AN EXPLICIT LOWER BOUND FOR SPECIAL VALUES OF DIRICHLET L-FUNCTIONS II

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Abstract. Let d be a fundamental discriminant, χ_d be the Dirichlet character associated to the quadratic field $\mathbb{Q}(\sqrt{d})$ and $L(s, \chi_d)$ be the Dirichlet L-function. In [Go], Goldfeld obtained an effective lower bound for $L(1, \chi_d)$ with uncalculated constants. For $d < 0$, the constants are computed in [Oe] and for $d > 0$, the constants are computed in [BK] by using elliptic curves with complex multiplication. In this paper, we show that the result of [BK] is worked out for elliptic curves without complex multiplication too and compute the corresponding constants.

1. Introduction and results

Let d be a fundamental discriminant, χ_d be the Dirichlet character associated to the quadratic field $\mathbb{Q}(\sqrt{d})$ and $L(s, \chi_d)$ be the Dirichlet L-function. In [Go], Goldfeld obtained an effective lower bound for $L(1, \chi_d)$.

Theorem 1.1. [Go, Theorem 1] *Let E be an elliptic curve over $\mathbb{Q}$ with conductor N. If E has complex multiplication and the L-function associated to E has a zero of order g at $s = 1$, then for any χ_d with $(d, N) = 1$ and $|d| > \exp\exp(c_1 N g^3)$, we have*

$$L(1, \chi_d) > \frac{c_2}{g^{4g} N^{13}} \frac{(\log |d|)^{g-\mu-1} \exp(-21\sqrt{g \log\log |d|})}{\sqrt{|d|}},$$

where $\mu = 1$ or 2 is suitably chosen so that $\chi_d(-N) = (-1)^{g-\mu}$, and the constants c_1, $c_2 > 0$ can be effectively computed and are independent of g, N and d.

In fact, Goldfeld proved Theorem 1.1 under the assumption that the associated base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order

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$\geq g$. In [BK], we explicitly computed the constants c_1, c_2 for $d > 0$ in Theorem 1.1 and proved the following theorem.

Theorem 1.2. [BK, Theorem 1.3] *Let $d > 0$ be a fundamental discriminant of a real quadratic field $\mathbb{Q}(\sqrt{d})$. Let E be an elliptic curve over $\mathbb{Q}$ with conductor N and $g \geq 4$ be a positive integer. If E has complex multiplication and the associated base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$ at $s = 1$, then for any such d with $(d, N) = 1$ and $d > \exp\exp(4000Ng^3)$, we have*

$$L(1, \chi_d) > \frac{10^{180}}{g^{4g}N^5} \frac{(\log d)^{g-3} \exp(-21\sqrt{g \log\log d})}{\sqrt{d}}.$$

In [Go], Goldfeld remarked that Theorem 1.1 also holds for elliptic curves E without complex multiplication provided that $L_E(s)$ comes from a cusp form of $\Gamma_0(N)$, which is now true for every elliptic curves E over $\mathbb{Q}$ with conductor N according to the modularity theorem (cf. [BCDT], [Wi]). But he did not give the proof. In this paper, we show that the result of [BK] is worked out for elliptic curves without complex multiplication too and explicitly compute the corresponding constants.

Theorem 1.3. *Let $d > 0$ be a fundamental discriminant of a real quadratic field $\mathbb{Q}(\sqrt{d})$. Let E be an elliptic curve over $\mathbb{Q}$ with conductor N of which the product of distinct prime factors is 13 or more and $g \geq 4$ be a positive integer. If the associated base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$ at $s = 1$, then for any such d with $(d, N) = 1$ and $d > \exp\exp(300Ng^3)$, we have*

$$L(1, \chi_d) > \frac{6 \times 10^{184}}{g^{4g}N} \frac{(\log d)^{g-3} \exp(-21\sqrt{g \log\log d})}{\sqrt{d}}.$$

The proof of Theorem 1.3 goes along similar lines as that of Theorem 1.2 in the previous paper [BK]. However, to remove complex multiplication condition, we will use the motivic symmetric square L-function (see (5) for the definition) and the modularity theorem instead of Hecke L-function and Deuring's Theorem for elliptic curves with complex multiplication (cf. [Go, Theorem 2]).

Remark 1.4. Let E be an elliptic curve with complex multiplication by an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-k})$. In the proof of Theorem 1.1, Goldfeld use the fact that $k \leq N$ as well as Deuring's theorem. In the proof of Theorem 1.2, we use the fact that $k \leq 163$ as well as Deuring's theorem. In the proof of Theorem 1.3, we use theory of the motivic symmetric square L-function instead of Deuring's theorem. That is why there is a difference for exponents of N among Theorem 1.1, Theorem 1.2 and Theorem 1.3.

