CLASS NUMBERS OF REAL QUADRATIC FIELDS

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Abstract. Let $d > 0$ be a fundamental discriminant of a real quadratic field. Let $h(d)$ be the class number and ε_d the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d})$. In this paper, we prove that if there is an elliptic curve E over $\mathbb{Q}$ whose Hasse-Weil L-function $L_{E/\mathbb{Q}}(s)$ has a zero of order g at $s = 1$, then there is an effectively computable constant $\kappa > 0$ satisfying

$$h(d)\log\varepsilon_d > \tfrac{1}{\kappa}(\log d)^{g-3} \prod_{p|d,\, p\neq d} \left(1 - \tfrac{\lfloor 2\sqrt{p}\rfloor}{p+1}\right).$$

1. Introduction and results

Let d be a fundamental discriminant, χ_d the Dirichlet character associated to the quadratic field $\mathbb{Q}(\sqrt{d})$ and $L(s, \chi_d)$ the Dirichlet L-function. The Dirichlet class number formula is as follows

$$L(1,\chi_d) = \begin{cases} \frac{2\pi h(d)}{\omega\sqrt{|d|}} & \text{if } d < 0, \\ \frac{2h(d)\log\varepsilon_d}{\sqrt{d}} & \text{if } d > 0, \end{cases}$$

where $h(d)$ is the class number of $\mathbb{Q}(\sqrt{d})$, ω the number of roots of unity in $\mathbb{Q}(\sqrt{d})$ $(d < 0)$ and ε_d the fundamental unit of $\mathbb{Q}(\sqrt{d})$ $(d > 0)$. Siegel [Si] proved that there is a positive constant $\kappa(\epsilon)$ such that

$$L(1,\chi_d) > \frac{1}{\kappa(\epsilon)}|d|^{-\epsilon} \ \ (\epsilon > 0).$$

Thus we have

$$|d|^{\frac{1}{2}-\epsilon} \leq \begin{cases} \kappa(\epsilon)\, h(d) & \text{for } \ d < 0, \\ \kappa(\epsilon)\, h(d)\log\varepsilon_d & \text{for } \ d > 0. \end{cases}$$

But there is no known method to compute the constant $\kappa(\epsilon)$.

The first and second author were supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (Grant No. 2016R1D1A1B03931404) and (Grant Nos. 2019R1A6A1A11051177 and 2020R1I1A1A01074746), respectively.

In [Go], Goldfeld proved that if there is an elliptic curve E over $\mathbb{Q}$ of conductor N whose Hasse-Weil L-function $L_{E/\mathbb{Q}}(s)$ has a zero of order g at $s=1$, then there is an effectively computable constant $\kappa>0$ satisfying

$$\exp(-21\sqrt{g\log\log|d|})(\log|d|)^{g-\mu-1} \le \begin{cases} \kappa\, h(d) & \text{for} \quad d<0, \\ \kappa\, h(d)\log\varepsilon_d & \text{for} \quad d>0, \end{cases}$$

where $\mu \in \{1,2\}$ satisfying $\chi_d(-N) = (-1)^{g-\mu}$. In [BK] and [BK1], we explicitly compute the constant κ for $d>0$.

In [Oe], Oesterlé simplified the method of Goldfeld [Go] by using definite binary quadratic forms and proved that there is an effectively computable constant $\kappa>0$ satisfying

$$\theta(d)\,\log d \le \kappa\, h(d)$$

for any fundamental discriminant $d<0$ of imaginary quadratic fields, where

$$\theta(d) = \prod_{p\in P(d)} \left(1 - \frac{\lfloor 2\sqrt{p}\rfloor}{p+1}\right) \tag{1.0.1}$$

and $P(d)$ is the set of primes p dividing d except for the largest of them. Moreover, using an elliptic curve E over $\mathbb{Q}$ of conductor 5077 whose Hasse-Weil L-function $L_{E/\mathbb{Q}}(s)$ has a zero of order 3, Oesterlé proved that for any $d<0$ with $(5077,d)=1$,

$$h(d) > \tfrac{1}{55}(\log|d|) \prod_{p|d,\, p\neq d} \left(1 - \tfrac{\lfloor 2\sqrt{p}\rfloor}{p+1}\right).$$

In this paper, we modify the method of Oesterlé [Oe] by using indefinite binary quadratic forms and prove the following theorem.

