this paper is to give an answer to the analogous question for cubic twists, i.e., to determine which elliptic curves have the same global root number over all its cubic twists.

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Theorem 1.1. *Let E/K be an elliptic curve with j-invariant 0 defined by the equation $y^2 = x^3 + a$, where $a \in K$. For an element $D \in K^\times/(K^\times)^3$, let $E_D : y^2 = x^3 + aD^2$ be the cubic twist of E. Then the root number $w(E_D/K)$ is constant for all $D \in K^\times/(K^\times)^3$ if and only if K contains $\sqrt{-3}$. In particular, if $K \ni \sqrt{-3}$, then $w(E_D/K) = +1$ for all $D \in K^\times/(K^\times)^3$, and if $K \not\ni \sqrt{-3}$, then there are infinitely many E_D/K such that $w(E_D/K) = +1$, and $w(E_D/K) = -1$, respectively.*

2. Preliminaries

Let us first collect a general list of local root number formulas. It is well-known that:

$$w(E/K_v) = \begin{cases} +1 & \text{if } E/K_v \text{ has good or non-split multiplicative reduction;} \\ -1 & \text{if } v \text{ is Archimedean or } E/K_v \text{ has split multiplicative reduction.} \end{cases}$$

At places of additive reduction, Rohrlich [R] gives the following formulas for $w(E/K_v)$ when $K_v = \mathbb{Q}_p$ with $p \geq 5$, and Conrad, Conrad and Helfgott [C-C-H] prove that the formulas can be extended for any K_v with $p \geq 5$.

Theorem 2.1. ([R] Proposition 2, 3, [C-C-H] Theorem 3.1.) *Let K_v be a local field, with residue field of characteristic $p \geq 5$ and normalized valuation v. Let E/K_v be an elliptic curve with additive reduction. We denote the usual discriminant of a Weierstrass model for E/K_v by Δ, and the quadratic residue symbol on the residue field of K_v by $\left(\frac{\cdot}{k_v}\right)$.*

(1) *Assume E has potentially good reduction. Define $e = 12/\gcd(v(\Delta), 12)$. We have $e \in \{1, 2, 3, 4, 6\}$, and the local root number $w(E/K_v)$ can be computed by the following formulas:*

$$w(E/K_v) = \begin{cases} 1 & \text{if } e = 1, \\ \left(\frac{-1}{k_v}\right) & \text{if } e = 2 \text{ or } 6, \\ \left(\frac{-3}{k_v}\right) & \text{if } e = 3, \\ \left(\frac{-2}{k_v}\right) & \text{if } e = 4. \end{cases}$$

(2) *If E has potentially multiplicative reduction, then $w(E/K_v) = \left(\frac{-1}{k_v}\right)$.*

In [K], Kobayashi gives a formulas for the local root number $w(E/K_v)$ with any local field K_v of *odd* residue characteristic.

Theorem 2.2. ([K], Theorem 1.1.) *Let K_v be a local field, with residue field of odd characteristic and normalized valuation v. Let E/K_v be an elliptic curve with potentially good reduction. We denote the Hilbert symbol of K_v by $(\ ,\ )_{K_v}$.*

(1) *If the Kodaira-Néron type of E is I_0 or I_0^*, then*

$$w(E/K_v) = \left(\frac{-1}{k_v}\right)^{v(\Delta)/2}.$$

(2) *If the Kodaira-Néron type of E is III or III^*, then*

$$w(E/K_v) = \left(\frac{-2}{k_v}\right).$$

(3) *If the Kodaira-Néron type of E is II, IV, IV^* or II^*, there exists a Weierstrass equation such that $y^2 = x^3 + ax^2 + bx + c$ with $3 \nmid v(c)$. Then for such equation, we have*

$$w(E/K_v) = \delta\ (\Delta,\ c)_{K_v} \left(\frac{v(c)}{k_v}\right)^{v(\Delta)} \left(\frac{-1}{k_v}\right)^{\frac{v(\Delta)(v(\Delta)-1)}{2}}$$

where $\delta = \pm 1$ and $\delta = 1$ if and only if $\Delta^{\frac{1}{2}} \in K_v$.

Thus, the local root numbers $w(E/K_v)$ have been classified except at places above 2. In [D-D2], Dokchitser and Dokchitser complete the remaining case. However for an elliptic curve $E : y^2 = x^3 + a$, the local root number $w(E/K_v)$ at places above 2 can be easily computed by Kobayashi. (Note that Dokchitser and Dokchitser [D-D2, Section 4] inform the misprints in [K, Proposition 6.1].)

Theorem 2.3. ([K], Proposition 6.1.) *Let K_v be a local field, with residue field of even characteristic and normalized valuation v. Let E/K_v be an elliptic curve $E : y^2 = x^3 + a$. Then*

$$w(E/K_v) = \begin{cases} (-1,\ a)_{K_v} & \text{if } v(\Delta) \equiv 0 \pmod 3 \text{ or } \sqrt{-3} \in K_v; \\ (-1)^{\frac{N(E/K_v)}{2}}\ (3,\ a)_{K_v} & \text{otherwise.} \end{cases}$$

3. Proof of Theorem 1.1

In [K, Corollary 6.3], Kobayashi shows that if $K \ni \sqrt{-3}$ then any elliptic curve with j-invariant 0 over K has global root number 1. Now we will show that the local root number calculations prevent this in the case that $K \not\ni \sqrt{-3}$.

