SUMS OF TWO RATIONAL CUBES WITH MANY PRIME FACTORS

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Abstract. In this paper, we show that for any given integer $k \geq 2$, there are infinitely many cube-free integers n having exactly k prime divisors such that n is a sum of two rational cubes. This is a cubic analogue of the work of Tian [Ti], which proves that there are infinitely many congruent numbers having exactly k prime divisors for any given integer $k \geq 1$.

1. Introduction and results

Let n be a cube-free integer and $E_n : x^3 + y^3 = n$ the elliptic curve defined over $\mathbb{Q}$. Let $L_{E_n}(s)$ be the Hasse-Weil L-function of E_n and $w_n \in \{1, -1\}$ its root number. Then $L_{E_n}(s)$ satisfies the functional equation

$$N^{s/2}(2\pi)^{-s}\Gamma(s)L_{E_n}(s) = w_n N^{(2-s)/2}(2\pi)^{-(2-s)}\Gamma(2-s)L_{E_n}(2-s),$$

where N is the conductor of E_n whose divisors are 3 and primes $p \mid n$. The *analytic rank* of E_n is the order of vanishing at the central point $s = 1$ of $L_{E_n}(s)$. The functional equation implies that $w_n = 1$ if and only if the analytic rank of E_n is even. The Birch and Swinnerton-Dyer(BSD) conjecture states that the rank of the Mordell-Weil group $E_n(\mathbb{Q})$ is equal to the analytic rank of E_n. So the BSD conjecture implies that if $w_n = -1$, then n is a sum of two rational cubes.

The root number w_n can be computed by the following way, due to Birch and Stephens [BS],

$$w_n = \prod_{p \text{ prime}} w_n(p),$$

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where for $p \neq 3$,

$$w_n(p) = \begin{cases} -1 & \text{if } p \,|\, n \text{ and } p \equiv 2 \pmod 3 \\ 1 & \text{otherwise} \end{cases}$$

and for $p = 3$,

$$w_n(p) = \begin{cases} -1 & \text{if } n \equiv 0, \pm 2, \pm 4, \pmod 9 \\ 1 & \text{otherwise.} \end{cases}$$

In [Sa], Satgé proved that $n = 2p$, where p is a prime congruent to 2 mod 9 (so $w_n = -1$), and also $n = 2p^2$, where p is a prime congruent to 5 mod 9 (so $w_n = -1$), are sums of two rational cubes. Coward [Co] proved that $n = 25p$, where p is a prime congruent to 2 mod 9 (so $w_n = -1$), and also $n = 25p^2$, where p is a prime congruent to 5 mod 9 (so $w_n = -1$), are sums of two rational cubes. In [DV], Dasgupta and Voight proved that if p is a prime congruent to 4 or 7 mod 9 (so $w_p = -1$) and 3 is not a cube mod p, then p is a sum of two rational cubes. We note that there are infinitely many such p. Here we mention Sylvester's conjecture which asserts if p is a prime congruent to 4, 7 or 8 mod 9 (so $w_p = -1$), then p is a sum of two rational cubes. For more details and history on Sylvester's conjecture, see [DV].

In this paper, we prove the following theorem.

Theorem 1.1. *For any given integer $k \geq 2$ and $e \in \{1, -1\}$, there are infinitely many cube-free integers (in fact, square-free integers) n having exactly k prime divisors such that n is a sum of two rational cubes and $w_n = e$.*

In [Ti], Tian has shown that for any given integer $k \geq 1$, there are infinitely many square-free positive integers m such that m is a congruent number and the corresponding elliptic curve $E : y^2 = x^3 - m^2 x$ has the root number -1. So Theorem 1.1 for the case $w_n = -1$ is a cubic analogue of the work of Tian.

On the other hand, Coates and Wiles [CW] proved that if n is a sum of two rational cubes, than the analytic rank of E_n is greater than zero. So we immediately have the following corollary from Theorem 1.1 for the case $w_n = 1$.

Corollary 1.2. *For any given integer $k \geq 2$, there are infinitely many cube-free integers n having exactly k prime divisors such that the analytic rank of E_n is at least 2.*

In [Ma], Mai proved that there are infinitely many cube-free integers n such that the analytic rank of E_n is at least 2, more precisely, the number of cube-free integers $n \leq X$ such that the analytic rank of E_n is even(i.e., $w_n = 1$) and ≥ 2 is at least $CX^{2/3-\epsilon}$, where ϵ is arbitrarily small and C is a positive constant, for X large enough, without consideration of the number of prime divisors of n.

2. Preliminaries

Let n be a cube-free integer and $E_n : x^3 + y^3 = n$. Then E_n has the Weierstrass form $E'_n : y^2 = x^3 - 2^4 3^3 n^2$. We know that all E'_n except for $n = \pm 1$ and ± 2 have no rational torsion(cf. [Si, Exercises 10.19]). In [Ma, Lemma 2.1], Mai proved the following lemma.

Lemma 2.1. *E'_n has integral points if and only if n has one of the following six forms:*

$$n = \pm \frac{b(a^2 - b^2)}{4} \text{ or } n = \pm \frac{3a^2 b - 3b^3}{24} \pm \frac{a^3 - 9ab^2}{24} \text{ for some } a, b \in \mathbb{Z}.$$

To control the root number w_n and the number of prime divisors of n in Lemma 2.1, we slightly modify the result of Brüdern, Kawada and Wooley [BKW, Theorem 1], which is a quantitative strengthening of a theorem of Perelli [Pe].