2. Proof of Theorem 1.3

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N. Assume the same conditions as in Theorem 1.3. When the associated Hasse-Weil L-function $L_E(s) = L_{E/\mathbb{Q}}(s)$ over $\mathbb{Q}$ is given by

$$L_E(s) = \sum_{n=1}^{\infty} E_n n^{-s},$$

define

$$L_E(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n) E_n n^{-s} \text{ and } L_E(s, \lambda) = \sum_{n=1}^{\infty} \lambda(n) E_n n^{-s},$$

where $\lambda(n) = \prod_{p^r || n} (-1)^r$. Let

$$\varphi(s) = L_E(s + \tfrac{1}{2}) L_E(s + \tfrac{1}{2}, \chi_d) = \sum_{n=1}^{\infty} a_n n^{-s}$$

and

$$\varphi_1(2s) = L_E(s + \tfrac{1}{2}) L_E(s + \tfrac{1}{2}, \lambda).$$

We note that $\varphi(s) = L_{E/\mathbb{Q}(\sqrt{d})}(s + \frac{1}{2})$ has a zero of order $\geq g$ at $s = \frac{1}{2}$. Let

$$G(s) = \frac{\varphi(s)}{\varphi_1(2s)} = \sum_{n=1}^{\infty} g_n n^{-s} \text{ and } G(s, x) = \sum_{n<x} g_n n^{-s}.$$

For $A = \frac{dN}{4\pi^2}$ and $U = (\log d)^{8g}$, let

$$H = \left(\tfrac{d}{ds}\right)^{g-\mu} \left[A^s \Gamma^2(s + \tfrac{1}{2}) G(s, U) \varphi_1(2s)\right]_{s=\frac{1}{2}}.$$

To prove Theorem 1.3, we need the following propositions.

Proposition 2.1. *Assume the same conditions as in Theorem 1.3. Then for any such* $d \geq \exp\exp(300Ng^3)$, *either* $L(1, \chi_d) > (\log d)^{g-\mu-1} \frac{1}{\sqrt{d}}$ *or else*

$$|H| \geq 1.2 \times 10^{-3} \cdot g\sqrt{N}(\log N)^{-1} \sqrt{d} (\log d)^{g-\mu-1} \prod_{\substack{\chi_d(p) \neq -1 \\ p<U}} \left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}} \right)^2 .$$

Proposition 2.2. *Assume the same conditions as in Theorem 1.3. Then for any such* $d \geq \exp\exp(300Ng^3)$, *either* $L(1,\chi_d) > (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}}$ *or else*

$$|H| \leq 2\times 10^9 \cdot (\tfrac{80}{e})^g g^{2g+4.5} L(1,\chi_d) A(\log\log A)^{g-\mu+6}.$$

We will prove Proposition 2.1 in Section 3 and Proposition 2.2 in Section 4. If we assume Proposition 2.1 and 2.2, then we can prove Theorem 1.3 as follows.

Proof of Theorem 1.3. Let $\mathcal{P}$ be the set of primes $p < (\log d)^{8g}$ for which $\chi_d(p) \neq -1$. If

$$L(1,\chi_d) > (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}} \quad (d \geq \exp\exp(300Ng^3) \text{ and } N \geq 13),$$

then Theorem 1.3 is true. Thus we may assume

$$L(1,\chi_d) \leq (\log d)^{g-\mu-1}\frac{1}{\sqrt{d}} \quad (d \geq \exp\exp(300Ng^3) \text{ and } N \geq 13).$$

From [BK, p. 276] we have

$$\log \prod_{p\in\mathcal{P}} \left(\frac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\right)^2 \leq 20 g^{\frac{1}{2}} (\log\log d)^{\frac{1}{2}}.$$

Let $f(N,g,d) = \exp\left(g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\right) \cdot (\frac{80}{e})^{-g} g^{2g-4.5} (\log\log \frac{dN}{4\pi^2})^{-g-5}$. We claim that if $N \geq 13$, $g \geq 3$ and $d \geq \exp\exp(300Ng^3)$, then

$$f(N,g,d) \geq \exp(450).$$

Since $\log\log\frac{dN}{4\pi^2} \leq \log\log d^e = \log\log d + 1$, we have

$$\begin{aligned} &\log f(N,g,d) \\ \geq\ & \left(g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\right) - g\log\tfrac{80}{e} + (2g-4.5)\log g - (g+5)\log(\log\log d + 1), \end{aligned}$$

which is an increasing function for d because its partial derivative with respect to d is

$$\begin{aligned} & \frac{\sqrt{g}}{2\sqrt{\log\log d}(\log d)d} - \frac{g+5}{(\log\log d+1)(\log d)d} \\ > \ & \frac{\sqrt{g(\log\log d)} - 2(g+5)}{2(\log\log d)(\log d)d} \\ > \ & 0. \end{aligned}$$