Theorem 1. *If there is an elliptic curve E over $\mathbb{Q}$ whose Hasse-Weil L-function $L_{E/\mathbb{Q}}(s)$ has a zero of order g at $s=1$, then there is an effectively computable constant $\kappa>0$ satisfying*

$$\theta(d)\,(\log d)^{g-3} \le \kappa\, h(d)\,\log\varepsilon_d$$

for any fundamental discriminant $d>0$ of a real quadratic field.

Theorem 1 immediately follows from Theorem 2.

Theorem 2. *Let E be an elliptic curve over $\mathbb{Q}$ and $\mathcal{D}(g)$ the set of fundamental discriminants $d > 0$ of real quadratic fields such that the base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$ at $s = 1$. Then there are effectively computable constants c_1 and $c_2 > 0$ such that for any $d \in \mathcal{D}(g)$ greater than c_1,*

$$h(d) \log \varepsilon_d \geq c_2 (\log d)^{g-3} \theta(d).$$

Since $\log \varepsilon_d \gg \log d$, it is required that $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order ≥ 5 at $s = 1$ to get a non-trivial lower bound. But there is no known elliptic curve E whose $L_{E/\mathbb{Q}}(s)$ has a zero of order ≥ 4 at $s = 1$. Let $E(\mathbb{Q})$ be the Mordell-Weil group of an elliptic curve E over $\mathbb{Q}$. Birch and Swinnerton-Dyer conjectured that if the rank of $E(\mathbb{Q})$ is equal to g, then $L_{E/\mathbb{Q}}(s)$ has a zero of order g at $s = 1$.

Among the known elliptic curves E whose Mordell-Weil group $E(\mathbb{Q})$ has rank 4, for example, choose the curve

$$E : y^2 + y = x^3 + x^2 - 72x + 210$$

with the smallest prime conductor $N = 501029$ (cf. [Cr]). Then $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order ≥ 5 at $s = 1$ for any d such that $(\frac{d}{N}) = -1$ under the assumption that the conjecture of Birch and Swinnerton-Dyer is true.

Let $\Delta = n^2 + r$ be a positive square free integer with $r \in \{\pm 1, \pm 4\}$. The real quadratic field $\mathbb{Q}(\sqrt{\Delta})$ is called a real quadratic field of narrow Richaud-Degert type (cf. [De]). Let d be the fundamental discriminant of the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$ of narrow Richaud-Degert type. Then we have

$$\varepsilon_d = \begin{cases} n + \sqrt{n^2 + r} & \text{if } r = \pm 1, \\ \frac{n + \sqrt{n^2 + r}}{2} & \text{if } r = \pm 4. \end{cases}$$

Thus $\log \epsilon_d \leq \log (2\sqrt{d})$. By numerically computing the constants c_1 and c_2 in Theorem 2, we can obtain the following lower bound for the class number of the real quadratic field of narrow Richaud-Degert type under the assumption that the conjecture of Birch and Swinnerton-Dyer is true.

