AN EXPLICIT LOWER BOUND FOR SPECIAL VALUES OF DIRICHLET L-FUNCTIONS

DONGHO BYEON AND JIGU KIM

Abstract. Let d be a fundamental discriminant, χ_d be the Dirichlet character associated to the quadratic field $\mathbb{Q}(\sqrt{d})$ and $L(s,\chi_d)$ be the Dirichlet L-function. In [Go], Goldfeld obtained an effective lower bound for $L(1,\chi_d)$ with uncalculated constants. For $d<0$, Oesterlé [Oe] computed the constants. However, for $d>0$, the constants are not computed yet. In this paper, we compute the constants for $d>0$ and give an explicit lower bound for $L(1,\chi_d)$ with $d>0$. Finally, as an application, we give an explicit lower bound for class numbers of certain real quadratic fields.

1. Introduction and results

Let d be a fundamental discriminant, χ_d be the Dirichlet character associated to the quadratic field $\mathbb{Q}(\sqrt{d})$ and $L(s,\chi_d)$ be the Dirichlet L-function. Dirichlet class number formula says

$$L(1,\chi_d)=\begin{cases}\frac{2\pi h(d)}{\omega\sqrt{|d|}} & \text{if } d<0,\\ \frac{2h(d)\log\epsilon_d}{\sqrt{d}} & \text{if } d>0,\end{cases}$$

where $h(d)$ is the class number of $\mathbb{Q}(\sqrt{d})$, ω the number of roots of unity in $\mathbb{Q}(\sqrt{d})$ $(d<0)$ and ϵ_d the fundamental unit of $\mathbb{Q}(\sqrt{d})$ $(d>0)$. Siegel [Si] proved that

$$L(1,\chi_d)>c(\epsilon)|d|^{-\epsilon}\ \ (\epsilon>0).$$

But there is no known method to compute the constant $c(\epsilon)>0$. In [Go], Goldfeld obtained an effective lower bound for $L(1,\chi_d)$.

Theorem 1.1. [Theorem 1, Go] *Let E be an elliptic curve over $\mathbb{Q}$ with conductor N. If E has complex multiplication and the L-function associated*

The authors were supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2016R1D1A1B03931404).

to E has a zero of order g at $s=1$, then for any χ_d with $(d,N)=1$ and $|d|>\exp\exp(c_1Ng^3)$, we have

$$L(1,\chi_d)>\frac{c_2}{g^{4g}N^{13}}\frac{(\log|d|)^{g-\mu-1}\exp(-21\sqrt{g\log\log|d|})}{\sqrt{|d|}},$$

where $\mu=1$ or 2 is suitably chosen so that $\chi_d(-N)=(-1)^{g-\mu}$, and the constants c_1, $c_2>0$ can be effectively computed and are independent of g, N and d.

In fact, Goldfeld proved Theorem 1.1 under assumption that the associated base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$. Thus the proof of Theorem 1.1 in [Go] also implies the following theorem (cf. [Theorem 1, Go1] and the remark below [Theorem 1, Go1]).

Theorem 1.2. *Let E be an elliptic curve over $\mathbb{Q}$ with conductor N and $g\geq 4$ be a positive integer. If E has complex multiplication and the associated base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$ at $s=1$, then for any such d with $(d,N)=1$ and $|d|>\exp\exp(c_1Ng^3)$, we have*

$$L(1,\chi_d)>\frac{c_2}{g^{4g}N^{13}}\frac{(\log|d|)^{g-3}\exp(-21\sqrt{g\log\log|d|})}{\sqrt{|d|}},$$

where the constants c_1, $c_2>0$ can be effectively computed and are independent of g, N and d.

In [Oe], Oesterlé explicitly computed the constants c_1, c_2 for $d<0$ in Theorem 1.1 or Theorem 1.2 and wrote the term $\exp(-21\sqrt{g\log\log|d|})$ as a simple product over primes dividing d. Finally he proved that for the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, $d<0$ and $(d,5077)=1$,

$$h(d)>\tfrac{1}{55}\log|d|\prod_{p|d,\,p\neq|d|}\left(1-\tfrac{\lfloor 2\sqrt{p}\rfloor}{p+1}\right).$$

However the constants c_1, c_2 for $d>0$ in Theorem 1.2 are not computed yet. In this paper, we will explicitly compute the constants c_1, c_2 for $d>0$ in Theorem 1.2 and prove the following theorem.

Theorem 1.3. *Let $d>0$ be a fundamental discriminant of a real quadratic field $\mathbb{Q}(\sqrt{d})$. Assume the same conditions as in Theorem 1.2. Then for any* *such d with $(d,N)=1$ and $d>\exp\exp(4000Ng^3)$, we have*

$$L(1,\chi_d) > \frac{10^{180}}{g^{4g}N^5}\cdot\frac{(\log d)^{g-3}\exp(-21\sqrt{g\log\log d})}{\sqrt{d}}.$$

Remark 1.4. *Let E be an elliptic curve with complex multiplication by an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-k})$. In the proof of Theorem 1.3, we use the fact that $-k$ is one of $-3,-4,-7,-8,-11,-19,-43,-67,-163$ (cf. [Example 11.3.1, Sil]), so $k\le 163$, instead of the fact $k\le N$ (because $k|N$), which is used in the proof of Theorem 1.1 or Theorem 1.2. That is why there is a difference for exponents of N between Theorem 1.2 and Theorem 1.3.*

Dirichlet class number formula and Theorem 1.3 imply the following theorem.

Theorem 1.5. *Let $d>0$ be a fundamental discriminant of a real quadratic field $\mathbb{Q}(\sqrt{d})$. Assume the same conditions as in Theorem 1.2. Then for any such $d>\exp\exp(4000Ng^3)$ with $(d,N)=1$, we have*

$$h(d)\log\epsilon_d > \frac{10^{180}}{2g^{4g}N^5}(\log d)^{g-3}\exp(-21\sqrt{g\log\log d}).$$

Finally, as an application, we give the following explicit lower bound for class numbers of certain real quadratic fields.

Theorem 1.6. *Let m be an integer and $d_m=4199^2(2m)^4-1$ be a square-free integer. Then for any $d_m\ge\exp\exp(3\times 10^{14})$, we have*

$$h(d_m) > 9\times 10^{121}\cdot(\log d_m)^{1-3\times 10^{-6}}.$$

Proof. Let $E: y^2=x^3-4199^2x$ be an elliptic curve over $\mathbb{Q}$ of conductor $N=32\cdot 4199^2$. It is known that E has complex multiplication by $\mathbb{Q}(\sqrt{-1})$ and analytic rank $g_1\ge 3$ (cf. [El]).

Let $d_m=4199^2(2m)^4-1$ be a square-free integer and $E_{d_m}: y^2=x^3-4199^2(d_m)^2x$ be the quadratic twist of E. Since E_{d_m} has a rational point $(4199^2(2m)^2d_m, 4199^2(2m)d_m^2)$ of infinite order (cf. Proposition 17 in [p.44, Ko]), E_{d_m} has analytic rank $g_{d_m}\ge 1$. We note that $4199d_m\equiv 1 \pmod 8$, so E_{d_m} has the root number 1 (cf. Theorem in [p.84, Ko]) and has even analytic rank. Thus E_{d_m} has analytic rank $g_{d_m}\ge 2$ and $L_{E/\mathbb{Q}(\sqrt{d_m})}$ has a zero of order $g_1+g_{d_m}\ge g=5$ at $s=1$.

It is known that the real quadratic field $\mathbb{Q}(\sqrt{d_m})$, so called Richaud-Degert type (cf. [De]), has the fundamental unit $\epsilon_{d_m} = \sqrt{d_m+1} + \sqrt{d_m} < d_m$. Thus from Theorem 1.5, we have for any $d_m > \exp\exp(4000 \cdot 32 \cdot 4199^2 \cdot 5^3)$,

$$h(d_m) \quad > \quad \frac{10^{180}}{2 \cdot 5^{20}(32 \cdot 4199^2)^5}(\log d_m)\exp(-21\sqrt{5\log\log d_m}).$$

We note that if $d_m > \exp\exp(4000 \cdot 32 \cdot 4199^2 \cdot 5^3)$, then for $\epsilon > 3 \times 10^{-6}$,

$$\exp(21\sqrt{5\log\log d_m})) < (\log d_m)^{\epsilon}.$$

Thus we have for any $d_m \geq \exp\exp(3 \times 10^{14})$,

$$h(d_m) > 9 \times 10^{121} \cdot (\log d_m)^{1-3\times 10^{-6}}.$$

□

Remark 1.7. *In [El], Elkies lists the* 75 *(*4199 *is the smallest integer.) values of* $n < 2 \cdot 10^5$ *with* $n \equiv 7 \,(mod\, 8)$ *for which the elliptic curve* E_n : $y^2 = x^3 - n^2x$ *has analytic rank at least* 3*. We can apply the proof of Theorem 1.6 to such* n*.*

Remark 1.8. *In [La], Lapkova did a similar work on lower bound for class numbers of certain real quadratic fields. But the constants in the lower bound were not computed.*

2. Proof of Theorem 1.3

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N with complex multiplication. Assume the same conditions as in Theorem 1.2. As [Go], let

$$\varphi(s) = L_E(s+\tfrac{1}{2})L_E(s+\tfrac{1}{2},\chi_d) = \sum_{n=1}^{\infty} a_n n^{-s}$$

and

$$\varphi_1(2s) = L_E(s+\tfrac{1}{2})L_E(s+\tfrac{1}{2},\lambda),$$

where $\lambda(n) = \prod_{p^r||n}(-1)^r$. We note that $\varphi(s) = L_{E/\mathbb{Q}(\sqrt{d})}(s+\frac{1}{2})$ and $\varphi(s)$ has a zero of order $\geq g$ at $s = \frac{1}{2}$. Let

$$G(s) = \frac{\varphi(s)}{\varphi_1(2s)} = \sum_{n=1}^{\infty} g_n n^{-s} \quad \text{and} \quad G(s,x) = \sum_{n<x} g_n n^{-s}.$$

For $A = \frac{dN}{4\pi^2}$ and $U = (\log d)^{8g}$, let

$$H = \left(\tfrac{d}{ds}\right)^{g-\mu}\left[A^s\Gamma^2(s+\tfrac{1}{2})G(s,U)\varphi_1(2s)\right]_{s=\frac{1}{2}}.$$

In [Go], Goldfeld proved that for $d > \exp\exp(cNg^3)$ and c sufficiently large, either $L(1, \chi_d) > (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}}$ or else

$$|H| \gg gN^{-12+\frac{1}{2}}(\log d)^{g-\mu-1} \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} (1+p^{-\frac{1}{2}})^{-4} \quad [\text{p.662, Go}] \tag{1}$$

and that for $d > \exp(500g^3)$, either $L(1, \chi_d) > (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}}$ or else

$$|H| \ll g^{4g} NL(1, \chi_d)A(\log\log A)^{g-\mu+6} \quad [(52), \text{Go}]. \tag{2}$$

We see that both $L(1, \chi_d) > (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}}$ and (1),(2) imply Theorem 1.1. or Theorem 1.2. To prove Theorem 1.3, we need the following propositions corresponding to (1) and (2), respectively.