So we have

$$\begin{aligned} & \log f(N,g,d) \\ \geq\ & (300N)^{\frac{1}{2}}g^2 - g\log\tfrac{80}{e} + (2g-4.5)\log g - (g+5)\log(300Ng^3+1), \end{aligned}$$

which is an increasing function for g because its partial derivative with respect to g is

$$\begin{aligned} & 2(300N)^{\frac{1}{2}}g - \log(\tfrac{80}{e}) + 2\log g + \frac{2g-4.5}{g} \\ & -\log(300Ng^3+1) - \frac{3\cdot 300Ng^2(g+5)}{300Ng^3+1} \\ >\ & 2(300N)^{\frac{1}{2}}g - \log(\tfrac{80}{e}) - \tfrac{4.5}{g} - \log g - \log(300N+1) - \tfrac{3(g+5)}{g} \\ >\ & 0. \end{aligned}$$

So we have

$$\begin{aligned} & \log f(N,g,d) \\ \geq\ & (300N)^{\frac{1}{2}}\cdot 3^2 - 3\log\tfrac{80}{e} + 1.5\log 3 - 8\log(300\cdot 3^3N+1), \end{aligned}$$

which is an increasing function for N because its derivative with respect to N is

$$\frac{\sqrt{300}\cdot 3^2}{2\sqrt{N}} - \frac{8\cdot 300\cdot 3^3}{300\cdot 3^3N+1} > \frac{\sqrt{300}\cdot 3^2}{2\sqrt{N}} - \frac{8}{N} > 0.$$

So we have

$$\begin{aligned} & \log f(N,g,d) \\ \geq\ & \sqrt{3900}\cdot 3^2 - 3\log\tfrac{80}{e} + 1.5\log 3 - 8\log(3900\cdot 3^3+1) \\ >\ & 450 \end{aligned}$$

and the claim is proved. Thus we have

$$\exp\big(g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\big)\cdot(\tfrac{80}{e})^{-g}g^{2g-4.5}(\log\log\tfrac{dN}{4\pi^2})^{-g-5} \geq \exp(450).$$

From Proposition 2.1 and Proposition 2.2, we have for $d \geq \exp\exp(300Ng^3)$,

$$\begin{aligned} & 2\times 10^9\cdot(\tfrac{80}{e})^g g^{2g+4.5}L(1,\chi_d)A(\log\log A)^{g-\mu+6} \\ \geq\ & 1.2\times 10^{-3}\cdot g\sqrt{N}(\log N)^{-1}\sqrt{d}(\log d)^{g-\mu-1}\exp\big(-20g^{\frac{1}{2}}(\log\log d)^{\frac{1}{2}}\big). \end{aligned}$$

Thus we have

$$\begin{aligned} L(1,\chi_d) &> \frac{1.2\times 10^{-3}\cdot g\sqrt{N}(\log N)^{-1}}{2\times 10^9\cdot (\frac{80}{e})^g g^{2g+4.5}}\cdot \frac{\sqrt{d}(\log d)^{g-\mu-1}\exp\big(-20g^{\frac12}(\log\log d)^{\frac12}\big)}{A(\log\log A)^{g-\mu+6}} \\ &= \frac{1.2\times 10^{-3}\cdot 4\pi^2\cdot g}{2\times 10^9\cdot (\frac{80}{e})^g g^{2g+4.5}\sqrt{N}\log N}\cdot \frac{(\log d)^{g-\mu-1}\exp\big(-20g^{\frac12}(\log\log d)^{\frac12}\big)}{\sqrt{d}(\log\log \frac{dN}{4\pi^2})^{g-\mu+6}} \\ &> \frac{1.2\times 10^{-3}\cdot 4\pi^2\cdot \exp(450)}{2\times 10^9\cdot g^{4g}N}\cdot \frac{(\log d)^{g-3}\exp\big(-21g^{\frac12}(\log\log d)^{\frac12}\big)}{\sqrt{d}} \\ &> \frac{6\times 10^{184}}{g^{4g}N}\cdot \frac{(\log d)^{g-3}\exp\big(-21g^{\frac12}(\log\log d)^{\frac12}\big)}{\sqrt{d}}. \end{aligned}$$

□

3. Proof of Proposition 2.1

Let $\kappa = g-\mu$ and define H_1 and H_2 by

$$\begin{aligned} H &= H_1 + H_2 \\ &= 2\kappa\sqrt{A}(\log A)^{\kappa-1}G(\tfrac12,U)\varphi_1'(1) \\ &\quad +\sqrt{A}\sum_{r=2}^{\kappa}\binom{\kappa}{r}(\log A)^{\kappa-r}\left(\tfrac{d}{ds}\right)^r\left[\Gamma^2(s+\tfrac12)G(s,U)\varphi_1(2s)\right]_{s=\frac12}. \end{aligned}$$

Since $|H|\ge |H_1|-|H_2|$, to get an explicit lower bound for $|H|$, we need an explicit upper bound for $|H_2|$ and an explicit lower bound for $|H_1|$.