Lemma 2.2. *Let $f(x) \in \mathbb{Z}[x]$ be a polynomial which has a positive leading coefficient with degree k. Let A, B be relatively prime odd integers and i, j positive integers with $0 < i, j < 9$ and $(i, 9) = (j, 9) = 1$. Suppose there is at least one integer m such that*

$$2f(m) \equiv Ai + Bj \pmod 9 \text{ and } (AB, 2f(m)) = 1.$$

Let $\mathcal{E}_k^{ABij}(N, f)$ be the number of positive integers $n \in [1, N]$ with $2f(n) \equiv Ai + Bj \pmod 9$ and $(AB, f(n)) = 1$ for which the equation $2f(n) = Ap_1 + Bp_2$ has no solution in primes $p_1 \equiv i, p_2 \equiv j \pmod 9$. Then there is an *absolute constant $c > 0$ such that*

$$\mathcal{E}_k^{ABij}(N, f) \ll_f N^{1-\frac{c}{k}},$$

so there are infinitely many integers n such that

$$2f(n) = Ap_1 + Bp_2,$$

for some primes $p_1 \equiv i$ and $p_2 \equiv j \pmod 9$.

Proof. Let N be a large positive integer, δ a sufficiently small positive real number to be chosen later, $X := 2f(N)$, $P := X^{6\delta}$, $Q := X/P$ and $\kappa := 2^{-\frac{1}{k}}$. Let A, B be positive odd integers and i, j positive integers with $0 < i, j < 9$ and $(i, 9) = (j, 9) = 1$. We define the exponential sum $S_i(\alpha)$ by

$$S_i(\alpha) := \sum_{\substack{P<p\le X \\ p\equiv i \pmod 9}} (\log p)e(\alpha p),$$

where $e(\alpha p) := e^{2\pi\alpha p i}$ and the summation is over primes p with $P < p \le X$ and $p \equiv i \pmod 9$. When $T \subseteq [0, 1]$, we write

$$r_{ABij}(n; T) := \int_T S_i(A\alpha)S_j(B\alpha)e(-\alpha n)d\alpha$$

and $r_{ABij}(n) := r_{ABij}(n; [0, 1])$. Then $r_{ABij}(2f(n))$ counts the number of solutions of the equation $2f(n) = Ap_1 + Bp_2$ in primes $p_1 \equiv i$, $p_2 \equiv j \pmod 9$ with weight $\log p_1 \log p_2$.

Let $\mathfrak{M} \subset [0, 1]$ be the major arc defined by

$$\mathfrak{M} = \bigcup_{\substack{0\le a\le q\le P \\ (a,q)=1}} \mathfrak{M}(q, a),$$

where

$$\mathfrak{M}(q, a) = \left\{ \alpha \in [0, 1] : \left|\alpha - \frac{a}{q}\right| \le \frac{P}{qX} \right\}$$

and $\mathfrak{m} \subset [0, 1]$ be the minor arc defined by

$$\mathfrak{m} = [0, 1] \setminus \mathfrak{M}.$$

First we consider minor arc. Let χ be a Dirichlet character of modulus 9. The orthogonality relations of Dirichlet characters $\sum_\chi \bar{\chi}(i)\chi(p) =$

$\varphi(9)\delta(i,p)$, where the sum is over all Dirichlet characters of modulus 9, and $\delta(i,p) = 1$ if $p \equiv i \pmod 9$ and $\delta(i,p) = 0$ if $p \not\equiv i \pmod 9$ imply that

$$\begin{aligned} S_i(A\alpha) &= \sum_{P<p\le X} \frac{1}{\varphi(9)} \sum_{\chi} \bar{\chi}(i)\chi(p)(\log p)e(\alpha Ap) \\ &\ll \sum_{\chi} \left| \sum_{P<p\le X} \chi(p)(\log p)e(\alpha Ap) \right|. \end{aligned}$$

From the proof of [Va, Theorem 3.1], we have *if* $(a,q) = 1$, $q \le X$ *and* $|\alpha - a/q| \le q^{-2}$, *then*

$$\sum_{P<p\le X} \chi(p)(\log p)e(\alpha Ap) \ll (\log X)^4(Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2}),$$

so

$$S_i(A\alpha) \ll (\log X)^4(Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2}).$$

Using this upper bound and the proof of [BKW, Lemma 1], we have the following same result for $r_{ABij}(m)$ on minor arcs;

There is a positive real number $a = a(\delta)$ *such that*

$$\sum_{\kappa N<n\le N} |r_{ABij}(2f(n); \mathfrak{m})| \ll XN^{1-\frac{a}{k}}. \tag{1}$$

Now we consider major arc. For a Dirichlet character χ, we define

$$\psi(x,\chi) := \sum_{n\le x} \chi(n)\Lambda(n) \ \text{ and } \ \psi(x,\chi,i) := \sum_{\substack{n\le x \\ n\equiv i \pmod 9}} \chi(n)\Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function which defined as follows;