Proposition 2.1. *Assume the same conditions as in Theorem 1.3. Then for any such $d \geq \exp\exp(4000Ng^3)$, either $L(1, \chi_d) > (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}}$ or else*

$$|H| \geq 1.8 \times 10^{-5} \cdot gN^{-4}\sqrt{d}(\log d)^{g-\mu-1} \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} \left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2.$$

Proposition 2.2. *Assume the same conditions as in Theorem 1.3. Then for any such $d \geq \exp\exp(4000Ng^3)$, either $L(1, \chi_d) > (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}}$ or else*

$$|H| \leq 2 \times 10^9 \cdot (\tfrac{80}{e})^g g^{2g+4.5} L(1, \chi_d)A(\log\log A)^{g-\mu+6}.$$

We will prove Proposition 2.1 in Section 3 and Proposition 2.2 in Section 4. If we assume Proposition 2.1 and 2.2, then we can prove Theorem 1.3 as follows.

Proof of Theorem 1.3. Let $\mathcal{P}$ be the set of primes $p < (\log d)^{8g}$ for which $\chi_d(p) \neq -1$. We may assume

$$L(1, \chi_d) \leq (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}} \quad (\ d \geq \exp\exp(4000Ng^3)).$$

From the inequality $2^{|\mathcal{P}|} \leq \frac{1}{4\log 2}(\log d)^{g-\mu-1}$ in the proof of [Lemma 9, Go], we see that $|\mathcal{P}| < \frac{1}{\log 2} g(\log\log d)$. So we have

$$\begin{aligned} & \log \prod_{p\in\mathcal{P}} \left(\tfrac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\right)^2 = \sum_{p\in\mathcal{P}} 2\log\left(\tfrac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\right) \\ \leq\ & \sum_{p\in\mathcal{P}} 2\Big(\tfrac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}} - 1\Big) = \sum_{p\in\mathcal{P}} \tfrac{4}{\sqrt{p}-1} \\ \leq\ & \int_2^{|\mathcal{P}|} \tfrac{4}{\sqrt{x}-1} dx = \Big[8x^{\frac{1}{2}} + 8\log(x^{\frac{1}{2}} - 1)\Big]_2^{|\mathcal{P}|} \\ \leq\ & 16|\mathcal{P}|^{\frac{1}{2}} \\ \leq\ & 20 g^{\frac{1}{2}} (\log\log d)^{\frac{1}{2}}. \end{aligned}$$

From Proposition 2.1 and Proposition 2.2, we have for $d \geq \exp\exp(4000Ng^3)$,

$$\begin{aligned} & 2\times 10^9 \cdot (\tfrac{80}{e})^g g^{2g+4.5} L(1,\chi_d) A(\log\log A)^{g-\mu+6} \\ \geq\ & 1.8\times 10^{-5} \cdot gN^{-4}\sqrt{d}(\log d)^{g-\mu-1} \exp\big(-20g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\big). \end{aligned}$$

Let $f(N,g,d) = \exp\big(g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\big) \cdot (\frac{80}{e})^{-g} g^{2g-4.5} (\log\log\frac{dN}{4\pi^2})^{-g-5}$. We claim that if $N \geq 1$, $g \geq 3$ and $d \geq \exp\exp(4000Ng^3)$, then

$$f(N,g,d) \geq \exp(450).$$

Since $\log\log\frac{dN}{4\pi^2} \leq \log\log d^e = \log\log d + 1$, we have

$$\begin{aligned} & \log f(N,g,d) \\ \geq\ & \big(g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\big) - g\log\tfrac{80}{e} + (2g-4.5)\log g - (g+5)\log(\log\log d + 1), \end{aligned}$$

which is an increasing function for d because its partial derivative with respect to d is

$$\begin{aligned} & \frac{\sqrt{g}}{2\sqrt{\log\log d}(\log d)d} - \frac{g+5}{(\log\log d+1)(\log d)d} \\ >\ & \frac{\sqrt{g(\log\log d)} - 2(g+5)}{2(\log\log d)(\log d)d} \\ >\ & 0. \end{aligned}$$

So we have

$$\begin{aligned} & \log f(N,g,d) \\ \geq\ & (4000N)^{\frac{1}{2}} g^2 - g\log\tfrac{80}{e} + (2g-4.5)\log g - (g+5)\log(4000Ng^3+1), \end{aligned}$$

which is an increasing function for g because its partial derivative with respect to g is

$$\begin{aligned} & 2(4000N)^{\frac{1}{2}}g - \log\left(\tfrac{80}{e}\right) + 2\log g + \frac{2g-4.5}{g} \\ & -\log\left(4000Ng^3+1\right) - \frac{3\cdot 4000Ng^2(g+5)}{4000Ng^3+1} \\ > \; & 2(4000N)^{\frac{1}{2}}g - \log\left(\tfrac{80}{e}\right) - \tfrac{4.5}{g} - 3\log g - \log\left(4000N+1\right) - \tfrac{3(g+5)}{g} \\ > \; & 0. \end{aligned}$$

So we have

$$\begin{aligned} & \log f(N,g,d) \\ \geq \; & (4000N)^{\frac{1}{2}}\cdot 3^2 - 3\log\tfrac{80}{e} + 1.5\log 3 - 8\log\left(4000\cdot 3^3N+1\right), \end{aligned}$$

which is an increasing function for N because its derivative with respect to N is

$$\frac{\sqrt{4000}\cdot 3^2}{2\sqrt{N}} - \frac{8\cdot 4000\cdot 3^3}{4000\cdot 3^3N+1} > \frac{\sqrt{4000}\cdot 3^2}{2\sqrt{N}} - \frac{8}{N} > 0.$$

So we have

$$\begin{aligned} & \log f(N,g,d) \\ \geq \; & \sqrt{4000}\cdot 3^2 - 3\log\tfrac{80}{e} + 1.5\log 3 - 8\log\left(4000\cdot 3^3+1\right) \\ > \; & 450 \end{aligned}$$

and the claim is proved. Thus we have

$$\exp\left(g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\right) > \exp(450)\cdot\left(\tfrac{80}{e}\right)^g g^{-2g+4.5}\left(\log\log\tfrac{dN}{4\pi^2}\right)^{g+5}.$$

Recall $A = \frac{dN}{4\pi^2}$. Then we have for $d \geq \exp\exp\left(4000Ng^3\right)$,

$$\begin{aligned} L(1,\chi_d) \; & > \frac{1.8\times 10^{-5}\cdot gN^{-4}}{2\times 10^9\cdot\left(\frac{80}{e}\right)^g g^{2g+4.5}}\cdot\frac{\sqrt{d}(\log d)^{g-\mu-1}\exp\left(-20g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\right)}{A(\log\log A)^{g-\mu+6}} \\ & > \frac{1.8\times 10^{-5}\cdot 4\pi^2\cdot gN^{-5}}{2\times 10^9\cdot\left(\frac{80}{e}\right)^g g^{2g+4.5}}\cdot\frac{(\log d)^{g-\mu-1}\exp\left(-20g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\right)}{\sqrt{d}(\log\log\frac{dN}{4\pi^2})^{g-\mu+6}} \\ & > \frac{1.8\times 10^{-5}\cdot 4\pi^2\cdot\exp(450)}{2\times 10^9\cdot g^{4g}N^5}\cdot\frac{(\log d)^{g-3}\exp\left(-21g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\right)}{\sqrt{d}} \\ & > \frac{10^{180}}{g^{4g}N^5}\cdot\frac{(\log d)^{g-3}\exp\left(-21g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\right)}{\sqrt{d}}. \end{aligned}$$

□

3. Proof of Proposition 2.1

In this section, we will prove Proposition 2.1. Let $\kappa = g - \mu$. From [(53), Go], we define H_1 and H_2 by

$$\begin{aligned} H &= H_1 + H_2 \\ &= 2\kappa\sqrt{A}(\log A)^{\kappa-1}G(\tfrac{1}{2},U)\varphi_1'(1) \\ &\quad +\sqrt{A}\sum_{r=2}^{\kappa}\binom{\kappa}{r}(\log A)^{\kappa-r}\left(\tfrac{d}{ds}\right)^r\left[\Gamma^2(s+\tfrac{1}{2})G(s,U)\varphi_1(2s)\right]_{s=\frac{1}{2}}. \end{aligned}$$

Since $|H| \geq |H_1| - |H_2|$, to get an explicit lower bound for $|H|$, we need an explicit upper bound for $|H_2|$ and an explicit lower bound for $|H_1|$.

Upper Bound for $|H_2|$. Using Leibniz' rule and Cauchy's Theorem (for detail, see [p.657 and p.658, Go]) we have

$$\begin{aligned} |H_2| &= |\sqrt{A}\sum_{r=2}^{\kappa}\binom{\kappa}{r}(\log A)^{\kappa-r} \\ &\quad \cdot(\sum_{h=0}^{r-1}\binom{r}{h}\left(\tfrac{d}{ds}\right)^{r-h}\left[\Gamma^2(s+\tfrac{1}{2})\varphi_1(2s)\right]_{s=\frac{1}{2}}\cdot\left(\tfrac{d}{ds}\right)^h\left[G(s,U)\right]_{s=\frac{1}{2}})| \\ &\leq \sqrt{A}\sum_{r=2}^{\kappa}\binom{\kappa}{r}(\log A)^{\kappa-r} \\ &\quad \cdot(\sum_{h=0}^{r-1}\binom{r}{h}2^{3(r-h)}(r-h)!\max_{s\in\mathbf{C_2}}|\Gamma^2(s+\tfrac{1}{2})\varphi_1(2s)|\cdot 2^{2h}h!\max_{s\in\mathbf{C_1}}|G(s,U)|) \\ &\leq \sqrt{A}\sum_{r=2}^{\kappa}8^r r!r\binom{\kappa}{r}(\log A)^{\kappa-r}\max_{s\in\mathbf{C_2}}|\Gamma^2(s+\tfrac{1}{2})|\max_{s\in\mathbf{C_2}}|\varphi_1(2s)|\max_{s\in\mathbf{C_1}}|G(s,U)|, \end{aligned} \tag{3}$$

where $\mathbf{C_1}$ is the circle of radius $\frac{1}{4}$ centered at $s = \frac{1}{2}$ and $\mathbf{C_2}$ is the circle of radius $\frac{1}{8}$ centered at $s = \frac{1}{2}$.

By [(46), Go], we have for $s = \sigma + it \in \mathbf{C_2}$,

$$\begin{aligned} \max_{s\in\mathbf{C_2}}|\Gamma^2(s+\tfrac{1}{2})| &\leq \max_{s\in\mathbf{C_2}}\left\{\sqrt{2\pi}\exp\left(\tfrac{1}{12(\sigma+\frac{1}{2})}\right)|s+\tfrac{1}{2}|^\sigma\exp\left(-\sigma-\tfrac{1}{2}\right)\right\}^2 \\ &\leq \left(\sqrt{2\pi}(\tfrac{9}{8})^{\frac{5}{8}}\exp\left(\tfrac{1}{12}\cdot\tfrac{8}{7}-\tfrac{7}{8}\right)\right)^2 \\ &\leq 1.6. \end{aligned} \tag{4}$$

We need the following lemma, which is an explicit version of [(49), Go].