Upper Bound for $|H_2|$. By [BK, (3)], we have

$$|H_2|\le \sqrt{A}\sum_{r=2}^{\kappa}8^r r! r\binom{\kappa}{r}(\log A)^{\kappa-r}\max_{|s-\frac12|=\frac18}|\Gamma^2(s+\tfrac12)|\max_{|s-\frac12|=\frac18}|\varphi_1(2s)|\max_{|s-\frac12|=\frac14}|G(s,U)|. \tag{1}$$

By [BK, (4)], we have

$$\max_{|s-\frac12|=\frac18}|\Gamma^2(s+\tfrac12)|\le 1.6. \tag{2}$$

Now we give bounds for $\varphi_1(s)$. We denote by $S_2^p(N)$ the set of normalized primitive holomorphic cusp forms for $\Gamma_0(N)$ of weight 2 with trivial nebentypus 1_N. For any $f\in S_2^p(N)$, f has a Fourier expansion at infinity of the form

$$f(z)=\sum_{n\ge 1}a_f(n)\sqrt{n}e^{2\pi i n z}$$

with $a_f(1) = 1$ and $a_f(n)$ denoting the n-th eigenvalue of the Hecke operator T_n. From the Modularity Theorem, there exists $f \in S_2^p(N)$ such that

$$L_f(s) = L_E(s)$$

(cf. [DS, Theorem 8.8.3]).

Then we have

$$\begin{aligned} L_E(s+\tfrac{1}{2}) = L_f(s+\tfrac{1}{2}) &= \sum_{n=1}^{\infty} a_f(n) n^{-s} \\ &= \prod_p (1 - a_f(p)p^{-s} + 1_N(p)p^{-2s})^{-1} \\ &= \prod_{p \mid N} (1 - a_f(p)p^{-s}) \prod_{p \nmid N} (1-\alpha_p p^{-s})(1-\beta_p p^{-s}), \end{aligned}$$

where for $(p, N) = 1$, $\alpha_p + \beta_p = a_f(p)$, $|\alpha_p| = |\beta_p| = 1$, $\alpha_p = \overline{\beta}_p$ and for $p \mid N$,

$$a_f(p) = \begin{cases} \frac{1}{\sqrt{p}} & \text{if } E \text{ has split multiplicative reduction at } p, \\ -\frac{1}{\sqrt{p}} & \text{if } E \text{ has nonsplit multiplicative reduction at } p, \\ 0 & \text{if } E \text{ has additive reduction at } p \end{cases}$$

(cf. [Sil, Appendix C.16]).

For convenience, we follow the notation

$$L_E(s+\tfrac{1}{2}) = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}, \tag{3}$$

where

$$\begin{cases} \text{for } p \nmid N, & \alpha_p + \beta_p = a_f(p),\ |\alpha_p| = |\beta_p| = 1,\ \alpha_p = \overline{\beta}_p, \\ \text{for } p \parallel N, & \alpha_p = \pm\frac{1}{\sqrt{p}},\ \beta_p = 0, \\ \text{for } p^2 \mid N, & \alpha_p = \beta_p = 0 \end{cases}$$

(cf. [Wa, p. 490]).

The analytic symmetric square L-function is defined as

$$\begin{aligned} L^A(\mathrm{Sym}^2 E, s) &= \prod_p L_p^A(\mathrm{Sym}^2 E, s) \\ &= \prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}. \end{aligned} \tag{4}$$

To satisfy the functional equation, we must adjust $L^A(\mathrm{Sym}^2E, f)$ by appropriate Euler factors when $p^2 \mid N$. We can define the Euler product $U(s)$ by the motivic symmetric square L-function

$$L^M(\mathrm{Sym}^2E, s) = L^A(\mathrm{Sym}^2E, s) \cdot U(s) \tag{5}$$

so that

$$\Lambda^M(\mathrm{Sym}^2E, s) = \widetilde{N}^s \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{s+2}{2}\right) L^M(\mathrm{Sym}^2E, s)$$

satisfies the functional equation given by

$$\Lambda^M(\mathrm{Sym}^2E, s) = \Lambda^M(\mathrm{Sym}^2E, 1-s) \tag{6}$$

(cf. [Wa, p. 490]).

