ELLIPTIC CURVES WITH ALL QUARTIC TWISTS OF THE SAME ROOT NUMBER

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Abstract. Let E/K be an elliptic curve with j-invariant 1728 defined over a number field K. In this note, we give a simple condition on K which determines whether all quartic twists of E/K have the same root number or not. This completes a series of works on the same root number of twists begun in [DD1] and [BK].

1. Introduction and results

Let K be a number field, E/K an elliptic curve defined over K, and $L(E/K,s)$ its Hasse-Weil L-function defined for $\Re(s) > \frac{3}{2}$. Then $L(E/K,s)$ conjecturally satisfies a functional equation under $s \leftrightarrow 2-s$ with the sign given by the (global) root number $w(E/K) = \pm 1$. The functional equation implies that $w(E/K) = (-1)^{\mathrm{ord}_{s=1} L(E/K,s)}$. The root number $w(E/K)$ is the product of the local root numbers over all places v of K,

$$w(E/K) = \prod_v w(E/K_v).$$

It is well known that there are four types of twists of elliptic curves;
Quadratic twist. For an elliptic curve $E/K : y^2 = x^3 + ax + b$ and $D \in K^\times/(K^\times)^2$, the quadratic twist of E/K by D is $E_D/K : y^2 = x^3 + aD^2x + bD^3$.
Cubic twist. For an elliptic curve E/K with j-invariant 0 defined by the equation $E/K : y^2 = x^3 + a$ and $D \in K^\times/(K^\times)^3$, the cubic twist of E/K by D is $E_D/K : y^2 = x^3 + aD^2$.
Quartic twist. For an elliptic curve E/K with j-invariant 1728 defined by the equation $E/K : y^2 = x^3 + ax$ and $D \in K^\times/(K^\times)^4$, the quartic twist of E/K by D is $E_D/K : y^2 = x^3 + aDx$.

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Sextic twist. For an elliptic curve E/K with j-invariant 0 defined by the equation $E/K : y^2 = x^3 + a$ and $D \in K^\times/(K^\times)^6$, the sextic twist of E/K by D is $E_D/K : y^2 = x^3 + aD$.

In [DD1], Dokchitser and Dokchitser give a sufficient and necessary condition on $E/K : y^2 = x^3 + ax + b$ that its quadratic twist $E_D/K : y^2 = x^3 + aD^2x + bD^3$ has the same root number for all $D \in K^\times/(K^\times)^2$. In [BK], using Kobayashi's computation of root numbers in [Ko], Byeon and Kim prove that for $E/K : y^2 = x^3 + a$, its cubic twist $E_D/K : y^2 = x^3 + aD^2$ has the same root number for all $D \in K^\times/(K^\times)^3$ if and only if $\sqrt{-3} \in K$. It is easily seen that this condition is also applied to sextic twist.

The aim of this note is to give a simple condition on K which determines whether all quartic twists of $E/K : y^2 = x^3 + ax$ have the same root number or not. This completes a series of works on the same root number of twists.

Theorem 1.1. *Let E/K be an elliptic curve with j-invariant 1728 defined by the equation $E/K : y^2 = x^3 + ax$, where $a \in K^\times$. For an element $D \in K^\times/(K^\times)^4$, let $E_D : y^2 = x^3 + aDx$ be the quartic twist of E. Then the root number $w(E_D/K)$ is constant for all $D \in K^\times/(K^\times)^4$ if and only if $\sqrt{-1} \in K$. In particular, if K contains $\sqrt{-1}$, then $w(E_D/K) = +1$ for all $D \in K^\times/(K^\times)^4$, and if K does not contain $\sqrt{-1}$, then there are infinitely many E_D/K such that $w(E_D/K) = +1$, and $w(E_D/K) = -1$, respectively.*

Remark. Varilly-Alvarado [Vá] and Desjardins [De] consider the behaviour of the root number in the family given by the twists of an elliptic curve $E/\mathbb{Q}$ by the rational values of a polynomial $f(T)$ and present a criterion for the family to have a constant root number over $\mathbb{Q}$.

2. Preliminaries

To prove Theorem 1.1, we need the following propositions. Before we state them, we introduce some notation for a place v of K above 2.

K_v: a local field with respect to a place $v|2$,

$L = K_v(E[3])$,

$G = \mathrm{Gal}(L/K_v)$,

$\gamma(x) = x^8 + 288ax^4 - 6912a^2$,

$(\deg \gamma_i)_i$: the tuple of degrees of irreducible factors of $\gamma(x) = \prod_i \gamma_i(x)$ over K_v,

$\mu_m \subset \bar{K}_v$: the set of m-th roots of unity.

Proposition 2.1. *Let K_v be a local field at a place $v|2$. Let E/K be an elliptic curve with j-invariant 1728 defined by the equation $E/K : y^2 = x^3 + ax$. Then the structure of G is given by the following table.*

$\mu_3 \subset K_v$		$\mu_3 \not\subset K_v$	
$(\deg \gamma_i)_i$	G	$(\deg \gamma_i)_i$	G
$(2,2,2,2)$	C_2	$(2,2,4)$	$C_2 \times C_2$
$(4,4)$	C_4	$(4,4)$	D_8
(8)	Q_8	(8)	$\begin{cases} C_8 & \text{if } \mu_4 \subset K_v \\ H_{16} & \text{if } \mu_4 \not\subset K_v \end{cases}$

Here, C_m is the cyclic group of order m, D_8 is the dihedral group of order 8, Q_8 is the quaternion group of order 8, and H_{16} is the 2-Sylow subgroup of $GL_2(\mathbb{Z}/3\mathbb{Z})$.