Theorem 3. *Let $E : y^2 + y = x^3 + x^2 - 72x + 210$ be an elliptic curve over $\mathbb{Q}$ of conductor $N = 501029$. If the conjecture of Birch and Swinnerton-Dyer is true for E, that is, the Hasse-Weil L-function $L_{E/\mathbb{Q}}(s)$ associated to E*

has a zero of order 4 *at* $s = 1$*, then for any fundamental discriminant* $d > 0$ *of the real quadratic field* $\mathbb{Q}(\sqrt{\Delta})$ *of narrow Richaud-Degert type such that* $(d, 501029) = 1$*, we have*

$$h(d) \geq \frac{1}{5200}(\log d) \prod_{p|d,\, p\neq d} \left(1 - \tfrac{\lfloor 2\sqrt{p} \rfloor}{p+1}\right).$$

2. Real quadratic fields and binary quadratic forms

In this section, we introduce Hecke's idea [He] which shows how a Dirichlet series involving an indefinite quadratic form can be written as an integral of a series involving a definite quadratic form. For more details, see Section 3, Zeta functions of quadratic fields in [Go] and Section 3, Hecke's Theorem in [Za].

Let $d > 0$ be a fundamental discriminant of a real quadratic field. Let $\zeta_K(s)$ be the Dedekind zeta function of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. Then we have

$$\zeta_K(s) = \sum_{\mathfrak{A}} \zeta(s, \mathfrak{A}),$$

where $\mathfrak{A}$ runs over the ideal class group of K and

$$\zeta(s, \mathfrak{A}) = \sum_{\mathfrak{a} \in \mathfrak{A}} \frac{1}{\mathcal{N}(\mathfrak{a})^s}$$

with the absolute norm $\mathcal{N}$ from nonzero integral ideals of the ring of integers $\mathcal{O}_K$ of K to $\mathbb{N}^*$ defined by $\mathfrak{a} \mapsto |\mathcal{O}_K/\mathfrak{a}|$.

If $\mathfrak{b} \in \mathfrak{A}^{-1}$, then the correspondence

$$\mathfrak{a} \mapsto \mathfrak{a}\mathfrak{b} = (v)$$

is a bijection between ideals $\mathfrak{a} \in \mathfrak{A}$ and principal ideals (v) with $v \in \mathfrak{b}$. Let $U_K = \{\pm {\varepsilon_d}^n \mid n \in \mathbb{Z}\}$ be the group of units of K. Then v_1 and v_2 in $\mathfrak{b}$ define the same principal ideal if and only if $v_1/v_2 \in U_K$. Hence we have

$$\zeta(s, \mathfrak{A}) = \sum_{v \in \mathfrak{b}/U_K}{}' \frac{\mathcal{N}(\mathfrak{b})^s}{|vv'|^s},$$

where (here and in the sequel) the prime on the summation sign indicates that the value 0 is to be omitted and v' is the conjugate of v in $K/\mathbb{Q}$.

By the reduction of indefinite binary quadratic forms, we can choose the basis of an ideal $\mathfrak{b}$

$$\mathfrak{b} = \left[a, \frac{-b+\sqrt{d}}{2}\right]$$

such that $b+\sqrt{d} > 2|a| > -b+\sqrt{d} > 0$ and $\mathcal{N}(\mathfrak{b}) = |a|$ (cf. [p. 633, Go]). Note that $\mathfrak{b}$ corresponds to an indefinite binary quadratic form $(a,b,c) = ax^2+bxy+cy^2$ with $d = b^2-4ac > 0$. Let

$$v = am + \frac{-b+\sqrt{d}}{2}n, \quad v' = am + \frac{-b-\sqrt{d}}{2}n$$

and

$$w = \frac{-b+\sqrt{d}}{2|a|}, \quad w' = \frac{-b-\sqrt{d}}{2|a|},$$

where m, n are rational integers. Since

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\phi}{(v^2e^{\phi}+v'^2e^{-\phi})^s} &= \frac{1}{|vv'|^s}\int_{-\infty}^{\infty}\frac{d\phi}{(e^{\phi}+e^{-\phi})^s} \\ &= \frac{1}{|vv'|^s}\frac{\Gamma(s/2)^2}{2\Gamma(s)}\end{aligned}$$