We denote by $\widetilde{N} = \prod_p p^{\widetilde{\delta}_p}$ the symmetric square conductor and denote by $U_p(s)$ the local factor of $U(s)$ at a prime p. Then we have

$$\begin{cases} \text{for } p \nmid N, & \widetilde{\delta}_p = 0,\ U_p(s) = 1, \\ \text{for } p \parallel N, & \widetilde{\delta}_p = 1,\ U_p(s) = 1, \\ \text{for } p^2 \mid N, & \widetilde{\delta}_p \geq 1, \text{ there are three possibilities for } U_p(s)\text{: } 1, (1 \pm p^{-s})^{-1}. \end{cases} \tag{7}$$

For additive reduction, that is, $p^2 \mid N$, both $\widetilde{\delta}_p$ and $U_p(s)$ were determined by Coates and Schmidt [CS] (with corrections by Watkins [Wa]). We note that $\widetilde{N}$ always divides N and is equal to N if the conductor is square-free (cf. [Wa, section 2.2]).

Remark 3.1. In this paper, we normalize the motivic symmetric square L-function in [Wa] so that $s = 1/2$ is the point of symmetry. In [Wa1], the symmetric square conductor is defined by $\widetilde{N}^2$.

From (3), (4), and (5), we have

$$\begin{aligned} \varphi_1(s) &= \prod_{p \mid N} (1 - a_f(p)^2 p^{-s})^{-1} \prod_{p \nmid N} (1 - \alpha_p^2 p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1} \\ &= \frac{L^A(\mathrm{Sym}^2E, s)}{\zeta(s)} \prod_{p \mid N} (1 - p^{-s})^{-1} \\ &= \frac{L^M(\mathrm{Sym}^2E, s)}{\zeta(s)} \prod_{p \mid N} (1 - p^{-s})^{-1} \prod_{p^2 \mid N} U_p(s)^{-1}. \end{aligned} \tag{8}$$

The following lemma is a generalization of [BK, Lemma 3.1].

Lemma 3.2. *For $s = \sigma + it \in \mathbb{C}$,*

$$|\varphi_1(s)| \le \begin{cases} 2 \times 10^{10} \cdot N^3 t^6 & \text{if } 1 - \frac{1}{100800 \log |t|} \le \sigma \le \frac{3}{2}, \quad |t| \ge 2 + \frac{1}{840}, \\ 2 \cdot N^3 |s+2|^3 & \text{if } \frac{3}{4} \le \sigma \le \frac{3}{2}, \quad |t| \le 2 + \frac{1}{840}. \end{cases}$$

Proof. From (4), (5), and (7), we have for $\sigma > 1$,

$$\begin{aligned} L^M(\mathrm{Sym}^2 E, s) &= \prod_{p \nmid N} (1 - \alpha_p^2 p^{-s})^{-1} (1 - p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1} \\ &\quad \cdot \prod_{p \| N} (1 - p^{-s-1})^{-1} \cdot \prod_{p^2 | N} U_p(s). \end{aligned}$$

By the Euler product, we have

$$\left| L^M(\mathrm{Sym}^2 E, \tfrac{3}{2} - it) \right| \le \zeta(\tfrac{3}{2})^3 < 18.$$

From (6) we have

$$\begin{aligned} \left| L^M(\mathrm{Sym}^2 E, -\tfrac{1}{2} + it) \right| &= \frac{\widetilde{N}^2}{\pi^3} \left| \frac{\Gamma(\frac{5}{4} - i\frac{t}{2})}{\Gamma(\frac{1}{4} + i\frac{t}{2})} \right|^2 \left| \frac{\Gamma(\frac{7}{4} - i\frac{t}{2})}{\Gamma(\frac{3}{4} + i\frac{t}{2})} \right| \\ &\quad \cdot \left| L^M(\mathrm{Sym}^2 E, \tfrac{3}{2} - it) \right| \\ &< 18 \frac{N^2}{\pi^3} \left| \frac{1}{4} + i\frac{t}{2} \right|^2 \left| \frac{3}{4} + i\frac{t}{2} \right| \\ &< 18 \frac{N^2}{8\pi^3} \left| \frac{3}{2} + it \right|^3. \end{aligned}$$

Since $N \ge 13$, the function

$$f(s) = L^M(\mathrm{Sym}^2 f, s)(s+2)^{-3}$$

is bounded by

$$B = \max \left\{ \frac{18}{(\frac{3}{2} + 2)^3}, 18 \frac{N^2}{8\pi^3} \right\} = 18 \frac{N^2}{8\pi^3}$$

on the lines $\sigma = -\frac{1}{2}$ and $\sigma = \frac{3}{2}$. By Lindelöf theorem (cf. [HR, p. 15]), this implies that

$$\left| L^M(\mathrm{Sym}^2 f, s) \right| \le 18 \frac{N^2}{8\pi^3} |s+2|^3 \quad (-\tfrac{1}{2} \le \sigma \le \tfrac{3}{2}). \tag{9}$$