Proof. The elliptic curve E/K has potentially good reduction because its j-invariant is integral (see [p.197, Proposition 5.5, Si]) and additive reduction because $\Delta = (-4a)^3$, $c_4 = -48a$, $c_6 = 0$. Since $\Delta \in (K_v^\times)^3$, G is determined by whether $\mu_3 \subset K_v$ or not and what the irreducible factors of $\gamma(x) = x^8 + 288ax^4 - 6912a^2$ are (see [Proposition 2, DD]).

When $\mu_3 \not\subset K_v$ and $\gamma(x)$ is irreducible, there are two possible Galois groups (see [Proposition 2, DD]). Since $\Delta \in (K_v^\times)^3$, $\mu_3 \not\subset K_v$ is equivalent to the condition that $x^3 - 12^3\Delta = x^3 + (48a)^3$ has exactly one root. And we find that the root is $\delta = -48a = c_4$. Therefore it follows that $-3(c_4 - \delta) = 0$ is a square and $-3(c_4^2 + c_4\delta + \delta^2) = -3^2(48a)^2$ is a square if and only if $\mu_4 \subset K_v$. From [Lemma 3, DD], one may verify that this is equivalent to $G = C_8$. Hence Proposition 2.1 follows from [Proposition 2, DD]. $\square$

Proposition 2.2. *Let K_v be a local field at a place $v|2$. Let E/K be an elliptic curve with j-invariant 1728 defined by the equation $E/K : y^2 = x^3 + ax$.*
(a) If $\mu_4 \subset K_v$, then $G = C_2$, C_4, or C_8. In particular, G is abelian.
(b) If $\mu_4 \not\subset K_v$, then $G = C_2 \times C_2$, D_8, Q_8, or H_{16}. In particular, G is not abelian except for the case that $G = C_2 \times C_2$ when $(\deg \gamma_i)_i = (2, 2, 4)$.

Proof. (a) Suppose that $\mu_4 \subset K_v$. If $\mu_3 \subset K_v$, then $\sqrt{3} \in K_v$, so $\gamma(x)$ is reducible over K_v and its factorization is

$$\gamma(x) = (x^4 + 144a - 96a\sqrt{3})(x^4 + 144a + 96a\sqrt{3}). \tag{1}$$

Hence $G = C_2$ or C_4 from Proposition 2.1. If $\mu_3 \not\subset K_v$, then $\gamma(x)$ is irreducible. From Proposition 2.1, we obtain $G = C_8$.
(b) Suppose that $\mu_4 \not\subset K_v$ and $\sqrt{3} \in K_v$. Then $\mu_3 \not\subset K_v$ but $\gamma(x)$ is reducible over K_v factoring as (1). If both factors of $\gamma(x)$ in (1) are irreducible, then we have $G = D_8$ from Proposition 2.1. If $\gamma(x)$ has an irreducible factor of degree 2, then the possible G is only $C_2 \times C_2$ when $(\deg \gamma_i)_i = (2, 2, 4)$ from Proposition 2.1. Suppose that $\mu_4 \not\subset K_v$ and $\sqrt{3} \notin K_v$. Then $\gamma(x)$ is irreducible. So we have $G = Q_8$ when $\mu_3 \subset K_v$ or H_{16} when $\mu_3 \not\subset K_v$ from Proposition 2.1. □

3. Proof of Theorem 1.1

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. In [Proposition 6.3, Če], Česnavičius proved that if $\sqrt{-1} \in K$, then any elliptic curve with j-invariant 1728 over K has root number 1. Now we will show that the structure of G prevent this in the case that $\sqrt{-1} \notin K$. We will use the fact that there are infinitely many principal prime ideals (of residue class degree 1) in K, which follows from the Frobenius density theorem.

Assume that $\sqrt{-1} \notin K$. Since the factorization of $\gamma(x)$ over $\bar{K}$ is following

$$\begin{aligned}\gamma(x) &= (x^2 + 4\cdot\sqrt{-1}\cdot\sqrt{a}\cdot\sqrt{9-6\sqrt{3}})(x^2 - 4\cdot\sqrt{-1}\cdot\sqrt{a}\cdot\sqrt{9-6\sqrt{3}}) \\ &\times\ (x^2 + 4\cdot\sqrt{-1}\cdot\sqrt{a}\cdot\sqrt{9+6\sqrt{3}})(x^2 - 4\cdot\sqrt{-1}\cdot\sqrt{a}\cdot\sqrt{9+6\sqrt{3}}),\end{aligned}$$

we may find infinitely many principal prime ideals (π_n) $(n \in \mathbb{N})$ of K such that $(\deg \gamma_{\pi_n i})_i \neq (2,2,4)$ for a place $v|2$, where $\gamma_{\pi_n}(x) = x^8 + 288a\pi_n x^4 - 6912a^2{\pi_n}^2$. Then G for E_{π_n}/K_v is not abelian by Proposition 2.2 (b). So E_{π_n}/K_v is chaotic and E_{π_n}/K is also chaotic, which means that there is a $\alpha_n \in K^\times/(K^\times)^2$ such that $w(E_{\pi_n {\alpha_n}^2}/K) = -w(E_{\pi_n}/K)$ (see [DD1]). We note that no $a\pi_n$, $a\pi_m$, $a\pi_n{\alpha_n}^2$, $a\pi_m{\alpha_m}^2$ $(n \neq m \in \mathbb{N})$ are congruent to each other modulo $(K^\times)^4$. This completes the proof. □

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