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{otherwise.} \end{cases}$$

Using the orthogonality relations of Dirichlet characters, we have

$$\psi(x,\chi,i) = \frac{1}{\varphi(9)} \sum_{n\le x} (\sum_{\chi'} \bar{\chi}'(i)\chi'(n))\chi(n)\Lambda(n) = \frac{1}{\varphi(9)} \sum_{\chi'} \bar{\chi}'(i)\psi(x, \chi\cdot\chi'),$$

where χ' varies in the set of Dirichlet characters of modulus 9 and $\chi\cdot\chi'(n) := \chi(n)\chi'(n)$. From the proof of [Ga, Theorem 7], we have for $q \le T \le x^{\frac{1}{2}}$,

$$\psi(x,\chi,i) = \frac{1}{\varphi(9)} \sum_{\chi'} \bar{\chi}'(i)(\delta_{\chi\cdot\chi'}x - \sum_{\rho} \frac{x^\rho}{\rho}) + O(\frac{x\log^2 x}{T})$$

and

$$\sum_{q\le P}\sum_{\chi}{}^{*}\sum_{\substack{x \\ p\equiv i \pmod 9}}^{x+h}\chi(p)\log p \ll h(\sum_{q\le P}\sum_{\chi}{}^{*}\sum_{\chi'}\sum_{\rho}x^{\beta-1}+\frac{P^4}{T}),$$

where $\rho=\beta+\gamma i$ varies in the set of zeros of $L(s,\chi\cdot\chi')$ with $0\le \mathrm{Re}(\rho)\le 1$, $|\mathrm{Im}(\rho)|\le T$ and $\sum^*$ denotes that the sum is taken over all primitive Dirichlet characters of modulus q. From the proof of [Ga, Theorem 7], additional computations and the argument below [MV, Lemma 4.3], we have the following modification of [MV, Lemma 4.3];

For suitable (small) positive absolute constants c_1, c_2,

$$\sum_{q\le P}\sum_{\chi}{}^{*}\max_{x\le N}\max_{h\le N}(h+\frac{N}{P})^{-1}\Big|\sum_{\substack{x-h \\ p\equiv i \\ (\mathrm{mod}\, 9)}}^{x}{}^{\sharp}\chi(p)\log p\Big| \ll \exp(-c_1\frac{\log N}{\log P}) \qquad (2)$$

provided $\exp(\log^{\frac12} N)\le P\le N^{c_2}$. *Here* $\sum^{\sharp}$ *indicates that the term with* $q=1$ *is to be*

$$\sum_{\substack{x-h \\ p\equiv i \pmod 9}}^{x}\log p-\sum_{\substack{x-h<n\le x \\ n>0 \\ n\equiv i \pmod 9}}1$$

and that if there is an exceptional character $\tilde{\chi}$ *then the corresponding term is*

$$\sum_{\substack{x-h \\ p\equiv i \pmod 9}}^{x}\tilde{\chi}(p)\log p+\sum_{\substack{x-h<n\le x \\ n>0 \\ n\equiv i \pmod 9}}n^{\tilde{\beta}-1},$$

where $\tilde{\beta}$ *is the (unique) exceptional zero of* $L(s,\tilde{\chi})$. *If the exceptional character occurs, the right hand side of (2) may be reduced by a factor of* $(1-\tilde{\beta})\log P$.

For a Dirichlet character χ of modulus q, we define

$$S_i(\chi,\eta):=\sum_{\substack{P<p\le X \\ p\equiv i \pmod 9}}(\log p)\chi(p)e(p\eta),$$

and

$$T_i(\eta):=\sum_{\substack{P<n\le X \\ n\equiv i \pmod 9}}e(n\eta), \qquad \widetilde{T}_i(\eta):=-\sum_{\substack{P<n\le X \\ n\equiv i \pmod 9}}n^{\tilde{\beta}-1}e(n\eta),$$

where the last one is defined only if there is an exceptional character.

Let χ_0 be the principal character modulo q. Define

$$W_i(\chi,\eta) := \begin{cases} S_i(\chi,\eta) - T_i(\eta) & \text{if } \chi = \chi_0, \\ S_i(\chi,\eta) - \widetilde{T}_i(\eta) & \text{if } \chi = \tilde{\chi}\chi_0, \\ S_i(\chi,\eta) & \text{otherwise.} \end{cases}$$

Suppose that a Dirichlet character $\chi \pmod q$ is induced by a primitive character $\chi^* \pmod r$. Put

$$W_i^A(\chi) = \Big(\int_{-\frac{1}{rQ}}^{\frac{1}{rQ}} |W_i(\chi, A\eta)|^2\, d\eta\Big)^{\frac{1}{2}} \qquad \text{and} \qquad W_i^A = \sum_{q\le P}\sum_{\chi}{}^{*} W_i^A(\chi).$$

Applying [MV, Lemma 4.2] to the real numbers

$$u_n := \begin{cases} \chi(p)\log p & \text{if } n = Ap,\ P < p \le X,\ p \equiv i \pmod 9, \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$\begin{aligned} W_i^A(\chi) &\ll \Big(\int_0^{2AX} \Big|\frac{1}{qQ} \sum_{\substack{P<p\le X \\ x-\frac{qQ}{2}\le Ap\le x \\ p\equiv i \pmod 9}} {}^{\sharp}\chi(p)\log p\Big|^2 dx\Big)^{\frac{1}{2}} \\ &\ll X^{\frac{1}{2}} \max_{x\le 2X}\max_{0<h\le X} (h + \frac{X}{P})^{-1} \Big|\sum_{\substack{x-h \\ p\equiv i \\ (\text{mod } 9)}}^{x} {}^{\sharp}\chi(p)\log p\Big|. \end{aligned}$$

Then using the above modification of [MV, Lemma 4.3], we have the following modification of [MV, (7.1) and $(7.\tilde{1})$];

If there is no exceptional character,

$$W_i^A \ll X^{\frac{1}{2}} \exp(-c_3 \frac{\log X}{\log P}),$$

and if the exceptional character occurs,

$$W_i^A \ll X^{\frac{1}{2}} (1-\tilde{\beta}) \log P \exp(-c_3 \frac{\log X}{\log P}).$$