In the proof of [BK, Lemma 3.1], we showed that

$$|\zeta(s)^{-1}| \le \begin{cases} 56 \cdot 840^2 \cdot 6^3 |t|^3 & \text{if } \sigma \ge 1 - \frac{1}{840 \cdot 6 \cdot 20 \log |t|}, \quad |t| \ge 2 + \frac{1}{840}, \\ 13 & \text{if } \frac{3}{4} \le \sigma \le \frac{3}{2}, \quad |t| \le 2 + \frac{1}{840}. \end{cases} \tag{10}$$

Since

$$\frac{2^{3/4}+1}{2^{3/4}-1} < 4 \text{ and } \frac{p^{3/4}+1}{p^{3/4}-1} < p \text{ for } p \geq 3,$$

from (7) we have for $\sigma \geq \frac{3}{4}$,

$$\begin{aligned} \left| \prod_{p|N} (1-p^{-s})^{-1} \prod_{p^2|N} U_p(s)^{-1} \right| &\leq \prod_{p\|N} \frac{1}{1-|p^{-s}|} \prod_{p^2|N} \frac{1+|p^{-s}|}{1-|p^{-s}|} \\ &\leq \prod_{p|N} \frac{p^{3/4}+1}{p^{3/4}-1} \\ &< 2N. \end{aligned} \tag{11}$$

Thus Lemma 3.2 follows from (8), (9), (10) and (11). □

Remark 3.3. In [Go, (49)] and [BK, Lemma 3.1], Deuring's Theorem and the functional equation for Hecke L-function are used to give upper bound for $\varphi_1(s)$ in the case of elliptic curves with complex multiplication. To remove complex multiplication condition, we use the functional equation for the motivic symmetric square L-function. We also note that Lemma 3.2 implies [BK, Lemma 3.1].

From Lemma 3.2, we have

$$\begin{aligned} \max_{|s-\frac{1}{2}|=\frac{1}{8}} |\varphi_1(2s)| &\leq \max_{|s-\frac{1}{2}|=\frac{1}{8}} (2 \cdot N^3 |2s+2|^3) \\ &\leq 70N^3. \end{aligned} \tag{12}$$

Moreover,

$$\max_{|s-\frac{1}{2}|=\frac{1}{4}} |G(s,U)| < \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} (1-p^{-\frac{1}{4}})^{-4} \text{ (cf. [Go, p.657]).} \tag{13}$$

Since $\log A > \frac{1}{2}\log d \geq \frac{1}{2}\exp(300Ng^3)$, we have

$$\sum_{r=2}^{\kappa} 8^r r! r \binom{\kappa}{r} (\log A)^{\kappa-r} \leq 2 \cdot 8^2 \cdot 2! \cdot 2 \binom{\kappa}{2} (\log A)^{\kappa-2},$$

thus from (1), (2), (12) and (13) we have

$$|H_2| \leq 6 \cdot 10^4 N^3 g^2 \sqrt{A} (\log A)^{\kappa-2} \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} (1-p^{-\frac{1}{4}})^{-4}. \tag{14}$$

Lower Bound for $|H_1|$. We need the following lemmas; one is [BK, Lemma 3.2] and the other is [Wa1, Lemma 3.4].

Lemma 3.4. [BK, Lemma 3.2] *If* $d > \exp(500g^3)$, *then either* $L(1, \chi_d) > (\log d)^{\kappa-1}\frac{1}{\sqrt{d}}$ *or else we have*

$$|G(\tfrac{1}{2}, U)| \quad \geq \prod_{\substack{\chi_d(p)\neq -1 \\ p<U}} \left(\tfrac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 - (\log d)^{-2g}.$$

Lemma 3.5. [Wa1, Lemma 3.4] *Let* E *be an elliptic curve over* $\mathbb{Q}$ *with* $\widetilde{N}^2 \geq 142$. *Then*

$$L^M(\mathrm{Sym}^2 E, 1) \geq \frac{0.033}{\log \widetilde{N}^2}.$$

Proof. See [Wa1, section 3]. Note that the definition of the symmetric square conductor in [Wa1] is different (cf. Remark 3.1). □

Lemma 3.5 implies the following lemma which is a generalization of [BK, Lemma 3.3].

Lemma 3.6. *Let* E *be an elliptic curve over* $\mathbb{Q}$ *with conductor* N *of which the product of distinct prime factors is* 13 *or more. Then*

$$\varphi_1'(1) \geq \frac{0.033}{2\log N}.$$

Proof. From (7) we have

$$\widetilde{N} \geq \text{ the product of distinct prime factors of } N \geq 13,$$

and so

$$\widetilde{N}^2 \geq 142.$$

From (7) and (8) we have

$$\prod_{p|N}(1-p^{-1})^{-1}\prod_{p^2|N}U_p(1)^{-1} \geq \prod_{p\|N}\frac{1}{1-p^{-1}}\prod_{p^2|N}\frac{1-p^{-1}}{1-p^{-1}} \geq 1,$$

and

$$\begin{aligned}\varphi_1'(1) &= L^M(\mathrm{Sym}^2 E, 1)\prod_{p|N}(1-p^{-1})^{-1}\prod_{p^2|N}U_p(1)^{-1} \\ &\geq L^M(\mathrm{Sym}^2 E, 1).\end{aligned}$$