Lemma 3.1. *For $s = \sigma + it \in \mathbb{C}$,*

$$|\varphi_1(s)| \leq \begin{cases} 3 \times 10^{12} \cdot N^3 t^6 & \text{if } 1 - \frac{1}{100800 \log |t|} \leq \sigma \leq \frac{3}{2}, \quad |t| \geq 2 + \frac{1}{840}, \\ 10^5 \cdot N^3 \frac{1}{|s-1|} & \text{if } \frac{3}{4} \leq \sigma \leq \frac{3}{2}, \quad |t| \leq 2 + \frac{1}{840}. \end{cases}$$

Proof. Let ψ be the primitive *Grössencharakter* of $K = \mathbb{Q}(\sqrt{-k})$ with conductor $\mathfrak{f}$ such that $L_E(s) = L_K(s, \psi)$ (cf. [Theorem 2, Go]). By [Lemma 2, Go], we have

$$\varphi_1(s) = L_K(s+1, \psi^2) \frac{L(s, \chi_k)}{\zeta(s)} \prod_{p|k} (1 - p^{-s})^{-1}, \tag{5}$$

where χ_k is a real, primitive, Dirichlet character (mod k).

From [p.654, Go], we have for $0 \leq \sigma \leq \frac{3}{2}$,

$$\left| L_K(s+1, \psi^2) \right| \leq \tfrac{10N^3}{4\pi^2} |s+3|^2. \tag{6}$$

Theorem 5.3.13 in [Ja] gives that if $|t| \geq 2 + \frac{1}{840}$ and $\sigma \geq 1 - \frac{1}{840 \cdot 6(\log |t| + 11)}$, then

$$|\zeta(s)^{-1}| \leq 56 \cdot 840^2 (\log |t| + 11)^3.$$

Proposition 3.1.16 in [Ja] gives that for $\sigma > -1$,

$$\zeta(s) = \tfrac{1}{s-1} + \tfrac{1}{2} + r_1^*(s),$$

where $|r_1^*(s)| \leq |\frac{s(s+1)}{8(\sigma+1)}|$. So we have if $|t| \leq 2 + \frac{1}{840}$ and $\frac{3}{4} \leq \sigma \leq \frac{3}{2}$, then

$$\begin{aligned} |\zeta(s)| &\geq |\tfrac{s+1}{2(s-1)}| - |r_1^*(s)| \\ &\geq \frac{|s+1|}{8|s-1|(\sigma+1)} (4(\sigma+1) - |s||s-1|) \\ &\geq \frac{1}{13}. \end{aligned}$$

Thus we have the following explicit version of a statement in [p.653, Go].

$$|\zeta(s)^{-1}| \leq \begin{cases} 56 \cdot 840^2 \cdot 6^3 |t|^3 & \text{if } \sigma \geq 1 - \frac{1}{840 \cdot 6 \cdot 20 \log |t|}, \quad |t| \geq 2 + \frac{1}{840}, \\ 13 & \text{if } \frac{3}{4} \leq \sigma \leq \frac{3}{2}, \quad |t| \leq 2 + \frac{1}{840}. \end{cases} \tag{7}$$

We note that

$$L(s, \chi_k) = \sum_{n=1}^{\infty} \tfrac{\chi_k(n)}{n^s} = \tfrac{1}{k^s} \sum_{l=1}^{k-1} \chi_k(l) \zeta(s, \tfrac{l}{k}),$$

where $\zeta(s,a)$ is the Hurwitz zeta function and $0 < a \leq 1$. Theorem 12.21 in [Ap] gives that for any integer $M \geq 0$ and $\sigma > 0$,

$$\zeta(s,a) = \sum_{n=0}^{M} \frac{1}{(n+a)^s} + \frac{(M+a)^{1-s}}{s-1} - s\int_M^\infty \frac{x-[x]}{(x+a)^{s+1}}dx.$$

So we have, for $\sigma \geq \frac{1}{2}$,

$$\left|\zeta(s,a) - a^{-s}\right| \leq \sum_{n=1}^{M} \frac{1}{\sqrt{n}} + \frac{(M+1)^{1-\sigma}}{\sqrt{(\sigma-1)^2+t^2}} + \frac{\sqrt{\sigma^2+t^2}}{\sigma M^\sigma}. \qquad (8)$$

By applying (8) with $M = \lfloor t \rfloor$ to the region; $\frac{1}{2} \leq \sigma \leq 2$ and $t \geq 2 + \frac{1}{840}$, we have

$$\begin{aligned}\left|\zeta(s,a) - a^{-s}\right| &\leq 1 + \int_1^{\lfloor t \rfloor} \frac{1}{\sqrt{x}}dx + \frac{\sqrt{t+1}}{t} + \frac{\sqrt{1+4t^2}}{\sqrt{t-1}} \\ &\leq 5\sqrt{t},\end{aligned}$$

which gives

$$\begin{aligned}|L(s,\chi_k)| &\leq k^{-\sigma}\sum_{l=1}^{k-1}((\tfrac{l}{k})^{-\sigma} + 5\sqrt{t}) \\ &\leq (\sum_{l=1}^{k-1} l^{-\frac{1}{2}}) + \frac{5(k-1)}{\sqrt{k}}\sqrt{t} \\ &< 7\sqrt{kt}.\end{aligned}$$

By applying (8) with $M = 1$ to the region; $\frac{1}{2} \leq \sigma \leq 2$ and $0 \leq t \leq 2 + \frac{1}{840}$, we have

$$\begin{aligned}\left|\zeta(s,a) - a^{-s}\right| &\leq 1 + \frac{\sqrt{2}}{|s-1|} + \sqrt{1+4t^2} \\ &< \frac{16}{|s-1|},\end{aligned}$$

which gives

$$\begin{aligned}|L(s,\chi_k)| &\leq k^{-\sigma}\sum_{l=1}^{k-1}((\tfrac{l}{k})^{-\sigma} + \frac{16}{|s-1|}) \\ &\leq (\sum_{l=1}^{k-1} l^{-\frac{1}{2}}) + \frac{16(k-1)}{\sqrt{k}}\frac{1}{|s-1|} \\ &< \frac{22\sqrt{k}}{|s-1|}.\end{aligned}$$

We note that $L(\bar{s},\chi_k) = \overline{L(s,\chi_k)}$. Then we have the following explicit version of a statement in [p.653, Go].

$$|L(s,\chi_k)| \leq \begin{cases} 7\sqrt{k|t|} & \text{if } \frac{1}{2} \leq \sigma \leq 2, \quad |t| \geq 2 + \frac{1}{840}, \\ 22\sqrt{k}|s-1|^{-1} & \text{if } \frac{1}{2} \leq \sigma \leq 2, \quad |t| \leq 2 + \frac{1}{840}. \end{cases} \tag{9}$$

Since $\sigma \geq \frac{1}{2}$ and $\{p : p|k\}$ is a set containing only one prime from Remark 1.4, we have $|\prod_{p|k}(1-p^{-s})^{-1}| \leq |(1-2^{-s})^{-1}| \leq \frac{\sqrt{2}}{\sqrt{2}-1}$. Thus Lemma 3.1 follows from (5), (6), (7), (9) and Remark 1.4. □

From Lemma 3.1, we have

$$\begin{aligned} \max_{s\in\mathbf{C_2}} |\varphi_1(2s)| &\leq \max_{s\in\mathbf{C_2}} \left(10^5 \tfrac{N^3}{|2s-1|}\right) \\ &\leq 4\cdot 10^5 N^3. \end{aligned} \tag{10}$$

Moreover,

$$\max_{s\in\mathbf{C_1}} |G(s,U)| < \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} (1-p^{-\frac{1}{4}})^{-4} \ \ (\text{cf. [p.657, Go]}). \tag{11}$$

Thus from (3), (4), (10) and (11) we have

$$|H_2| \leq 4\cdot 10^8 N^3 g^2 \sqrt{A} (\log A)^{\kappa-2} \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} (1-p^{-\frac{1}{4}})^{-4}. \tag{12}$$

Lower Bound for $|H_1|$. We need the following lemma, which is an explicit version of [(55), Go]. (We use $\prod \left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2$ in Lemma 3.2 instead of $\prod(1+p^{-\frac{1}{2}})^{-4}$ in [(55), Go].)

Lemma 3.2. *If $d > \exp(500g^3)$, then either $L(1,\chi_d) > (\log d)^{\kappa-1}\frac{1}{\sqrt{d}}$ or else we have*

$$|G(\tfrac{1}{2},U)| \ \ \geq \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} \left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 - (\log d)^{-2g}.$$

Proof. We denote by $P(s,U)$ the partial Euler product of $G(s)$ for primes $p \leq U$ and write

$$G(s,U) = P(s,U) - R(s,U).$$

From [Lemma 1, Go], we see that

$$|P(\tfrac{1}{2},U)| \geq \prod_{\substack{\chi_d(p)\neq -1\\ p<U}} \left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2.$$

So we only need to show that

$$|R(\tfrac{1}{2},U)| \leq (\log d)^{-2g}.$$

If

$$\mathcal{N}_U = \{n \text{ such that } p \mid n \ \Rightarrow\ p < U\}$$

then

$$R(s,U) = \sum_{n>U,\ n\in\mathcal{N}_U} g_n n^{-s}.$$

We write

$$|R(\tfrac{1}{2},U)| \leq \sum_{U<n\leq\frac{1}{4}\sqrt{d}} |g_n| n^{-\frac{1}{2}} + \sum_{\frac{1}{4}\sqrt{d}<n,\ n\in\mathcal{N}_U} |g_n| n^{-\frac{1}{2}} = R_1 + R_2.$$

We may assume

$$L(1,\chi_d) \leq (\log d)^{\kappa-1}\tfrac{1}{\sqrt{d}} \ \ (d > \exp(500g^3)).$$

Let $\frac{\zeta(s)L(s,\chi_d)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{\nu_n}{n^s}$. Then by [Lemma 1 and Lemma 4, Go], we have

$$\begin{aligned} R_1 &\leq U^{-\frac{1}{2}}\Big(\sum_{n\leq\frac{1}{4}\sqrt{d}} \nu_n\Big)^2 \\ &\leq U^{-\frac{1}{2}}\big(\tfrac{1}{4\log 2}\big)^2(\log d)^{2(\kappa-1)} \\ &= \big(\tfrac{1}{4\log 2}\big)^2(\log d)^{-2(g+\mu+1)}. \end{aligned}$$

Now we estimate R_2. Let

$$P_1(s,U) = \prod_{\substack{\chi_d(p)\neq -1\\ p<U}} (1-p^{-s})^{-4}.$$

Since $|\mathcal{P}| < \frac{1}{\log 2} g(\log\log d)$ (cf. Proof of Theorem 1.3), we have

$$\begin{aligned} \log P_1(\tfrac{1}{6}, U) &= \log \prod_{p\in\mathcal{P}} \left(\tfrac{1}{1-p^{-\frac{1}{6}}}\right)^4 \\ &\le \sum_{p\in\mathcal{P}} \tfrac{4}{\sqrt[6]{p}-1} \\ &\le \int_2^{|\mathcal{P}|} \tfrac{4}{\sqrt[6]{x}-1} dx \\ &= \left[\tfrac{24}{5}x^{\frac{5}{6}} + 6x^{\frac{2}{3}} + 8x^{\frac{1}{2}} + 12x^{\frac{1}{3}} + 24x^{\frac{1}{6}} + 24\log(x^{\frac{1}{6}}-1)\right]_2^{|\mathcal{P}|} \\ &\le 58|\mathcal{P}|^{\frac{5}{6}} \\ &\le 80(g\log\log d)^{\frac{5}{6}}. \end{aligned}$$