For $\alpha \in \mathfrak{M}(q,a)$ we write $\alpha = \frac{a}{q} + \eta$ for $(a,q) = 1$, $-\frac{1}{qQ} \le \eta < \frac{1}{qQ}$ and $q < P$. For a character χ of modulus q, let $\tau(\chi) = \sum_{n=1}^{q} \chi(n) e(\frac{n}{q})$ be the Gaussian sum. For integers $C, D \in \{A, B, q, n, 2f(n)\}$, we define $C_D := \frac{C}{(C,D)}$. Using arguments in [MV, Section 6], we have

$$S_i(A\alpha) = \frac{\mu(q_A)}{\varphi(q_A)} T_i(A\eta) + \frac{1}{\varphi(q_A)} \sum_{\chi} \chi(A_q a)\tau(\bar{\chi}) W_i(\chi, A\eta), \tag{3}$$

where the sum is over all Dirichlet characters χ of modulus q_A, unless there is an exceptional character of modulus $\tilde{r}$, in which case $\tilde{r}|q_A$ then we obtain an additional term

$$\frac{\tilde{\chi}(A_q a)\tau(\tilde{\chi}\chi_0)}{\varphi(q_A)}\widetilde{T}_i(A\eta)$$

on the right hand side of (3).

First we assume that there is no exceptional character. Let $n \in (\kappa N, N]$ be an integer with $2f(n) \equiv Ai + Bj \pmod 9$ and $(AB, f(n)) = 1$. For simplicity, we define

$$\begin{aligned}
t_i^A t_j^B(\eta) &:= T_i(A\eta)T_j(B\eta)e(-2f(n)\eta),\\
t_i^A w_j^B(\eta) &:= T_i(A\eta)W_j(\chi', B\eta)e(-2f(n)\eta),\\
t_j^B w_i^A(\eta) &:= T_j(B\eta)W_i(\chi, A\eta)e(-2f(n)\eta),\\
w_i^A w_j^B(\eta) &:= W_i(\chi, A\eta)W_j(\chi', B\eta)e(-2f(n)\eta),
\end{aligned}$$

where χ and χ' are characters of modulus of q_A and q_B, respectively. Then we have

$$\begin{aligned}
& r_{ABij}(2f(n); \mathfrak{M})\\
= & \sum_{q\le P}\frac{\mu(q_A)\mu(q_B)}{\varphi(q_A)\varphi(q_B)}c_q(-2f(n))\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} t_i^A t_j^B(\eta)d\eta && (4)\\
+ & \sum_{q\le P}\frac{\mu(q_A)}{\varphi(q_A)\varphi(q_B)}\sum_{\chi'}\chi'(B_q)c_{\chi'}(-2f(n))\tau(\bar{\chi'})\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} t_i^A w_j^B(\eta)d\eta && (5)\\
+ & \sum_{q\le P}\frac{\mu(q_B)}{\varphi(q_A)\varphi(q_B)}\sum_{\chi}\chi(A_q)c_{\chi}(-2f(n))\tau(\bar{\chi})\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} t_j^B w_i^A(\eta)d\eta && (6)\\
+ & \sum_{q\le P}\frac{1}{\varphi(q_A)\varphi(q_B)}\Big(\sum_{\chi,\chi'}\chi(A_q)\chi'(B_q)c_{\chi\chi'}(-2f(n))\tau(\bar{\chi})\tau(\bar{\chi'})\\
\times & \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} w_i^A w_j^B(\eta)d\eta\Big), && (7)
\end{aligned}$$

where $c_q(m) = \sum_{(a,q)=1}^{q} e(\frac{am}{q})$ and $c_*(m) := \sum_{h=1}^{q} *(h)e(\frac{hm}{q})$ (We remark that h goes from 1 to q though the modulus of $*$ is a divisor of q.).

Using [MV, Lemma 5.5] and arguments in [MV, Section 6], we have

$$\begin{aligned}
(5) \ll\ & X^{\frac{1}{2}}\sum_{q\le P}\sum_{\chi'}\Big|\frac{\mu(q_A)\chi'(B_q)c_{\chi'}(-2f(n))\tau(\bar{\chi'})}{\varphi(q_A)\varphi(q_B)}\Big|\Big(\int_{-\frac{1}{rQ}}^{\frac{1}{rQ}} |W_j(\chi', B\eta)|^2 d\eta\Big)^{\frac{1}{2}}\\
\ll\ & X^{\frac{1}{2}}\sum_{q\le P}\sum_{\chi'}\Big|\frac{c_{\chi'}(-2f(n))\tau(\bar{\chi'})}{\varphi(q)^2}\Big|\Big(\int_{-\frac{1}{rQ}}^{\frac{1}{rQ}} |W_j(\chi', B\eta)|^2 d\eta\Big)^{\frac{1}{2}}\\
\ll\ & \frac{2f(n)}{\varphi(2f(n))}W_j^B X^{\frac{1}{2}}.
\end{aligned}$$

By the same method, we have

$$(6) \ll \frac{2f(n)}{\varphi(2f(n))} W_i^A X^{\frac{1}{2}} \quad \text{and} \quad (7) \ll \frac{2f(n)}{\varphi(2f(n))} W_i^A W_j^B.$$