for nonzero real numbers v and v', we have

$$\begin{aligned}\zeta_K(s) &= \sum_{(a,b,c)}\sum_{v\in\mathfrak{b}/U_K}{}' \frac{\mathcal{N}(\mathfrak{b})^s}{|vv'|^s} \\ &= \sum_{(a,b,c)} \frac{2\Gamma(s)}{\Gamma(s/2)^2}\mathcal{N}(\mathfrak{b})^s \left(\sum_{v\in\mathfrak{b}/U_K}{}' \int_{-\infty}^{\infty}(v^2e^{\phi}+v'^2e^{-\phi})^{-s}d\phi\right) \\ &= \sum_{(a,b,c)} \frac{\Gamma(s)}{\Gamma(s/2)^2}\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\left(\sum_{v\in\mathfrak{b}}{}' \mathcal{N}(\mathfrak{b})^s(v^2e^{\phi}+v'^2e^{-\phi})^{-s}\right)d\phi \\ &= \sum_{(a,b,c)} \frac{\Gamma(s)}{\Gamma(s/2)^2}\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\sum_{(m,n)}{}' \Big(A(\phi)m^2+B(\phi)mn+C(\phi)n^2\Big)^{-s}d\phi,\end{aligned}$$

where (a,b,c) runs over basis of ideals representing the ideal class group of K and

$$A(\phi) = |a|(e^{\phi}+e^{-\phi}),\ B(\phi) = 2a(we^{\phi}+w'e^{-\phi}),\ C(\phi) = |a|(w^2e^{\phi}+w'^2e^{-\phi})$$

for a variable $\phi\in\mathbb{R}$ (cf. [p. 161, Za]). Note that $A(\phi)m^2+B(\phi)mn+C(\phi)n^2$ is a positive-definite binary quadratic form.

Let $\check{A}m^2+\check{B}mn+\check{C}n^2$ be the reduced form of the positive-definite binary quadratic form $Am^2+Bmn+Cn^2$ with real coefficients, that is, it satisfies simultaneously

(i) $|\check{A}| \le \check{B} \le \check{C}$
(ii) $\check{B} \ge 0$ if $\check{A} = |\check{B}|$ or $\check{C}$

with

$$\check{B}^2 - 4\check{A}\check{C} = B^2 - 4AC = -4d < 0.$$

Note that $\check{A}$, $\check{B}$, $\check{C}$ are piecewise continuous real-valued functions of a variable $\phi \in \mathbb{R}$ and the integral

$$\int_{-\log \varepsilon_d}^{\log \varepsilon_d} (\check{A}m^2 + \check{B}mn + \check{C}n^2)^{-s} d\phi$$

is well-defined. Therefore we represent $\zeta_K(s)$ as an integral of a series involving definite quadratic forms

$$\zeta_K(s) = \sum_{(a,b,c)} \frac{\Gamma(s)}{\Gamma(s/2)^2} \int_{-\log \varepsilon_d}^{\log \varepsilon_d} \sum_{(m,n)}{}' (\check{A}m^2 + \check{B}mn + \check{C}n^2)^{-s} d\phi. \tag{2.0.1}$$

Also we have,

$$\begin{aligned}\frac{\zeta_K(s)}{\zeta(2s)} &= \sum_{(a,b,c)} \frac{2\Gamma(s)}{\Gamma(s/2)^2} \\ &\cdot \left(\int_{-\log \varepsilon_d}^{\log \varepsilon_d} \check{A}^{-s} d\phi + \int_{-\log \varepsilon_d}^{\log \varepsilon_d} \sum_{\substack{(m,n)\in \mathbb{Z}\times\mathbb{N}^* \\ (m,n)=1}} (\check{A}m^2 + \check{B}mn + \check{C}n^2)^{-s} d\phi \right).\end{aligned} \tag{2.0.2}$$

These two identities (2.0.1) and (2.0.2) will be used in Section 4, Proof of Proposition 4, which is stated in Section 3, Proof of Theorem 2.