So we have

$$\begin{aligned} R_2 &\le \lim_{N\to\infty} \int_{2-i\infty}^{2+i\infty} P_1(\tfrac{1}{2}+z, U) \tfrac{N^z - (\sqrt{d}/4)^z}{z(z+1)} dz \\ &= \lim_{N\to\infty} \int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} P_1(\tfrac{1}{2}+z, U) \tfrac{N^z - (\sqrt{d}/4)^z}{z(z+1)} dz \\ &\le \lim_{N\to\infty} \int_{-\infty}^{\infty} P_1(\tfrac{1}{6}, U) \tfrac{N^{-\frac{1}{3}} + (\sqrt{d}/4)^{-\frac{1}{3}}}{|(-\frac{1}{3}+it)(\frac{2}{3}+it)|} dt \\ &\le P_1(\tfrac{1}{6}, U)(\tfrac{\sqrt{d}}{4})^{-\frac{1}{3}} \int_{-\infty}^{\infty} \tfrac{1}{2/9+t^2} dt \\ &\le 3\sqrt[6]{2}\pi \exp(80(g\log\log d)^{\frac{5}{6}}) \cdot \tfrac{1}{\sqrt[6]{d}}. \end{aligned}$$

Thus we have for $d \ge \exp(500g^3)$,

$$\begin{aligned} \left|R(\tfrac{1}{2}, U)\right| &\le (\tfrac{1}{4\log 2})^2 (\log d)^{-2(g+\mu+1)} + 3\sqrt[6]{2}\pi \cdot \exp(80(g\log\log d)^{\frac{5}{6}}) \cdot \tfrac{1}{\sqrt[6]{d}} \\ &\le (\log d)^{-2g}. \end{aligned}$$

□

By [Lemma 2, Go], we have

$$\begin{aligned} \left|\varphi_1'(1)\right| &= |L_K(2,\psi^2)L(1,\chi_k)\prod_{p|k}(1-p^{-1})^{-1}| \\ &\ge |L_K(2,\psi^2)L(1,\chi_k)|. \end{aligned} \tag{13}$$

Recall that ψ is the primitive *Grössencharakter* of $K = \mathbb{Q}(\sqrt{-k})$ such that $L_E(s) = L_K(s,\psi)$ (cf. [Theorem 2, Go]). So, to get an explicit lower bound for $|H_1|$, we need the following lemma, which is an explicit version of [Lemma

12, Go]. (We note that the inequality in [Lemma 12, Go] is in the wrong direction.)

Lemma 3.3.

$$|L_K(2,\psi^2)L(1,\chi_k)| \geq 0.98(kN^2)^{-2}.$$

We will prove Lemma 3.3 in Section 5. If we assume Lemma 3.3, then by Lemma 3.2 and (13) we have *for* $d > \exp(500g^3)$, *either* $L(1,\chi_d) > (\log d)^{\kappa-1}\frac{1}{\sqrt{d}}$ *or else*

$$|H_1| \geq 2\kappa\tfrac{0.98}{k^2N^4}\cdot\sqrt{A}(\log A)^{\kappa-1}\left(\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\left(\tfrac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 - (\log d)^{-2g}\right). \quad (14)$$

Now we can prove Proposition 2.1.

Proof of Proposition 2.1. We may assume

$$L(1,\chi_d) \leq (\log d)^{\kappa-1}\tfrac{1}{\sqrt{d}} \ \ (d > \exp(500g^3)).$$

From (12) and (14), we have

$$\begin{aligned}
|H| &\geq |H_1| - |H_2| \\
&\geq \Big[2\kappa\tfrac{0.98}{k^2N^4}\cdot\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\big(\tfrac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\big)^2\Big] \\
&\quad -\Big[2\kappa\tfrac{0.98}{k^2N^4}\cdot\sqrt{A}(\log A)^{\kappa-1}(\log d)^{-2g} \\
&\qquad +4\cdot 10^8N^3g^2\sqrt{A}(\log A)^{\kappa-2}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}(1-p^{-\frac{1}{4}})^{-4}\Big] \\
&= \tilde{H}_1 - \tilde{H}_2.
\end{aligned}$$

If $\frac{1}{2}\tilde{H}_1 \geq \tilde{H}_2$, then we have

$$\begin{aligned}
|H| &\geq \frac{\tilde{H}_1}{2} \\
&\geq \kappa\tfrac{0.98}{k^2N^4}\cdot\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi(p)\neq -1\\ p<U}}\left(\tfrac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 \\
&\geq \tfrac{0.98}{2\cdot 163^2}\cdot gN^{-4}\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi(p)\neq -1\\ p<U}}\left(\tfrac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 \quad (cf.\ Remark\,1.4)
\end{aligned}$$

as desired.

We see that

$$\begin{aligned}
\frac{\tilde{H}_2}{\tilde{H}_1} &= \frac{4\cdot 10^8 N^3 g^2 \sqrt{A}(\log A)^{\kappa-2}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}(1-p^{-\frac14})^{-4}}{2\kappa\frac{0.98}{k^2N^4}\cdot\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\big(\frac{1-p^{-\frac12}}{1+p^{-\frac12}}\big)^2} \\
&\quad + \frac{(\log d)^{-2g}}{\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\big(\frac{1-p^{-\frac12}}{1+p^{-\frac12}}\big)^2} \\
&\leq \frac{4\cdot 10^8}{2\cdot 0.98(g-2)}\cdot 163^2\cdot N^7 g^2(\log d)^{-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac12}}{1-p^{-\frac12}}\Big)^2\cdot\Big(\frac{1}{1-p^{-\frac14}}\Big)^4 \\
&\quad +(\log d)^{-2g}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac12}}{1-p^{-\frac12}}\Big)^2 \\
&\leq 2\cdot(\frac{4\cdot 10^8}{2\cdot 0.98(g-2)}\cdot 163^2\cdot N^7 g^2(\log d)^{-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac12}}{1-p^{-\frac12}}\Big)^2\cdot\Big(\frac{1}{1-p^{-\frac14}}\Big)^4).
\end{aligned}$$

Let $\mathcal{P}$ be the set of primes $p<(\log d)^{8g}$ for which $\chi_d(p)\neq -1$. Since $|\mathcal{P}|<\frac{g}{\log 2}(\log\log d)$, we have

$$\begin{aligned}
&\log\prod_{p\in\mathcal{P}}\Big(\frac{1+p^{-\frac12}}{1-p^{-\frac12}}\Big)^2\cdot\Big(\frac{1}{1-p^{-\frac14}}\Big)^4 \\
\leq\ & \sum_{p\in\mathcal{P}}\Big(\frac{4}{\sqrt{p}-1}+\frac{4}{\sqrt[4]{p}-1}\Big) \\
\leq\ & \int_2^{|\mathcal{P}|}\frac{4}{\sqrt{x}-1}+\frac{4}{\sqrt[4]{x}-1}dx \\
=\ & \Big[\tfrac{16}{3}x^{\frac34}+16x^{\frac12}+16x^{\frac14}+8\log(x^{\frac12}-1)+16\log(x^{\frac14}-1)\Big]_2^{|\mathcal{P}|} \\
\leq\ & 6|\mathcal{P}|^{\frac34} \\
\leq\ & 6\Big(\tfrac{g}{\log 2}\log\log d\Big)^{\frac34}.
\end{aligned}$$

Thus the sufficient condition of $\frac12\tilde{H}_1\geq\tilde{H}_2$ is that

$$\log\log d-6(\tfrac{g}{\log 2}\log\log d)^{\frac34}\geq\log\Big(4\cdot\tfrac{4\cdot 10^8}{2\cdot 0.98}\cdot 163^2\cdot N^7\tfrac{g^2}{g-2}\Big). \tag{15}$$

We write $d\geq\exp\exp(c_1Ng^3)$ and assume $g\geq 3$. If c_1 is sufficiently large, the left hand in (15) is greater than

$$c_1Ng^3-6(\tfrac{1}{\log 2}c_1Ng^4)^{\frac34}=g^3(c_1N-\tfrac{6}{(\log 2)^{3/4}}c_1^{3/4}N^{3/4}),$$

and the right hand in (15) is less than

$$31 + 7\log N + \log \tfrac{g^2}{g-2}.$$

Since $g \geq 3$ and $N \geq 1$, a sufficient condition of $\frac{1}{2}\tilde{H}_1 \geq \tilde{H}_2$ is that $c_1 \geq 3897$. For convenience, if we choose $c_1 = 4000$, then Proposition 2.1 follows. □

4. Proof of Proposition 2.2

In this section, we will prove Proposition 2.2. From [(24), (26) and (51), Go] and the assumption that $\varphi(s)$ has a zero of order $\geq g$ at $s = \frac{1}{2}$, we can write

$$0 = (\tfrac{d}{ds})^\kappa \Big[A^s \Gamma^2(s+\tfrac{1}{2})\varphi(s)\Big]_{s=\frac{1}{2}} = T_1 + T_2, \tag{16}$$

where

$$T_1 = \delta \sum_{r=0}^{\kappa} \Big(\sum_{n \leq A_1} a_n \sqrt{A/n}(\log A/n)^{\kappa - r} I_r(n/A)\Big),$$

$$T_2 = \delta \sum_{r=0}^{\kappa} \Big(\sum_{n > A_1} a_n \sqrt{A/n}(\log A/n)^{\kappa - r} I_r(n/A)\Big),$$

$$\delta = 1 + (-1)^\kappa \chi_d(-N),$$

$$A_1 = A((8+2\kappa)\log A)^2,$$

and

$$I_r(M) = \int_{u_1=0}^{\infty}\int_{u_2=M/u_1}^{\infty} \exp(-(u_1+u_2))(\log u_1 u_2)^r du_1 du_2 \ (M \geq 0).$$

By [Lemma 10, Go], we have

$$|T_2| \leq 1.$$

Thus by (16) and [(27), (30), (31) and (39), Go], we have

$$\begin{aligned} & |2H| \\ = \ & |2H - T_1 - T_2| \\ \leq \ & |2H - T(G(s,U))| + |T(g(s))| + |S_1| + |S_2| + 1, \end{aligned} \tag{17}$$

where

$$T(F(s)) = (\tfrac{d}{ds})^\kappa \Big[\tfrac{\delta}{2\pi i}\int_{2-i\infty}^{2+i\infty} A^{s+z}\Gamma^2(s+z+\tfrac{1}{2})F(s+z)\varphi_1(2s+2z)\tfrac{dz}{z}\Big]_{s=\frac{1}{2}},$$

$$g(s) = G(s, A_0) - G(s, U),$$

$$A_0 = A(\log A)^{-20g},$$
$$S_1 = 2\sum_{r=0}^{\kappa}\binom{\kappa}{r}\Big(\sum_{A_0\le n\le J} b_n\sqrt{A/n}(\log A/n)^{\kappa-r}I_r(n/A)\Big),$$
$$S_2 = 2\sum_{r=0}^{\kappa}\binom{\kappa}{r}\Big(\sum_{J\le n\le A_1} b_n\sqrt{A/n}(\log A/n)^{\kappa-r}I_r(n/A)\Big),$$
$$J = A((\kappa+6)\log\log A)^2,$$

and

$$\sum_{n=1}^{\infty} b_n n^{-s} = G(s,A_1)\varphi_1(2s) - G(s,A_0)\varphi_1(2s).$$

So, to obtain an explicit upper bound for $|H|$, we need explicit upper bounds for $|S_1|$, $|S_2|$, $|T(g(s))|$ and $|2H - T(G(s,U))|$.