Suppose that $K \not\ni \sqrt{-3}$. Then there are infinitely many prime numbers p with $p \equiv 2 \pmod 3$ such that some $\mathfrak{p} \mid p$ in K has the residual degree $f(\mathfrak{p} \mid p) = 1$. So we can pick $\mathfrak{p} \mid p$ in K satisfying the following three conditions:

(a1) The prime number p is $\neq 2, 3$, and $p \equiv 2 \pmod 3$,
(a2) The residual degree $f(\mathfrak{p} \mid p)$ is 1,
(a3) $\mathfrak{p}$ doesn't contain the coefficient a of E.

Let v' be the finite place of K corresponding to the prime ideal $\mathfrak{p}$, and let $\pi \in K^\times$ be a local uniformizer of v' such that for all the other finite places v of K, $v(\pi) = 0$. Then we can easily find $D \in K^\times/(K^\times)^3$ satisfying the following three properties:

(b1) $3 \nmid v(aD^2)$ for all places v over 3 of K,
(b2) $v(\Delta(E_D)) \equiv 0 \pmod 3$ for all places v over 2 of K,
(b3) $v'(D) = 0$,

and we can show that the following two twists of E have different global root numbers:

$$E_D : y^2 = x^3 + aD^2, \quad E_{\pi D} : y^2 = x^3 + a\pi^2 D^2.$$

First we compare their local root numbers over K_v with the residue characteristic ≥ 5. For all these $v(\neq v')$, the assumption $v(\pi) = 0$ implies that the numbers e in Theorem 2.1(1) for E_D and $E_{\pi D}$ are same, and thus

$$w(E_D/K_v) = w(E_{\pi D}/K_v).$$

For the v', E_D has good reduction at v' by the conditions (a1), (a3) and (b3), and so $w(E_D/K_{v'}) = 1$. Also, since π is a local uniformizer, the number e in Theorem 2.1(1) is 3, and hence we have

$$w(E_{\pi D}/K_{v'}) = \left(\frac{-3}{k_{v'}}\right) = \left(\frac{-3}{p}\right)^{f(\mathfrak{p}|p)} = \left(\frac{p}{3}\right) = -1$$

by the conditions (a1) and (a2). So we have

$$w(E_D/K_{v'}) \neq w(E_{\pi D}/K_{v'}).$$

For a place v with the residue characteristic 3, E_D/K_v and $E_{\pi D}/K_v$ are of type II, IV, IV^* or II^* by the conditions (a1) and (b1). Then the formulas in Theorem 2.2 (3) for E_D and $E_{\pi D}$ are same, and thus

$$w(E_D/K_v) = w(E_{\pi D}/K_v).$$

Finally, for a place v with the residue characteristic 2, we have

$$w(E_D/K_v) = (-1\,,\, aD^2)_{K_v} = (-1\,,\, a)_{K_v} = (-1\,,\, a\pi^2 D^2)_{K_v} = w(E_{\pi D}/K_v)$$

by the condition (b2) and Theorem 2.3. So we conclude that

$$w(E_D/K) \neq w(E_{\pi D}/K).$$

Furthermore, for a fixed π, there are infinitely many $D \in K^\times/(K^\times)^3$ satisfying the property (b1)-(b3), and the involution $D \leftrightarrow \pi D$ on $K^\times/K^{\times 3}$ changes the sign of $w(E_D/K)$. This completes the proof of Theorem 1.1. □

Remark. The parity conjecture and Theorem 1.1 imply that if $K \ni \sqrt{-3}$, every cubic twist $E_D/K : y^2 = x^3 + aD^2$ of E/K has the Mordell-Weil group $E_D(K)$ of even rank. On the other hand, using the complex multiplication $[\zeta_3]$ of E_D/K, where ζ_3 is a primitive cubic root of unity and $[\zeta_3](x, y) = (\zeta_3 x, y)$, we can show this without the parity conjecture, because $E_D(K)$ is a direct sum of two-dimensional subspaces with the bases of the form $\{P, [\zeta_3]P\}$.

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References

[C-C-H] B. Conrad, K. Conrad and H. Helfgott, *Root numbers and ranks in positive characteristic*, Adv. Math. 198 (2005), 684-731.

[D-D] T. Dokchitser and V. Dokchitser, *Elliptic curves with all quadratic twists of positive rank*, Acta Arith. 137 (2009), 193-197.

[D-D2] T. Dokchitser and V. Dokchitser, *Root numbers of elliptic curves in residue characteristic 2*, Bull. Lond. Math. Soc. 40 (2008), 516-524.

[K] S. Kobayashi, *The local root number of elliptic curves with wild ramification*, Math. Ann. 323 (2002), 609-623.

[R] D. Rohrlich, *Variation of the root number in families of elliptic curves*, Compositio Math. 87 (1993) 119-151.

Department of Mathematics, Seoul National University, Seoul, 151-747, Republic of Korea

E-mail address: dhbyeon@math.snu.ac.kr

Department of Mathematics, Seoul National University, Seoul, 151-747, Republic of Korea

E-mail address: na0@snu.ac.kr