Since $\widetilde{N}^2 \geq 142$ and $\widetilde{N} \mid N$, Lemma 3.5 implies

$$\varphi_1'(1) \geq \frac{0.033}{2\log \widetilde{N}} \geq \frac{0.033}{2\log N}.$$

□

By Lemma 3.4 and Lemma 3.6, we have *for* $d > \exp(500g^3)$, *either* $L(1,\chi_d) > (\log d)^{\kappa-1}\frac{1}{\sqrt{d}}$ *or else*

$$|H_1| \geq 2\kappa\frac{0.033}{2\log N}\cdot\sqrt{A}(\log A)^{\kappa-1}\left(\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 - (\log d)^{-2g}\right). \quad (15)$$

Now we can prove Proposition 2.1.

Proof of Proposition 2.1. We may assume

$$L(1,\chi_d) \leq (\log d)^{\kappa-1}\frac{1}{\sqrt{d}} \quad (d > \exp(500g^3)).$$

From (14) and (15), we have

$$\begin{aligned}
|H| &\geq |H_1| - |H_2| \\
&\geq \Big[2\kappa\tfrac{0.033}{2\log N}\cdot\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}(\tfrac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}})^2\Big] \\
&\quad -\Big[2\kappa\tfrac{0.033}{2\log N}\cdot\sqrt{A}(\log A)^{\kappa-1}(\log d)^{-2g} \\
&\qquad +6\cdot 10^4 N^3 g^2\sqrt{A}(\log A)^{\kappa-2}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}(1-p^{-\frac{1}{4}})^{-4}\Big] \\
&= \tilde{H}_1 - \tilde{H}_2.
\end{aligned}$$

Since $g \geq 4$, we have $\kappa \geq g-2 \geq \frac{g}{2}$. If $\frac{1}{2}\tilde{H}_1 \geq \tilde{H}_2$, then we have

$$\begin{aligned}
|H| &\geq \frac{\tilde{H}_1}{2} \\
&\geq \kappa\frac{0.033}{2\log N}\cdot\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi(p)\neq -1\\ p<U}}\left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 \\
&\geq \frac{0.033}{4}\cdot g(\log N)^{-1}\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi(p)\neq -1\\ p<U}}\left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2 \\
&\geq 1.2\times 10^{-3}\cdot g\sqrt{N}(\log N)^{-1}\sqrt{d}(\log d)^{\kappa-1}\prod_{\substack{\chi(p)\neq -1\\ p<U}}\left(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\right)^2
\end{aligned}$$

as desired.

We see that

$$\begin{aligned}
\frac{\tilde{H}_2}{\tilde{H}_1} &= \frac{6\cdot 10^4 N^3 g^2\sqrt{A}(\log A)^{\kappa-2}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}(1-p^{-\frac{1}{4}})^{-4}}{2\kappa\frac{0.033}{2\log N}\cdot\sqrt{A}(\log A)^{\kappa-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\big(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\big)^2}\\
&\quad+\frac{(\log d)^{-2g}}{\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\big(\frac{1-p^{-\frac{1}{2}}}{1+p^{-\frac{1}{2}}}\big)^2}\\
&\le \frac{6\cdot 10^4}{0.033(g-2)}\cdot N^3(\log N)g^2(\log d)^{-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\Big)^2\cdot\Big(\frac{1}{1-p^{-\frac{1}{4}}}\Big)^4\\
&\quad+(\log d)^{-2g}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\Big)^2\\
&\le 2\cdot\big(\frac{6\cdot 10^7}{33(g-2)}\cdot N^3(\log N)g^2(\log d)^{-1}\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\Big)^2\cdot\Big(\frac{1}{1-p^{-\frac{1}{4}}}\Big)^4\big).
\end{aligned}$$

Thus the sufficient condition of $\frac{1}{2}\tilde{H}_1 \ge \tilde{H}_2$ is that

$$\Big(4\cdot\frac{6\cdot 10^7}{33}N^3(\log N)\tfrac{g^2}{g-2}\Big)\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\Big)^2\cdot\Big(\frac{1}{1-p^{-\frac{1}{4}}}\Big)^4\big)\le\log d$$

From [BK, p. 286] we have

$$\log\prod_{\substack{\chi_d(p)\neq -1\\ p<U}}\Big(\frac{1+p^{-\frac{1}{2}}}{1-p^{-\frac{1}{2}}}\Big)^2\cdot\Big(\frac{1}{1-p^{-\frac{1}{4}}}\Big)^4\le 6\Big(\frac{g}{\log 2}\log\log d\Big)^{\frac{3}{4}}.$$

Thus the sufficient condition of $\frac{1}{2}\tilde{H}_1 \ge \tilde{H}_2$ is that

$$\log\log d - 6(\tfrac{g}{\log 2}\log\log d)^{\frac{3}{4}} \ge \log\Big(4\cdot\frac{6\cdot 10^7}{33}N^3(\log N)\tfrac{g^2}{g-2}\Big). \tag{16}$$

We note that the left hand in (16) is an increasing function with respect to $d \ge \exp\exp\Big((\frac{3}{4}\cdot 6)^4\cdot(\frac{g}{\log 2})^3\Big)$.