Now we consider the term (4). Assume harmless conditions $qQ > 18A$ and $A \geq B$. Then we have

$$\begin{aligned} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} t_i^A t_j^B(\eta) d\eta &= \int_{-\frac{1}{18}}^{\frac{1}{18}} t_i^A t_j^B(\eta) d\eta - \int_{\frac{1}{18A}}^{\frac{1}{18}} t_i^A t_j^B(\eta) d\eta - \int_{-\frac{1}{18}}^{-\frac{1}{18A}} t_i^A t_j^B(\eta) d\eta \\ &- \int_{\frac{1}{qQ}}^{\frac{1}{18A}} t_i^A t_j^B(\eta) d\eta - \int_{-\frac{1}{18A}}^{-\frac{1}{qQ}} t_i^A t_j^B(\eta) d\eta. \end{aligned}$$

By the same argument for [MV, (6.10)], we have

$$\int_{\frac{1}{qQ}}^{\frac{1}{18A}} t_i^A t_j^B(\eta) d\eta \ll qQ.$$

By elementary computation, we have

$$\int_{-\frac{1}{18}}^{\frac{1}{18}} t_i^A t_j^B(\eta) d\eta = \sum_{\substack{P<k,l\leq X \\ k\equiv i, l\equiv j \\ Ak+Bl=2f(n)}} \frac{1}{9} = \frac{2f(n)}{9^3 AB} + O(P)$$

and

$$\begin{aligned} \int_{\frac{1}{18A}}^{\frac{1}{18}} t_i^A t_j^B(\eta) d\eta &= \sum_{\substack{P<k,l\leq X \\ k\equiv i, l\equiv j \\ Ak+Bl=2f(n)}} \left(\frac{1}{18} - \frac{1}{18A}\right) + O(\log X) \\ &= \left(\frac{1}{18} - \frac{1}{18A}\right) \frac{2f(n)}{9^2 AB} + O(P). \end{aligned}$$

Thus we have

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} t_i^A t_j^B(\eta) d\eta = \frac{2f(n)}{9^3 AB} - 2\left(\frac{1}{18} - \frac{1}{18A}\right) \frac{2f(n)}{9^2 AB} + O(qQ).$$

Using the above estimation for the integral in (4) and arguments for [MV, (6.12), (6.13), (6.14)], we have the the following estimation for the term (4);

$$(4) = \mathfrak{S}_{A,B}(2f(n)) \frac{2f(n)}{9^3 A^2 B} + O(X^{1+\delta} P^{-1}),$$

where $\mathfrak{S}_{A,B}(n) = \sum_{q=1}^{\infty} \frac{\mu(q_A)\mu(q_B)}{\varphi(q_A)\varphi(q_B)} c_q(-n)$.

Finally, combining the above bounds for (5), (6), (7) and the above estimation for (4), we have the following modification of [MV, (6.17)];

$$\begin{aligned} r_{ABij}(2f(n);\mathfrak{M}) &= \mathfrak{S}_{A,B}(2f(n))\frac{2f(n)}{9^3A^2B} + O(X^{1+\delta}P^{-1}) \\ &+ O(\frac{2f(n)}{\varphi(2f(n))}(W_i^A X^{\frac{1}{2}} + W_j^B X^{\frac{1}{2}} + W_i^A W_j^B)). \quad (8) \end{aligned}$$

Since A, B and $2f(n)$ are pairwise relatively prime, we have

$$\begin{aligned} \mathfrak{S}_{A,B}(2f(n)) &= \prod_p (1 + \frac{\mu(p_A)\mu(p_B)\mu(p_{(2f(n))})\varphi(p)}{\varphi(p_A)\varphi(p_B)\varphi(p_{(2f(n))})}) \\ &= \prod_{p|AB(2f(n))} (1 + \frac{1}{\varphi(p)}) \prod_{p\nmid AB(2f(n))} (1 - \frac{1}{\varphi(p)^2}) \end{aligned}$$

and by [BKW, (15)]

$$\mathfrak{S}_{A,B}(2f(n)) = c_{A,B}\mathfrak{S}_{1,1}(2f(n)) \geq c_{A,B}\frac{2f(n)}{\varphi(2f(n))}$$

for a constant $c_{A,B}$ depending only on A, B. From the above modification of [MV, (7.1)], the third term of the right hand side of (8) is less then $\frac{6f(n)}{\varphi(2f(n))}Xe^{-\frac{c_3}{6\delta}}$. If we choose a sufficiently small positive real number δ, then $r_{ABij}(2f(n);\mathfrak{M}) \geq (\frac{c_{A,B}}{9^3A^2B} - c_4)\mathfrak{S}(2f(n))(2f(n))$. This implies that $r_{ABij}(2f(n);\mathfrak{M}) \gg X$.

Next we assume that there is the exceptional character. Let $n \in (\kappa N, N]$ be an integer with $2f(n) \equiv Ai + Bj \pmod 9$ and $(AB, f(n)) = 1$. For simplicity, we define

$$\begin{aligned} \widetilde{t}_i^A\widetilde{t}_j^B(\eta) &:= \widetilde{T}_i(A\eta)\widetilde{T}_j(B\eta)e(-2f(n)\eta), \\ t_i^A\widetilde{t}_j^B(\eta) &:= T_i(A\eta)\widetilde{T}_j(B\eta)e(-2f(n)\eta), \\ \widetilde{t}_i^A t_j^B(\eta) &:= \widetilde{T}_i(A\eta)T_j(B\eta)e(-2f(n)\eta), \\ \widetilde{t}_i^A w_j^B(\eta) &:= \widetilde{T}_i(A\eta)W_j(\chi', B\eta)e(-2f(n)\eta), \\ \widetilde{t}_j^B w_i^A(\eta) &:= \widetilde{T}_j(B\eta)W_i(\chi, A\eta)e(-2f(n)\eta), \end{aligned}$$