Upper Bound for $|S_1|$*.* From [p.649, Go], we have

$$|S_1| \le 4^{\kappa+1}\kappa!(\log \tfrac{A}{A_0})^{\kappa}\sqrt{A}\sum_{A_0\le n\le J}\tfrac{|b_n|}{\sqrt{n}}. \tag{18}$$

We may assume

$$L(1,\chi_d) \le (\log d)^{\kappa-1}\tfrac{1}{\sqrt{d}} \quad (d > \exp\exp(4000Ng^3)).$$

Then we can choose

$$\begin{aligned} y &= L(1,\chi_d)^2 J \\ &\le (\log A)^{2\kappa-2}\tfrac{J}{d} \\ &\le A_0. \end{aligned}$$

Recall $\frac{\zeta(s)L(s,\chi_d)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{\nu_n}{n^s}$. By [(36), Go], we have

$$\begin{aligned} \sum_{A_0\le n\le J}\tfrac{|b_n|}{\sqrt{n}} &\le \sum_{k^2\le\frac{J}{A_0}}\tfrac{d(k)}{k}\sum_{A_0\le m\le\frac{J}{k^2}}\tfrac{1}{\sqrt{m}}\sum_{f|m}\nu_f\nu_{m/f} \\ &\le \Big(\sum_{k\le\sqrt{\frac{J}{A_0}}}\tfrac{d(k)}{k}\Big)\Big(\sum_{y\le m\le J}\tfrac{1}{\sqrt{m}}\sum_{f|m}\nu_f\nu_{m/f}\Big), \end{aligned} \tag{19}$$

where $d(k) = \sum_{f|k} 1$.

Lemma 4.1. (cf. [Problem 3, p.70, Ap]) *For* $x \ge 3$*,*

$$\sum_{n\le x}\tfrac{d(n)}{n} \le \tfrac{1}{2}\log^2 x + 2C\log x + 10$$

where $C(< 0.6)$ *is the Euler constant.*

Proof. By Euler's summation formula,

$$\begin{aligned}
\sum_{n\le x}\frac{1}{n} &= \int_1^x \frac{dt}{t} - \int_1^x \frac{t-[t]}{t^2}dt + 1 - \frac{x-[x]}{x} \\
&= \log x + \Big(1 - \int_1^\infty \frac{t-[t]}{t^2}dt\Big) + \Big(\int_x^\infty \frac{t-[t]}{t^2}dt - \frac{x-[x]}{x}\Big) \\
&\le \log x + C + \frac{1}{x}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n\le x}\frac{\log n}{n} &= \int_1^x \frac{\log t}{t}dt + \int_1^x (t-[t])\frac{1-\log t}{t^2}dt - (x-[x])\frac{\log x}{x} \\
&= \frac{1}{2}\log^2 x + A(x).
\end{aligned}$$

We note that

$$\begin{aligned}
|A(x)| &\le \int_1^x \frac{\log t + 1}{t^2}dt + (x-[x])\frac{\log x}{x} \\
&\le \Big[-\frac{\log t + 2}{t}\Big]_1^x + (x-[x])\frac{\log x}{x} \\
&\le 2.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n\le x}\frac{d(n)}{n} &= \sum_{d\le x}\frac{1}{d}\sum_{q\le \frac{x}{d}}\frac{1}{q} \le \sum_{d\le x}\frac{1}{d}\Big(\log\frac{x}{d} + C + \frac{d}{x}\Big) \\
&\le \sum_{d\le x}\Big(\frac{\log x + C}{d} - \frac{\log d}{d} + \frac{1}{x}\Big) \\
&\le (\log x + C)\sum_{d\le x}\frac{1}{d} - \sum_{d\le x}\frac{\log d}{d} + 1 \\
&\le (\log x + C)\big(\log x + C + \frac{1}{x}\big) - \big(\frac{1}{2}\log^2 x + A(x)\big) + 1 \\
&\le \frac{1}{2}\log^2 x + 2C\log x + C^2 + 2 - A(x) + 1 \\
&\le \frac{1}{2}\log^2 x + 2C\log x + 10.
\end{aligned}$$

□

Using (19), Lemma 4.1 and [Lemma 7, Go], we have

$$\begin{aligned}
&\sum_{A_0\le n\le J}\tfrac{|b_n|}{\sqrt{n}}\\
\le\ &\Big(\tfrac12(\log\sqrt{\tfrac{J}{A_0}})^2+2C\log\sqrt{\tfrac{J}{A_0}}+10\Big)\\
&\times 1500\Big(L(1,\chi)^2Jy^{-\frac12}+L(1,\chi_d)J^{\frac12}\Big)(\log y)^3\\
\le\ &(\log\tfrac{J}{A_0})^2\big(2\cdot 1500L(1,\chi_d)J^{\frac12}\big)(\log y)^3\\
\le\ &\big(20g\log\log A+2\log\log\log A+2\log(\kappa+6)\big)^2\\
&\times(3000L(1,\chi_d)\sqrt{A}(\kappa+6)\log\log A)\\
&\times\big((2\kappa-2)\log\log A+2\log\log\log A+\log\tfrac{N}{4\pi^2}+2\log(\kappa+6)\big)^3\\
\le\ &(3\cdot 20g\log\log A)^2\\
&\times(3000L(1,\chi_d)\sqrt{A}(\kappa+6)\log\log A)\\
&\times(4\cdot(2\kappa-2)\log\log A)^3. \qquad (20)
\end{aligned}$$

Using $\kappa\le g-1$, (18), (20) and the fact $n!\le e\sqrt{n}(\frac{n}{e})^n$, we have for $d\ge \exp\exp(4000Ng^3)$,

$$\begin{aligned}
|S_1|\ \le\ &4^{\kappa+1}\kappa!(20g\log\log A)^{\kappa}\sqrt{A}\sum_{A_0\le n\le J}\tfrac{|b_n|}{\sqrt{n}}\\
\le\ &3^2\cdot4^3\cdot3000(20g)^{\kappa+2}4^{\kappa+1}\kappa!(\kappa+6)(2\kappa-2)^3L(1,\chi_d)A(\log\log A)^{\kappa+6}\\
\le\ &3^2\cdot4^3\cdot3000\cdot(20\cdot g\cdot(20g)^g)\cdot4^g\cdot(g-1)!\cdot(2^3g^4)L(1,\chi_d)A(\log\log A)^{\kappa+6}\\
\le\ &2^3\cdot3^2\cdot4^3\cdot20\cdot3000\cdot(20g)^g\cdot4^g\cdot g!\cdot g^4L(1,\chi_d)A(\log\log A)^{\kappa+6}\\
\le\ &2^3\cdot3^2\cdot4^3\cdot20\cdot3000\cdot e\cdot(\tfrac{80}{e})^g\cdot g^{2g+4.5}L(1,\chi_d)A(\log\log A)^{\kappa+6}\\
=\ &S_1^*. \qquad (21)
\end{aligned}$$

Upper Bound for $|S_2|$. From [(32), Go], we have

$$|S_2|\ \le\ 4^{\kappa+1}(\kappa+1)!(\log\tfrac{A_1}{A})^{\kappa}\exp\big(-(\kappa+6)\log\log A\big)\sqrt{A}\sum_{J\le n\le A_1}\tfrac{|b_n|}{\sqrt{n}}. \qquad (22)$$

(We note that the term $\sqrt{A}$ is missed in [(32), Go].)

We may assume

$$L(1,\chi_d)\le(\log d)^{\kappa-1}\tfrac{1}{\sqrt{d}}\quad(d>\exp\exp(4000Ng^3)).$$

Then we can choose

$$\begin{aligned} y &= L(1,\chi_d)^2 A_1 \\ &\le (\log A)^{2\kappa-2}\tfrac{A_1}{d} \\ &\le A_0. \end{aligned}$$

From [(33), Go], we have

$$\begin{aligned} \sum_{J\le n\le A_1} \tfrac{|b_n|}{\sqrt{n}} &\le \sum_{k^2\le \frac{A_1}{A_0}} \tfrac{d(k)}{k} \sum_{A_0\le m\le \frac{A_1}{k^2}} \tfrac{1}{\sqrt{m}} \sum_{f|m} \nu_f \nu_{m/f} \\ &\le \Big(\sum_{k\le\sqrt{\frac{A_1}{A_0}}} \tfrac{d(k)}{k}\Big)\Big(\sum_{y\le m\le A_1} \tfrac{1}{\sqrt{m}} \sum_{f|m} \nu_f \nu_{m/f}\Big). \end{aligned} \tag{23}$$

(We note that we use $\frac{A_1}{A_0}$ instead of $\frac{A_1}{J}$ in [(33), Go].)
Using (23), Lemma 4.1 and [Lemma 7, Go], we have

$$\begin{aligned} &\sum_{J\le n\le A_1} \tfrac{|b_n|}{\sqrt{n}} \\ \le\ & \Big(\tfrac{1}{2}(\log\sqrt{\tfrac{A_1}{A_0}})^2 + 2C\log\sqrt{\tfrac{A_1}{A_0}} + 10\Big) \\ & \times 1500\Big(L(1,\chi_d)^2 A_1 y^{-\frac{1}{2}} + L(1,\chi_d) A_1^{\frac{1}{2}}\Big)(\log y)^3 \\ \le\ & (\log\tfrac{A_1}{A_0})^2\big(2\cdot 1500 L(1,\chi_d) A_1^{\frac{1}{2}}\big)(\log y)^3 \\ \le\ & \big((20g+2)\log\log A + \log(2\kappa+8)\big)^2 \\ & \times\big(3000 L(1,\chi_d)\sqrt{A}(2\kappa+8)\log A\big) \\ & \times\big(2\kappa\log\log A + \log\tfrac{N}{4\pi^2} + 2\log(2\kappa+8)\big)^3 \\ \le\ & \big(2\cdot(20g+2)\log\log A\big)^2 \\ & \times\big(3000 L(1,\chi_d)\sqrt{A}(2\kappa+8)\log A\big) \\ & \times\big(3\cdot 2\kappa\log\log A\big)^3. \end{aligned} \tag{24}$$

Using $g-2 \le \kappa \le g-1$, (22), (24) and the fact $n! \le e\sqrt{n}(\frac{n}{e})^n$, we have for $d \ge \exp\exp(4000Ng^3)$,

$$\begin{aligned}
|S_2| &\le 4^{\kappa+1}(\kappa+1)!\big(2\log\log A + 2\log(2\kappa+8)\big)^{\kappa}(\log A)^{-(\kappa+6)}\sqrt{A} \\
&\quad \times \sum_{J\le n\le A_1} \tfrac{|b_n|}{\sqrt{n}} \\
&\le 4^{\kappa+1}(\kappa+1)!\big(2\cdot 2\log\log A\big)^{\kappa}(\log A)^{-(\kappa+6)}\sqrt{A}\sum_{J\le n\le A_1}\tfrac{|b_n|}{\sqrt{n}} \\
&\le 2^2\cdot 3^3\cdot 3000\cdot 4\cdot 16^{\kappa}(\kappa+1)!(20g+2)^2(2\kappa+8)(2\kappa)^3 \\
&\quad \times L(1,\chi_d)A(\log A)^{-(\kappa+6)}(\log\log A)^{\kappa+5} \\
&\le 3^3\cdot 3000\cdot 16^g\cdot g!\cdot(20^2\cdot 2^7 g^6)\cdot(\log A)^{-(g+4)} \\
&\quad \times L(1,\chi_d)A(\log\log A)^{\kappa+5} \\
&\le 2^7\cdot 3^3\cdot 20^2\cdot 3000\cdot e\cdot(\tfrac{16}{e})^g\cdot g^{g+6.5}\cdot(4000Ng^3)^{-(g+4)} \\
&\quad \times L(1,\chi_d)A(\log\log A)^{\kappa+5} \\
&< S_1^*.
\end{aligned} \tag{25}$$

Upper Bound for $|\mathbf{T}(g(s))|$. From [p.651, Go], we have

$$|\mathbf{T}(g(s))| \le \kappa!\epsilon^{-\kappa}\cdot\max_{s\in\mathbf{C}}\left|\tfrac{1}{2\pi i}\int_{2\epsilon-i\infty}^{2\epsilon+i\infty} A^{s+z}\Gamma^2(s+z+\tfrac12)g(s+z)\varphi_1(2s+2z)\tfrac{dz}{z}\right|, \tag{26}$$

where $\mathbf{C}$ is the circle of radius $\epsilon = (\log d)^{-1}$ centered at $s=\frac12$.