Since we are assuming that $d \ge \exp\exp(c_1Ng^3)$ and $g \ge 3$, if c_1 is sufficiently large, the left hand in (16) is greater than

$$c_1Ng^3 - 6(\tfrac{1}{\log 2}c_1Ng^4)^{\frac{3}{4}} = g^3(c_1N - \tfrac{6}{(\log 2)^{3/4}}c_1^{3/4}N^{3/4}),$$

and the right hand in (16) is less than

$$16 + 3\log N + \log\log N + \log\tfrac{g^2}{g-2}.$$

Since $g \geq 3$ and $N \geq 13$, a sufficient condition of $\frac{1}{2}\tilde{H}_1 \geq \tilde{H}_2$ is that $c_1 \geq 299.7$. For convenience, if we choose $c_1 = 300$, then Proposition 2.1 follows. □

4. Proof of Proposition 2.2

From [BK, (17)] we have

$$|2H| \leq |2H - T(G(s,U))| + |T(g(s))| + |S_1| + |S_2| + 1, \tag{17}$$

where

$$T(F(s)) = (\tfrac{d}{ds})^\kappa \Big[\tfrac{\delta}{2\pi i}\int_{2-i\infty}^{2+i\infty} A^{s+z}\Gamma^2(s+z+\tfrac{1}{2})F(s+z)\varphi_1(2s+2z)\tfrac{dz}{z}\Big]_{s=\frac{1}{2}},$$

$$\delta = 1 + (-1)^\kappa \chi_d(-N) = 2,$$

$$g(s) = G(s, A_0) - G(s, U),$$

$$A_0 = A(\log A)^{-20g},$$

$$S_1 = 2\sum_{r=0}^{\kappa}\binom{\kappa}{r}\Big(\sum_{A_0 \leq n \leq J} b_n\sqrt{A/n}(\log A/n)^{\kappa-r}I_r(n/A)\Big),$$

$$S_2 = 2\sum_{r=0}^{\kappa}\binom{\kappa}{r}\Big(\sum_{J \leq n \leq A_1} b_n\sqrt{A/n}(\log A/n)^{\kappa-r}I_r(n/A)\Big),$$

$$J = A((\kappa+6)\log\log A)^2,$$

$$A_1 = A((8+2\kappa)\log A)^2,$$

$$\sum_{n=1}^{\infty} b_n n^{-s} = G(s, A_1)\varphi_1(2s) - G(s, A_0)\varphi_1(2s),$$

and

$$I_r(M) = \int_{u_1=0}^{\infty}\int_{u_2=M/u_1}^{\infty} \exp(-(u_1+u_2))(\log u_1u_2)^r du_1 du_2 \ (M \geq 0).$$

As [BK, (21)], let

$$S_1^* = 2^3 \cdot 3^2 \cdot 4^3 \cdot 20 \cdot 3000 \cdot e \cdot (\tfrac{80}{e})^g \cdot g^{2g+4.5} L(1,\chi_d) A(\log\log A)^{\kappa+6}.$$

Proof of Proposition 2.2. We may assume

$$L(1,\chi_d) \leq (\log d)^{\kappa-1}\frac{1}{\sqrt{d}} \quad (d > \exp\exp(300Ng^3) \text{ and } N \geq 13).$$

By [BK, section 4], we have for $d > \exp\exp(300Ng^3)$,

$$\begin{cases} |S_1| \leq S_1^*, \\ |S_2| \leq S_1^*, \\ |T(g(s))| \leq S_1^*. \end{cases} \tag{18}$$

Since Lemma 3.2 implies [BK, Lemma 3.1] (cf. Remark 3.3), we have for $d > \exp\exp(300Ng^3)$,

$$|2H - T(G(s,U))| \leq S_1^*. \tag{19}$$

By (17), (18) and (19) we have

$$\begin{aligned} & |2H| \\ \leq\ & 5S_1^* \\ <\ & 4\times 10^9 \cdot (\tfrac{80}{e})^g g^{2g+4.5} L(1,\chi) A (\log\log A)^{\kappa+6} \end{aligned}$$

and Proposition 2.2 immediately follows. □

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