where χ and χ' are characters of modulus of q_A and q_B, respectively. Then we have the following possible additional terms in $r_{ABij}(2f(n);\mathfrak{M})$;

$$\sum_{\substack{q\le P\\ \tilde{r}|q_A,q_B}} \frac{\tau(\tilde{\chi}\chi_0)\tau(\tilde{\chi}\chi_0')}{\varphi(q_A)\varphi(q_B)}\tilde{\chi}(A_qB_q)c_q(-2f(n))\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \widetilde{t}_i^A\widetilde{t}_j^B(\eta)d\eta \tag{9}$$

$$+\sum_{\substack{q\le P\\ \tilde{r}|q_B}} \frac{\mu(q_A)\tau(\tilde{\chi}\chi_0')}{\varphi(q_A)\varphi(q_B)}\tilde{\chi}(B_q)c_{\tilde{\chi}\chi_0'}(-2f(n))\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} t_i^A\widetilde{t}_j^B(\eta)d\eta \tag{10}$$

$$+\sum_{\substack{q\le P\\ \tilde{r}|q_A}} \frac{\mu(q_B)\tau(\tilde{\chi}\chi_0)}{\varphi(q_A)\varphi(q_B)}\tilde{\chi}(A_q)c_{\tilde{\chi}\chi_0}(-2f(n))\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \widetilde{t}_i^A t_j^B(\eta))d\eta \tag{11}$$

$$+\sum_{\substack{q\le P\\ \tilde{r}|q_A}} \frac{\tilde{\chi}(B_q)\tau(\tilde{\chi}\chi_0)}{\varphi(q_A)\varphi(q_B)}(\sum_{\chi} c_{\tilde{\chi}\chi}(-2f(n))\tau(\bar{\chi})\chi(A_q)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \widetilde{t}_j^B w_i^A(\eta)d\eta) \tag{12}$$

$$+\sum_{\substack{q\le P\\ \tilde{r}|q_B}} \frac{\tilde{\chi}(A_q)\tau(\tilde{\chi}\chi_0')}{\varphi(q_A)\varphi(q_B)}(\sum_{\chi'} c_{\tilde{\chi}\chi'}(-2f(n))\tau(\bar{\chi'})\chi'(B_q)\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \widetilde{t}_i^A w_j^B(\eta)d\eta). \tag{13}$$

By the same arguments for (5) and (6), we have

$$(12) \ll \frac{2f(n)}{\varphi(2f(n))}W_i^A X^{\frac{1}{2}} \quad \text{and} \quad (13) \ll \frac{2f(n)}{\varphi(2f(n))}W_j^B X^{\frac{1}{2}}.$$

Now we consider the first three terms (9), (10), (11). For the integral in (9), we have

$$\int_{\frac{1}{18A}}^{\frac{1}{18}} \widetilde{t}_i^A\widetilde{t}_j^B(\eta)d\eta = \sum_{\substack{P<k,l\le X\\ k\equiv i,l\equiv j\\ Ak+Bl=2f(n)}} (kl)^{\tilde{\beta}-1}(\frac{1}{18}-\frac{1}{18A}) + O(\log X)$$

and by the same argument for the estimation of the integral in (4), we have

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \widetilde{t}_i^A\widetilde{t}_j^B(\eta)d\eta = \widetilde{I}_{ij}^{AB} - 2\sum_{\substack{P<k,l\le X\\ k\equiv i,l\equiv j\\ Ak+Bl=2f(n)}} (kl)^{\tilde{\beta}-1}(\frac{1}{18}-\frac{1}{18A}) + O(qQ),$$

where

$$\widetilde{I}_{ij}^{AB} := \int_{-\frac{1}{18}}^{\frac{1}{18}} \widetilde{t}_i^A\widetilde{t}_j^B(\eta)d\eta = \sum_{\substack{P<k,l\le X\\ k\equiv i,l\equiv j\\ Ak+Bl=2f(n)}} (kl)^{\tilde{\beta}-1}(\frac{1}{9}).$$

Similarly, for the integral in (10), we have

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} t_i^A \widetilde{t}_j^B(\eta)d\eta = \widetilde{J}_{ij}^{AB} - 2 \sum_{\substack{P<k,l\le X \\ k\equiv i, l\equiv j \\ Ak+Bl=2f(n)}} (l)^{\tilde{\beta}-1}(\frac{1}{18} - \frac{1}{18A}) + O(qQ),$$

where $\widetilde{J}_{ij}^{AB} := \int_{-\frac{1}{18}}^{\frac{1}{18}} t_i^A \widetilde{t}_j^B(\eta)d\eta$ and for the integral in (11), we have

$$\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \widetilde{t}_i^A t_j^B(\eta)d\eta = \widetilde{J}_{ji}^{BA} - 2 \sum_{\substack{P<k,l\le X \\ k\equiv i, l\equiv j \\ Ak+Bl=2f(n)}} (k)^{\tilde{\beta}-1}(\frac{1}{18} - \frac{1}{18A}) + O(qQ),$$

where $\widetilde{J}_{ji}^{BA} := \int_{-\frac{1}{18}}^{\frac{1}{18}} \widetilde{t}_i^A t_j^B(\eta)d\eta$.