By the same argument in the proof of [Lemma 7, Go], we have for $x<d$ and $10^{10} < y < \min(\frac14\sqrt{d}, x/10)$,

$$\sum_{y\le n\le x} n^{-\frac12}\sum_{m|n}\nu_m\nu_{n/m} \le 1500(L(1,\chi_d)^2 dy^{-\frac12} + L(1,\chi_d)x^{\frac25}d^{\frac{1}{10}})(\log y)^3$$

instead of for $x<d$ and $10<y<\min(\frac14\sqrt{d}, x/10)$,

$$\sum_{y\le n\le x} n^{-\frac12}\sum_{m|n}\nu_m\nu_{n/m} \ll (L(1,\chi_d)^2 dy^{-\frac12} + L(1,\chi_d)x^{\frac25}d^{\frac{1}{10}})(\log y)^3$$

in [Lemma 8, Go].

We may assume

$$L(1,\chi_d) \le (\log d)^{\kappa-1}\tfrac{1}{\sqrt{d}} \quad (d > \exp\exp(4000Ng^3)).$$

Then by [(40), Go], we have

$$\begin{aligned}\max_{\substack{s\in\mathbf{C},\\ Re(z)=2\epsilon}} |g(s+z)| &\leq \sum_{U\leq n\leq A_0} n^{-\frac{1}{2}} \sum_{f|n} \nu_f \nu_{n/f} \\ &\leq 1500\big(L(1,\chi_d)^2 dU^{-\frac{1}{2}} + L(1,\chi_d) A_0^{\frac{2}{5}} d^{\frac{1}{10}}\big)(\log U)^3 \\ &\leq 1500 L(1,\chi_d)\sqrt{A} \\ &\quad \times\Big((\log d)^{\kappa-1}\tfrac{2\pi}{\sqrt{NU}} + (\log A)^{-8g}\big(\tfrac{4\pi^2}{N}\big)^{\frac{1}{10}}\Big)(\log U)^3. \quad (27)\end{aligned}$$

(We use $dU^{-\frac{1}{2}}$ instead of $A_0 u^{-\frac{1}{2}}$ in [(40), Go], so that it is a direct consequence of [Lemma 8, Go].)

By [(41), Go], we have

$$\max_{\substack{s\in\mathbf{C},\\ Re(z)=2\epsilon}} |\varphi_1(2s+2z)| \leq \zeta^2(1-2\epsilon+4\epsilon) < \tfrac{1}{2}\epsilon^{-2}. \quad (28)$$

To estimate integral of Gamma function, using [(4.6), Go1],

$$\begin{aligned} &\max_{s\in\mathbf{C}} \left| \frac{1}{2\pi i}\int_{2\epsilon-i\infty}^{2\epsilon+i\infty} A^{s+z}\Gamma^2(s+z+\tfrac{1}{2})\frac{dz}{z}\right| \\ \leq\ & A^{\frac{1}{2}+3\epsilon}\max_{s\in\mathbf{C}}\left|\int_0^\infty\int_0^\infty \Big(\frac{1}{2\pi i}\int_{2\epsilon-i\infty}^{2\epsilon+i\infty}(u_1u_2)^z\frac{dz}{z}\Big)e^{-u_1-u_2}(u_1u_2)^{s+\frac{1}{2}}\frac{du_1du_2}{u_1u_2}\right| \\ \leq\ & A^{\frac{1}{2}+3\epsilon}\int\int_{u_1u_2>1} e^{-u_1-u_2}(u_1u_2)^{1+\epsilon}\frac{du_1du_2}{u_1u_2} \\ <\ & A^{\frac{1}{2}+3\epsilon}. \quad (29)\end{aligned}$$

Since $A \leq d^2$, we have $A^{3\epsilon} \leq d^{6\log_d e} \leq e^6$. Thus by (26), (27), (28) and (29), we have for $d \geq \exp\exp(4000Ng^3)$,

$$\begin{aligned}
|\mathbf{T}(g(s))| &\leq \kappa!\epsilon^{-\kappa} \cdot \max_{\substack{s\in\mathbf{C},\\ Re(z)=2\epsilon}} |g(s+z)\varphi_1(2s+2z)| \\
&\quad \times \max_{s\in\mathbf{C}} \left| \tfrac{1}{2\pi i} \int_{2\epsilon-i\infty}^{2\epsilon+i\infty} A^{s+z}\Gamma^2(s+z+\tfrac{1}{2})\tfrac{dz}{z} \right| \\
&\leq \tfrac{1}{2}\cdot 1500\kappa!\epsilon^{-\kappa-2}\cdot L(1,\chi_d)A^{1+3\epsilon} \\
&\quad \times \Big((\log d)^{\kappa-1}\tfrac{2\pi}{\sqrt{NU}} + (\log A)^{-8g}\big(\tfrac{4\pi^2}{N}\big)^{\frac{1}{10}} \Big)(\log U)^3 \\
&\leq \tfrac{1}{2}\cdot 1500\cdot e^6\cdot \kappa!\cdot L(1,\chi_d)A\cdot(\log d)^{\kappa+2} \\
&\quad \times \Big((\log d)^{\kappa-1-4g}\tfrac{2\pi}{\sqrt{N}} + (\log A)^{-8g}\big(\tfrac{4\pi^2}{N}\big)^{\frac{1}{10}} \Big)(8g\log\log d)^3 \\
&\leq \tfrac{1}{2}\cdot 8^3\cdot 1500\cdot e^6\cdot g!\cdot g^3\cdot L(1,\chi_d)A\cdot(\log d)^{g+1} \\
&\quad \times \Big(2\cdot(\log d)^{-3g-2}\tfrac{2\pi}{\sqrt{N}} \Big)\cdot(\log\log d)^3 \\
&\leq 8^3\cdot 1500\cdot \tfrac{2\pi}{\sqrt{N}}\cdot e^{7-g}\cdot g^{g+3.5}\cdot(4000Ng^3)^{-2g-1} \\
&\quad \times L(1,\chi_d)A(\log\log A)^3 \\
&< S_1^*. \qquad (30)
\end{aligned}$$

Upper Bound for $|2H - \mathbf{T}\big(G(s,U)\big)|$. We note that κ is determined so that $\delta = 1 + (-1)^\kappa\chi_d(-N) = 2$. Then from [(45), Go], we have

$$\mathbf{T}(G(s,U)) = 2\cdot\tfrac{\kappa!}{2\pi i}\Big[\int_{\mathbf{C}}(s-\tfrac{1}{2})^{-\kappa-1}\sum_{r=1}^{5} I_r(s)ds\Big] \;+\; 2H, \qquad (31)$$

where $\mathbf{C}$ is the circle of radius $\frac{1}{2}\epsilon$ centered at $s=\frac{1}{2}$ and

$$I_1 = \int_{\frac{1}{8}+iM}^{\frac{1}{8}+i\infty},\ I_2 = \int_{\frac{1}{8}-i\infty}^{\frac{1}{8}-iM},\ I_3 = \int_{-\epsilon+iM}^{\frac{1}{8}+iM},\ I_4 = \int_{\frac{1}{8}-iM}^{-\epsilon-iM},\ I_5 = \int_{-\epsilon-iM}^{-\epsilon+iM}$$

of which the integrands are $\frac{1}{2\pi i}A^{s+z}\Gamma^2(s+z+\frac{1}{2})G(s+z,U)\varphi_1(2s+2z)\frac{dz}{z}$ and M is a large number to be determined later.

By [(46), Go], for $\sigma > 0$,

$$|\Gamma(s)| \leq \sqrt{2\pi}\exp\big(\tfrac{1}{12\sigma}\big)|s|^{\sigma-\frac{1}{2}} \begin{cases} \exp(-\sigma) & \text{if } |\frac{\sigma}{t}| \geq \frac{\pi}{2} \\ \exp(-\frac{\pi}{2}|t|) & \text{if } |\frac{\sigma}{t}| \leq \frac{\pi}{2}. \end{cases} \qquad (32)$$

From [(47), Go], we have for $\text{Re}(s+z) \geq 0$,

$$|G(s+z,U)| \leq (\log d)^{32g}. \tag{33}$$

To estimate $|\varphi_1(2s+2z)|$, we will use Lemma 3.1. Put $M = \log A$ and $\epsilon = (4 \cdot 10^5 \log\log A)^{-1}$. Then we have

$$1 - \frac{1}{100800 \log|\text{Im}(2s+2z)|} \leq \text{Re}(2s+2z) \quad \text{for} \quad z \in I_j \ \ (j = 1,2,3,4,5).$$

To estimate I_1, I_2, I_3 and I_4, we will use the fact that for $y > 1000$,

$$3 \cdot (2y)^2 \cdot (3y)^6 \cdot y^{-1} \cdot e^{-3y} \leq 10^{-830} \cdot e^{-y}. \tag{34}$$

Firstly, we consider the integral I_1. For $z = \frac{1}{8} + iy, \quad M \leq y < \infty$, we write

$$\sigma = \text{Re}(s+z+\tfrac{1}{2}) = \tfrac{9}{8} + \text{Re}(\tfrac{\epsilon}{2}e^{i\theta}), \quad t = \text{Im}(s+z+\tfrac{1}{2}) = y + \text{Im}(\tfrac{\epsilon}{2}e^{i\theta}).$$

By applying (32), (33), (34) and Lemma 3.1 to the integral I_1, we have

$$\begin{aligned}
\max_{s\in\mathbf{C}} |I_1| \;\leq\; & \max_{\substack{s\in\mathbf{C},\\ Re(z)=\frac{1}{8}}} |A^{s+z}G(s+z,U)| \\
& \cdot \max_{s\in\mathbf{C}} \left| \int_{\frac{1}{8}+iM}^{\frac{1}{8}+i\infty} \tfrac{1}{2\pi i}\Gamma^2(s+z+\tfrac{1}{2})\varphi_1(2s+2z)\tfrac{dz}{z} \right| \\
\leq\; & 3\times 10^{12} \cdot N^3(\log d)^{32g} A^{\frac{5}{8}+\frac{\epsilon}{2}} \\
& \cdot \max_{s\in\mathbf{C}} \int_{\frac{1}{8}+iM}^{\frac{1}{8}+i\infty} \exp(\tfrac{1}{6\sigma})|s+z+\tfrac{1}{2}|^{2\sigma-1}\exp(-\pi t)(2t)^6 |\tfrac{dz}{z}| \\
\leq\; & 3\times 10^{12} \cdot N^3(\log d)^{32g} A^{\frac{5}{8}+\frac{\epsilon}{2}} \\
& \cdot \int_M^\infty 3(2y)^2 \exp(-3y)(3y)^6 y^{-1} dy \\
\leq\; & 10^{-800} \cdot N^3(\log d)^{32g} A^{\frac{5}{8}+\frac{\epsilon}{2}} \int_M^\infty e^{-y} dy \\
\leq\; & 10^{-800} \cdot N^3(\log d)^{32g} A^{\frac{5}{8}+\frac{\epsilon}{2}} e^{-M}.
\end{aligned} \tag{35}$$