Using the arguments for [MV, (6.19), (6.$\tilde{1}$6)], we have

$$\begin{aligned}(9)+(10)+(11) &= \widetilde{\mathfrak{S}}_{A,B}(2f(n))\frac{\widetilde{I}_{ij}^{AB}}{A} \\ &+ O(\frac{\tilde{\chi}(2f(n))^2\tilde{r}\cdot 2f(n)X}{\varphi(\tilde{r})^2\varphi(2f(n))}) + O(X^{1+\delta}P^{-1}(2f(n),\tilde{r})),\end{aligned}$$

where $\widetilde{\mathfrak{S}}_{A,B}(n) := \sum_{\substack{q=1 \\ \tilde{r}|q_A,q_B}}^{\infty} \frac{\tau(\tilde{\chi}\chi_0)\tau(\tilde{\chi}\chi_0')}{\varphi(q_A)\varphi(q_B)}\tilde{\chi}(A_qB_q)c_q(-n)$. Since A, B and $2f(n)$ are pairwise relatively prime and one of the Gaussian sums in $\widetilde{\mathfrak{S}}_{A,B}(n)$ vanishes when $q/\tilde{r}$ and $\tilde{r}$ are not relatively prime, we have

$$\begin{aligned}&\widetilde{\mathfrak{S}}_{A,B}(2f(n)) \\ = &\frac{\tilde{\chi}(-1)\tilde{r}}{\varphi(\tilde{r})\varphi(\tilde{r}_{2f(n)})} \prod_{\substack{p\nmid\tilde{r} \\ p|AB}} (1+\frac{\tilde{\chi}(p)}{\varphi(p)}) \prod_{\substack{p\nmid\tilde{r} \\ p|2f(n)}} (1+\frac{1}{\varphi(p)}) \prod_{\substack{p\nmid\tilde{r} \\ p\nmid(2f(n))AB}} (1-\frac{1}{\varphi(p)^2}).\end{aligned}$$

Finally, combining all the above estimations, we have the following modification of [MV, (6.$\tilde{1}$7)];

$$\begin{aligned}r_{ABij}(2f(n);\mathfrak{M}) &= \mathfrak{S}_{A,B}(2f(n))\frac{2f(n)}{9^3A^2B} + \widetilde{\mathfrak{S}}_{A,B}(2f(n))\frac{\widetilde{I}_{ij}^{AB}}{A} \\ &+ O(\frac{\tilde{\chi}(2f(n))^2\tilde{r}\cdot 2f(n)X}{\varphi(\tilde{r})^2\varphi(2f(n))}) + O(X^{1+\delta}P^{-1}(2f(n),\tilde{r})) \\ &+ O(\frac{2f(n)}{\varphi(2f(n))}(W_i^AX^{\frac{1}{2}} + W_j^BX^{\frac{1}{2}} + W_i^AW_j^B)). \quad (14)\end{aligned}$$

If $(2f(n), \tilde{r}) = 1$, the fourth term of the right hand side of (14) is less then $X^{1-5\delta}$ and

$$\widetilde{\mathfrak{S}}_{A,B}(2f(n)) \ll \frac{\tilde{r}}{\varphi(\tilde{r})^2} \prod_{\substack{p\nmid \tilde{r} \\ p|2f(n)}} (1 + \frac{1}{\varphi(p)}) \ll \frac{\tilde{r} \cdot 2f(n)}{\varphi(\tilde{r})^2 \varphi(2f(n))} = o(1),$$

so using arguments in [MV, Section 8] and the above modification of [MV, (7.$\tilde{1}$)], we have $r_{ABij}(2f(n); \mathfrak{M}) \gg X$. Since

$$|\widetilde{\mathfrak{S}}_{A,B}(2f(n))| \quad \leq \quad \mathfrak{S}_{A,B}(2f(n)) \prod_{\substack{p\nmid \tilde{r} \\ p|AB \\ \tilde{\chi}(p)=-1}} (\frac{p-2}{p}) \prod_{\substack{p|\tilde{r} \\ p\nmid 2f(n)AB}} \frac{1}{(p-2)}, \quad (15)$$

using arguments in [BKW, p.122–123], we have that if $1 < (2f(n), \tilde{r}) \leq Y$,

$$r_{ABij}(2f(n); \mathfrak{M}) \gg \begin{cases} X & \text{if } \mathcal{P} \neq \varnothing, \\ XY^{-\frac{1}{2}}(\log X)^{-1} & \text{if } \mathcal{P} = \varnothing, \end{cases}$$

where $\mathcal{P}$ is the set of primes in the products of (15). Finally there are at most $O(N^{1+\epsilon}Y^{-1})$ possible exceptions n with $(2f(n), \tilde{r}) > Y$. Thus we have the following analogue of [BKW, Lemma 2];

Suppose that Y is a real number with $1 \leq Y \leq X^{\frac{\delta}{k}}$. Then one has

$$r_{ABij}(2f(n); \mathfrak{M}) \gg XY^{-\frac{1}{2}}(\log X)^{-1} \quad (16)$$

for all $n \in (\kappa N, N]$ with $2f(n) \equiv Ai + Bj \pmod 9$ and $(AB, f(n)) = 1$ with the possible exception of $O(N^{1+\epsilon}Y^{-1})$ values of n.

We note that if there is at least one integer m such that $2f(m) \equiv Ai + Bj \pmod 9$ and $(AB, 2f(m)) = 1$, the set of $n \in (\kappa N, N]$ with $2f(n) \equiv Ai + Bj \pmod 9$ and $(AB, f(n)) = 1$ has a positive density in the set of $n \in (\kappa N, N]$. Now Lemma 2.2 follows from (1), (16) and the proof of [BKW, Theorem 1]. □

3. Proof of Theorem 1.1

In this section, for convenience' sake, we assume that n is a square-free. However, the method used in this section can easily be modified to cube-free integers n though it is more complicated to state.