Similarly

$$\max_{s\in\mathbf{C}} |I_2| \leq 10^{-800} \cdot N^3(\log d)^{32g} A^{\frac{5}{8}+\frac{\epsilon}{2}} e^{-M}. \tag{36}$$

Secondly, we consider the integral I_3. For $z = x + iM, \quad -\epsilon \leq x < \frac{1}{8}$, we write

$$\sigma = \text{Re}(s+z+\tfrac{1}{2}) = x + 1 + \text{Re}(\tfrac{\epsilon}{2}e^{i\theta}), \quad t = \text{Im}(s+z+\tfrac{1}{2}) = M + \text{Im}(\tfrac{\epsilon}{2}e^{i\theta}).$$

By applying (32), (33), (34) and Lemma 3.1 to the integral I_3, we have

$$\begin{aligned}
\max_{s\in\mathbf{C}} |I_3| &\le \max_{\substack{s\in\mathbf{C}, \\ -\epsilon\le Re(z)\le\frac{1}{8}}} |A^{s+z}G(s+z,U)| \\
&\quad \cdot \max_{s\in\mathbf{C}} \left| \int_{-\epsilon+iM}^{\frac{1}{8}+iM} \tfrac{1}{2\pi i}\Gamma^2(s+z+\tfrac{1}{2})\varphi_1(2s+2z)\tfrac{dz}{z} \right| \\
&\le 3\times 10^{12}\cdot N^3(\log d)^{32g}A^{\frac{5}{8}+\frac{\epsilon}{2}} \\
&\quad \cdot \max_{s\in\mathbf{C}} \int_{-\epsilon+iM}^{\frac{1}{8}+iM} \exp\left(\tfrac{1}{6\sigma}\right)|s+z+\tfrac{1}{2}|^{2\sigma-1}\exp\left(-\pi t\right)(2t)^6|\tfrac{dz}{z}| \\
&\le 3\times 10^{12}\cdot N^3(\log d)^{32g}A^{\frac{5}{8}+\frac{\epsilon}{2}} \\
&\quad \cdot \int_{-\epsilon}^{\frac{1}{8}} 3(2M)^2\exp\left(-3M\right)(3M)^6M^{-1}dx \\
&\le 10^{-800}\cdot N^3(\log d)^{32g}A^{\frac{5}{8}+\frac{\epsilon}{2}}e^{-M}. \qquad (37)
\end{aligned}$$

Similarly

$$\max_{s\in\mathbf{C}} |I_4| \le 10^{-800}\cdot N^3(\log d)^{32g}A^{\frac{5}{8}+\frac{\epsilon}{2}}e^{-M}. \qquad (38)$$

Finally, we will estimate the integral I_5. For $z=-\epsilon+iy,\ \ -M\le y\le M$, we write

$$\sigma=\mathrm{Re}(s+z+\tfrac{1}{2})=1-\epsilon+\mathrm{Re}(\tfrac{\epsilon}{2}e^{i\theta}),\quad t=\mathrm{Im}(s+z+\tfrac{1}{2})=y+\mathrm{Im}(\tfrac{\epsilon}{2}e^{i\theta}).$$

By applying (33) to the integral I_5, we have

$$\begin{aligned}
\max_{s\in\mathbf{C}} |I_5| &\le \max_{\substack{s\in\mathbf{C}, \\ Re(z)=-\epsilon}} |A^{s+z}G(s+z,U)| \\
&\quad \cdot \max_{s\in\mathbf{C}} \left| \int_{-\epsilon-iM}^{-\epsilon+iM} \tfrac{1}{2\pi i}\Gamma^2(s+z+\tfrac{1}{2})\varphi_1(2s+2z)\tfrac{dz}{z} \right| \\
&\le (\log d)^{32g}A^{\frac{1}{2}(1-\epsilon)} \\
&\quad \cdot \max_{s\in\mathbf{C}} \int_{-\epsilon-iM}^{-\epsilon+iM} \tfrac{1}{2\pi}|\Gamma^2(s+z+\tfrac{1}{2})|\cdot|\varphi_1(2s+2z)|\cdot|\tfrac{dz}{z}|. \qquad (39)
\end{aligned}$$

To apply (32) and Lemma 3.1 to the integral I_5, we consider the following four integrals. Let y_1, y_2 and y_3 as follows.

$$\begin{aligned}
& \max_{s\in\mathbf{C}} \int_0^M \tfrac{1}{2\pi}|\Gamma^2(s+z+\tfrac{1}{2})|\cdot|\varphi_1(2s+2z)|\cdot \tfrac{dy}{\sqrt{\epsilon^2+y^2}} \\
\leq\ & \max_{s\in\mathbf{C}} \Big(\int_0^{\frac{1}{2\pi}(4-(6+\pi)\epsilon)} * + \int_{\frac{1}{2\pi}(4-(6+\pi)\epsilon)}^{\frac{1}{2\pi}(4+(\pi-2)\epsilon)} * + \int_{\frac{1}{2\pi}(4+(\pi-2)\epsilon)}^{2+\frac{1}{900}} * + \int_{2+\frac{1}{900}}^{M} * \Big) \\
=\ & \max_{s\in\mathbf{C}} \int_0^{y_1} * + \max_{s\in\mathbf{C}} \Big(\int_{y_1}^{y_2} * + \int_{y_2}^{y_3} * \Big) + \max_{s\in\mathbf{C}} \int_{y_3}^{M} *, \qquad (40)
\end{aligned}$$

where $* = \frac{1}{2\pi}|\Gamma^2(s+z+\frac{1}{2})|\cdot|\varphi_1(2s+2z)|\cdot\frac{dy}{\sqrt{\epsilon^2+y^2}}$.

We note that for $0 \leq y \leq y_1$,

$$\frac{\sigma}{t} \geq \frac{1-\frac{3\epsilon}{2}}{y_1+\frac{\epsilon}{2}} = \frac{\pi}{2}.$$

Thus, by applying (32) and Lemma 3.1 to the first interval, we have

$$\begin{aligned}
\max_{s\in\mathbf{C}} \int_0^{y_1} * \ \leq\ & 10^5\cdot N^3 \max_{s\in\mathbf{C}} \int_0^{y_1} \exp\big(\tfrac{1}{6\sigma}\big)|s+z+\tfrac{1}{2}|^{2\sigma-1}\exp(-2\sigma) \\
& \times \tfrac{1}{|2s+2z-1|}\tfrac{dy}{\sqrt{\epsilon^2+y^2}} \\
\leq\ & 10^5\cdot N^3 \int_0^{\frac{2}{\pi}} (y+1)\cdot\epsilon^{-2}dy \\
<\ & 10^5\cdot N^3\epsilon^{-2}. \qquad (41)
\end{aligned}$$

We need the following observation to apply (32) to the second and third intervals. For $y_1 \leq y \leq y_2$, we have

$$\begin{aligned}
& \max\big\{\exp(-\sigma), \exp(-\tfrac{\pi}{2}|t|)\big\} \\
\leq\ & \max\big\{\exp\big(-(1-\tfrac{3\epsilon}{2})\big), \exp\big(-\tfrac{\pi}{2}(y_1-\tfrac{\epsilon}{2})\big)\big\} \\
=\ & \exp(-1+\tfrac{3+\pi}{2}\epsilon).
\end{aligned}$$

For $y_2 \leq y \leq y_3$, we have

$$\frac{\sigma}{t} \leq \frac{1-\frac{\epsilon}{2}}{y_2-\frac{\epsilon}{2}} = \frac{\pi}{2}$$

and

$$\begin{aligned}
& \exp(-\tfrac{\pi}{2}|t|) \\
\leq\ & \exp(-\tfrac{\pi}{2}(y_2-\tfrac{\epsilon}{2})) \\
<\ & \exp(-1+\tfrac{3+\pi}{2}\epsilon).
\end{aligned}$$

Thus, by applying (32) and Lemma 3.1 to the second and third interval, we have

$$
\begin{aligned}
\max_{s\in\mathbf{C}}\int_{y_1}^{y_3} * &\leq 10^5\cdot N^3 \max_{s\in\mathbf{C}}\int_{y_1}^{y_3}\exp\left(\tfrac{1}{6\sigma}\right)|s+z+\tfrac{1}{2}|^{2\sigma-1} \\
&\quad \times\exp\big(-2+(3+\pi)\epsilon\big)\tfrac{1}{|2s+2z-1|}\tfrac{dy}{\sqrt{\epsilon^2+y^2}} \\
&\leq 10^5\cdot N^3\int_{\frac{1}{\pi}}^{y_3}(y+1)\cdot\pi\cdot\pi dy \\
&< 5\times 10^6\cdot N^3. \qquad (42)
\end{aligned}
$$

To estimate the fourth integral, we will use the fact that for $y\geq y_3$,

$$(y+1)\cdot(3y)^6\cdot y^{-1}\cdot e^{-3y}\leq 2000\cdot e^{-y}.$$

Thus, by applying (32) and Lemma 3.1 to the fourth interval, we have

$$
\begin{aligned}
\max_{s\in\mathbf{C}}\int_{y_3}^{M} * &\leq 3\cdot 10^{12}\cdot N^3 \\
&\quad \times\max_{s\in\mathbf{C}}\int_{y_3}^{M}\exp\left(\tfrac{1}{6\sigma}\right)|s+z+\tfrac{1}{2}|^{2\sigma-1}\exp\left(-\pi t\right)(2t)^6\tfrac{dy}{y} \\
&\leq 3\cdot 10^{12}\cdot N^3\int_{y_3}^{M}(y+1)\exp\left(-3y\right)(3y)^6 y^{-1}dy \\
&\leq 3\cdot 10^{12}\cdot N^3\int_{y_3}^{M}2000e^{-y}dy \\
&< 9\times 10^{14}\cdot N^3. \qquad (43)
\end{aligned}
$$

From (39), (40), (41), (42) and (43), we have

$$
\begin{aligned}
|I_5| &\leq 2\cdot\Big(N^3(\log d)^{32g}A^{\frac{1}{2}(1-\epsilon)}\big(10^5\cdot\epsilon^{-2}+5\times 10^6+9\times 10^{14}\big)\Big) \\
&< N^3(\log d)^{32g}A^{\frac{1}{2}(1-\epsilon)}\cdot 2\cdot\big(10^5\cdot\epsilon^{-2}+10^{15}\big) \qquad (44)
\end{aligned}
$$

Finally, by (31), (35), (36), (37), (38) and (44), we have

$$\begin{aligned}
&|2H - \mathbf{T}(G(s,U))| \\
= &\left| 2 \cdot \tfrac{\kappa!}{2\pi i} \int_{\mathbf{C}} (s - \tfrac{1}{2})^{-\kappa-1} \sum_{r=1}^{5} I_r(s) ds \right| \\
\leq &\ 2^{\kappa+1}\kappa!\epsilon^{-\kappa} \sum_{r=1}^{5} \max_{s\in\mathbf{C}} |I_r(s)| \\
< &\ 2^{\kappa+1}\kappa!\epsilon^{-\kappa} N^3 (\log d)^{32g} \sqrt{A} \\
&\cdot \big(4 \times 10^{-800} \cdot A^{\frac{1}{8}+\frac{\epsilon}{2}} e^{-M} + 2 \times 10^5 \cdot A^{-\frac{\epsilon}{2}} \epsilon^{-2} + 2 \times 10^{15} \cdot A^{-\frac{\epsilon}{2}}\big) \\
< &\ 2^{\kappa+1}\kappa!\epsilon^{-\kappa} N^3 (\log d)^{32g} \sqrt{A} \cdot 3 \cdot \big(2 \times 10^5 \cdot A^{-\frac{\epsilon}{2}} \epsilon^{-2}\big). \qquad (45)
\end{aligned}$$

For $d \geq \exp\exp(4000Ng^3)$, we see that

$$2^{\kappa+1}\kappa!\epsilon^{-\kappa} N^3 (\log d)^{32g} \cdot 3 \cdot \big(2 \times 10^5 \cdot A^{-\frac{\epsilon}{2}} \epsilon^{-2}\big) < 1,$$

so by (45), we have

$$|2H - \mathbf{T}(G(s,U))| < \sqrt{A} < S_1^*, \qquad (46)$$

as desired (cf. [p.656, Go]).

Now we can prove Proposition 2.2.

Proof of Proposition 2.2. We may assume

$$L(1, \chi_d) \leq (\log d)^{\kappa-1} \tfrac{1}{\sqrt{d}} \quad (d > \exp\exp(4000Ng^3)).$$

From (17), (21), (25), (30) and (46), we have for $d \geq \exp\exp(4000Ng^3)$,

$$\begin{aligned}
&|2H| \\
\leq &\ |2H - \mathbf{T}(G(s,U))| + |T(g(s))| + |S_1| + |S_2| + 1 \\
\leq &\ 5S_1^* \\
< &\ 4 \times 10^9 \cdot (\tfrac{80}{e})^g g^{2g+4.5} L(1,\chi) A (\log\log A)^{\kappa+6}
\end{aligned}$$

and Proposition 2.2 immediately follows. □

5. Proof of Lemma 3.3

In this section, we will prove Lemma 3.3.

Proof of Lemma 3.3. Let ψ' be a primitive *Grössencharakter* with conductor $\mathfrak{f}'$ of $K = \mathbb{Q}(\sqrt{-k})$ which induces ψ^2. Then $\psi'((\alpha)) = \alpha^2$ for $\alpha \equiv 1 \mod \mathfrak{f}'$. Since $L_E(s) = L_K(s, \psi)$, $L_K(s, \psi')$ is entire and has real coefficients.

We define (cf. [p.661, Go])

$$F(s) = \zeta(s)L(s, \chi_k)L_K(s+1, \psi') = \sum_{n=1}^{\infty} c_n n^{-s},$$

where

$$c_1 = 1, \;\; c_n \geq 0 \;\; (\text{for } n > 1).$$

Since the Dirichlet series expansion of $F(s)$ is majorised by that of $\zeta(s)^4$, we have

$$c_n \leq \sum_{lm=n} d(l)d(m) \leq \sum_{lm=n} 4\sqrt{n} \leq 8n \;\; (\text{for } n \geq 1) \tag{47}$$

where $d(k) = \sum_{f|k} 1 \leq 2\sqrt{k}$.

For fixed $x > 0$, we see that

$$\begin{aligned}
& \tfrac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s+1)F(s)x^s ds \\
= \;& \tfrac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \int_0^{\infty} e^{-u}u^s (\sum_{n=1}^{\infty} \tfrac{c_n}{n^s}) x^s du ds \\
= \;& \tfrac{1}{2\pi i} \sum_{n=1}^{\infty} c_n \int_0^{\infty} \int_{2-i\infty}^{2+i\infty} (\tfrac{ux}{n})^s ds \cdot e^{-u} du \\
= \;& \sum_{n=1}^{\infty} \tfrac{c_n}{e^{n/x}} \\
\geq \;& e^{-1/x},
\end{aligned}$$

so we have

$$\begin{aligned}
e^{-1/x} \;\; & \leq \;\; \tfrac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s+1)F(s)x^s ds \\
& = \;\; \Gamma(2)L(1, \chi_k)L_K(2, \psi')x + \tfrac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s+1)F(s)x^s ds.
\end{aligned} \tag{48}$$

The last integral in (48) can be estimated by using the following functional equations;

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s),$$
$$L(s,\chi_k) = (\tfrac{k}{\pi})^{\frac{1}{2}-s} \frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})} L(1-s,\chi_k),$$
$$L_K(s+1,\psi') = w(\tfrac{\sqrt{kN(\mathfrak{f}')}}{2\pi})^{1-2s} \frac{\Gamma(2-s)}{\Gamma(s+1)} L_K(2-s,\psi')$$

for some $w \in \mathbb{C},\ |w| = 1$.

Let $y = \frac{16\pi^4 x}{k^2 N(\mathfrak{f}')}$. Then by the duplication formula of Gamma function,

$$\begin{aligned} & \tfrac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s+1)F(s)x^s ds \\ = \ & w\tfrac{k\sqrt{N(\mathfrak{f}')}}{4\pi^2} \tfrac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \tfrac{\Gamma(1-s)}{\Gamma(s)} \Gamma(2-s)F(1-s)y^s ds. \end{aligned} \tag{49}$$

Using (47) and the following properties of Bessel function $J_0(2\sqrt{t}) = \sum_{n=0}^{\infty}(-1)^n \frac{t^n}{(n!)^2}$;

$$0 \le J_0(2\sqrt{t}) \le \exp(-t) \quad \text{for} \quad t \ge 0,$$
$$\int_0^\infty J_0(2\sqrt{t})t^{-s} dt = \tfrac{\Gamma(1-s)}{\Gamma(s)},$$

we have

$$\begin{aligned} & \tfrac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \tfrac{\Gamma(1-s)}{\Gamma(s)} \Gamma(2-s)F(1-s)y^s ds \\ = \ & \tfrac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \tfrac{\Gamma(s)}{\Gamma(1-s)} \Gamma(s+1)F(s)y^{1-s} ds \\ = \ & \tfrac{1}{2\pi i} \sum_{n=1}^{\infty} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \int_0^\infty \int_0^\infty J_0(2\sqrt{t})t^{s-1} \cdot u^s e^{-u} \cdot \frac{c_n}{n^s} \cdot y^{1-s} du\ dt\ ds \\ = \ & \sum_{n=1}^{\infty} c_n \int_0^\infty \int_0^\infty \tfrac{1}{2\pi i} \int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} (\frac{ut}{ny})^s ds \cdot J_0(2\sqrt{t}) e^{-u} t^{-1} y\ du\ dt \\ = \ & \sum_{n=1}^{\infty} c_n \int\int_{ut=ny} J_0(2\sqrt{t}) e^{-u} t^{-1} y\ du\ dt \\ \le \ & \sum_{n=1}^{\infty} \frac{c_n}{n} \int\int_{ut=ny} \exp(-t)\exp(-u) \frac{ny}{t}\ du\ dt \\ \le \ & 8\sum_{n=1}^{\infty} \int_0^\infty \exp(-t - \tfrac{ny}{t}) \tfrac{ny}{t} dt. \end{aligned} \tag{50}$$

Dividing integration with respect to t into two intervals $(0, \sqrt{ny})$ and $(\sqrt{ny}, \infty)$, we have

$$
\begin{aligned}
& 8\sum_{n=1}^{\infty}\int_0^{\infty} \exp\,(-t - \tfrac{ny}{t})\tfrac{ny}{t}dt \\
= \;& 8\sum_{n=1}^{\infty}\Big(\int_0^{\sqrt{ny}} \exp\,(-t - \tfrac{ny}{t})\tfrac{ny}{t}dt + \int_{\sqrt{ny}}^{\infty} \exp\,(-t - \tfrac{ny}{t})\tfrac{ny}{t}dt\Big) \\
\leq \;& 8\sum_{n=1}^{\infty}\Big(\int_0^{\sqrt{ny}} \exp\,(-\tfrac{ny}{t})\tfrac{ny}{t}dt + \int_{\sqrt{ny}}^{\infty} \exp\,(-t)\tfrac{ny}{t}dt\Big) \\
= \;& 16\sum_{n=1}^{\infty}\int_{\sqrt{ny}}^{\infty} \exp\,(-t)\tfrac{ny}{t}dt \\
\leq \;& 16\sum_{n=1}^{\infty}\int_{\sqrt{ny}}^{\infty} \sqrt{ny}\exp\,(-t)dt \\
= \;& 16\sum_{n=1}^{\infty} \sqrt{ny}\exp\,(-\sqrt{ny}). \qquad (51)
\end{aligned}
$$

Now let $x = k^4 N(\mathfrak{f}')^2$ so that $y = \frac{16\pi^4 x}{k^2 N(\mathfrak{f}')} = 16\pi^4 k^2 N(\mathfrak{f}')$. Then by (49), (50) and (51), we have

$$
\begin{aligned}
& \left|\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s+1)F(s)x^s ds\right| \\
\leq \;& \frac{k\sqrt{N(\mathfrak{f}')}}{4\pi^2}\cdot 16\sum_{n=1}^{\infty}\sqrt{ny}\exp\,(-\sqrt{ny}) \\
\leq \;& \frac{k\sqrt{N(\mathfrak{f}')}}{4\pi^2}\cdot 16\cdot 5!\sum_{n=1}^{\infty}\frac{\sqrt{ny}}{(\sqrt{ny})^5} \\
\leq \;& \frac{k\sqrt{N(\mathfrak{f}')}}{4\pi^2}\cdot \frac{1}{(4\pi^2 k\sqrt{N(\mathfrak{f}')}\,)^4}\cdot 16\cdot 5!\sum_{n=1}^{\infty}\frac{1}{n^2} \\
< \;& (4\pi^2)^{-5}\cdot 16\cdot 5!\cdot\frac{\pi^2}{6} \\
< \;& 4\cdot 10^{-5}. \qquad (52)
\end{aligned}
$$

Since $x = k^4 N(\mathfrak{f}')^2 \geq 3^4$, (48) and (52) give

$$|L_K(2,\psi')L(1,\chi_k)| \geq \frac{e^{-1/x} - 4\cdot 10^{-5}}{x} \geq \frac{e^{-1/81} - 4\cdot 10^{-5}}{k^4 N(\mathfrak{f}')^2} \geq \frac{0.98}{k^4 N(\mathfrak{f}')^2}.$$

From [(4) and Theorem 2, Go], we have

$$kN(\mathfrak{f}') \leq kN(\mathfrak{f}) = N$$

and by [(59), Go], we have

$$|L_K(2,\psi^2)L(1,\chi_k)| \geq N^{-2}|L_K(2,\psi')L(1,\chi_k)| \geq \frac{0.98}{k^2 N^4}.$$

□

Acknowledgment. The authors thank the referees for their careful readings and many valuable suggestions.

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Department of Mathematical Sciences,
Seoul National University,
Seoul, Korea
E-mail: dhbyeon@snu.ac.kr

Department of Mathematical Sciences,
Seoul National University,
Seoul, Korea
E-mail: potter@snu.ac.kr