Lemma 3.1. *Let $n > 0$ be a square-free integer. Then*

$$w_n = \begin{cases} +1 & \text{if } n \equiv 1, 2, 3 \text{ or } 5 \pmod 9 \\ -1 & \text{if } n \equiv 4, 6, 7 \text{ or } 8 \pmod 9. \end{cases}$$

Proof. Write the prime factorization of n in the form

$$n = 3^\alpha \prod_{p_i \equiv 1 \pmod 3} p_i \prod_{q_j \equiv 2 \pmod 3} q_j,$$

where $\alpha = 0$ or 1 and p_i, q_j are distinct primes.

Let a_n be the number of $q_j \equiv 2 \pmod 3$. Then the computation of root number w_n of E_n in Section 1 gives the following condition.

$$w_n = -1 \text{ if and only if } \begin{array}{ll} (i) & n \equiv \pm 1, \pm 3 \pmod 9 \text{ and } a_n \text{ is odd, or} \\ (ii) & n \equiv \pm 2, \pm 4 \pmod 9 \text{ and } a_n \text{ is even.} \end{array}$$

We note that $(\prod_{p_i \equiv 1 \pmod 3} p_i \prod_{q_j \equiv 2 \pmod 3} q_j) \equiv 2 \pmod 3$ if and only if a_n is odd. Then we have the following table and complete the proof of the lemma.

$n \pmod 9$	1	2	3	4	5	6	7	8
a_n	even	odd	even	even	odd	odd	even	odd
w_n	$+1$	$+1$	$+1$	-1	$+1$	-1	-1	-1

□

Proposition 3.2. *For any given $k \geq 2$ and $r \in \{1, 2, 4, 5, 7, 8\}$, there are infinitely many square-free integer $n > 0$ having exactly k prime divisors such that n is a sum of two rational cubes and $n \equiv r \pmod 9$. For $r \in \{3, 6\}$, the same statement holds for $k \geq 3$.*

Proof. By Lemma 2.1, we know that for nonzero $a, b \in \mathbb{Z}$, $16b^6 - a^2$ is a sum of two rational cubes because $b^3(16b^6 - a^2) = -\frac{(4b^3)(a^2-(4b^3)^2)}{4}$. Let

$$A = \prod_{i=1}^{l} p_i, \text{ for fixed primes } p_i \equiv 1 \pmod 9, \; B = 27,$$

where $l \geq 0$ is a fixed integer (if $l = 0$, then $A = 1$).

We note that $b^3 \equiv 0$ or $\pm 1 \pmod 9$ for any integer b. Since there is an integer b such that $8b^3 \equiv 8A + 8B \pmod 9$ and $(AB, 8b^3) = 1$, Lemma 2.2 ensures that there are infinitely many integers b that satisfy the equation

$$4b^3 = \frac{Ap + Bq}{2},$$

for some primes $p \equiv 8$ and $q \equiv 8 \pmod 9$. If $p = q$, then $8b^3 = Ap + 27p$, so $8p^2c^3 = A + 27$ for some positive integer c. Thus there are only finitely many p, q such that $p = q$ and we may assume $p \neq q$.

Let $a = \frac{Ap-Bq}{2} \in \mathbb{Z}$. Then $16b^6 - a^2 = ABpq = 27Apq$ is a sum of two rational cubes having exactly $(l+3)$ prime divisors because $p, q \nmid A, B$. Hence Apq is a square-free integer having exactly $(l+2)$ prime divisors such that Apq is sum of two rational cubes and $Apq \equiv 1 \pmod 9$. This proves the theorem for the case of $r = 1$. If we set $q \equiv 7, 5, 4$ and $2 \pmod 9$, then the theorem for the cases of $r = 2$, 4, 5, and 7 follows.

For the case $r = 8$ and $k \geq 3$, set

$$A = \prod_{i=1}^{l} p_i, \quad \text{for fixed primes } p_1 \equiv 2,\ p_2, \cdots, p_l \equiv 1 \,(\text{mod } 9),\ B = 27$$

and let p, q be primes such that $p \equiv 5,\ q \equiv 8 \,(\text{mod } 9)$. For the case $r = 8$ and $k = 2$, set

$$A = 1,\ B = 27$$

and let p, q be primes such that $p \equiv 8,\ q \equiv 1 \pmod 9$. Then the theorem for the case $r = 8$ follows.

For the case $r = 3$, let

$$A = \prod_{i=1}^{l} p_i, \ \text{for fixed primes } p_i \equiv 1 \pmod 9,\ B = 81,$$

where $l \geq 0$ is a fixed integer (if $l = 0$, then $A = 1$) and let p, q be primes such that $p \equiv 8,\ q \equiv 8 \pmod 9$ Then $3Apq$ is a square-free integer having exactly $(l+3)$ prime divisors such that $3Apq$ is sum of two rational cubes and $3Apq \equiv 3 \pmod 9$. This proves the theorem for the case of $r = 3$. Finally, if we set $q \equiv 7 \pmod 9$, then the theorem for the case $r = 6$ follows and the proof of the theorem is completed. □

Proof of Theorem 1.1. Lemma 3.1 and Proposition for the case $r = 4, 7, 8$ implies Theorem 1.1 for the case $w_n = -1$. Lemma 3.1 and Proposition for the case $r = 1, 2, 5$ implies Theorem 1.1 for the case $w_n = 1$. □

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