3. Proof of Theorem 2

3.1. **Associated L-function.** Let E be an elliptic curve over $\mathbb{Q}$ of conductor N such that the base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$ at $s = 1$.

We denote by $S_2^p(N)$ the set of normalized primitive holomorphic cusp forms for the congruence subgroup $\Gamma_0(N)$ of weight 2 with trivial nebentypus 1_N. From the Modularity Theorem, there exists $f = \sum_{n=1}^{\infty} a_n q^n$ ($q = e^{2\pi i\tau}$) $\in S_2^p(N)$ such that the associated L-function $L(f, s)$ satisfies

$$L_{E/\mathbb{Q}}(s) = L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

(cf. [Theorem 8.8.3, DS]). If necessary, we denote a_n by $a_n(f)$. Thus $L_{E/\mathbb{Q}}(s)$ has an analytic continuation to an entire function satisfying the functional equation

$$\Lambda(f, 2 - s) = W(f)\Lambda(f, s),$$

where $\Lambda(f,s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f,s)$ and $W(f) = \pm 1$ is the root number.

3.2. **Twisting by a quadratic Dirichlet character.** For a Dirichlet character χ_d, there exists an integer $N_{\chi_d} \geq 1$ and $f \otimes \chi_d \in S_2^p(N_{\chi_d})$ such that the p-th Fourier coefficient is given by

$$a_p(f \otimes \chi_d) = a_p(f)\chi_d(p)$$

for almost all primes p. This condition uniquely determines N_{χ_d} and $f \otimes \chi_d$. Let

$$M_d = \frac{\sqrt{N N_{\chi_d}}}{|d|}. \tag{3.2.1}$$

and

$$M = 2^{n_2} \cdot 3^{n_3} \cdot N, \tag{3.2.2}$$

where

$$\begin{cases} n_2 = \max_{\chi_d} \left\{0, \frac{\mathrm{ord}_2(N_{\chi_d}) - \mathrm{ord}_2(N)}{2} - 2\right\}, \\ n_3 = \max_{\chi_d} \left\{0, \frac{\mathrm{ord}_3(N_{\chi_d}) - \mathrm{ord}_3(N)}{2} - 1\right\}. \end{cases}$$

For any conductor N' of an elliptic curve $E'/\mathbb{Q}$, it is well known that

$$\begin{cases} \mathrm{ord}_2(N') \leq 8 \\ \mathrm{ord}_3(N') \leq 5 \\ \mathrm{ord}_p(N') \leq 2 \quad \text{for } p \neq 2, 3 \end{cases}$$

(cf. [p. 385 and p. 388, Sil]). Thus we have upper bounds for n_2 and n_3 as follows:

$$n_2 \leq \tfrac{8-0}{2} - 2 \leq 2,$$
$$n_3 \leq \tfrac{5-0}{2} - 1 \leq 1.5.$$

We note that $\mathrm{ord}_p(N) = \mathrm{ord}_p(N_{\chi_d})$ for any prime $p \nmid d$. Since $4 \mid d$ for even d, we have

$$M \geq M_d.$$

3.3. **Symmetric square L-functions.** The Hasse-Weil L-function $L_{E/\mathbb{Q}}(s)$ can be expanded as an Euler product:

$$\begin{aligned} L_{E/\mathbb{Q}}(s) &= \prod_p (1 - a_p p^{-s} + 1_N(p) p^{-2s})^{-1} \\ &= \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}, \end{aligned}$$

where

$$\begin{cases} \text{for } \ p \nmid N, & \alpha_p + \beta_p = a_p, \ |\alpha_p| = |\beta_p| = \sqrt{p}, \ \alpha_p = \bar{\beta}_p, \\ \text{for } \ p \parallel N, & \alpha_p = \pm 1, \ \beta_p = 0, \\ \text{for } \ p^2 \mid N, & \alpha_p = \beta_p = 0. \end{cases}$$

If necessary, we denote α_p and β_p by $\alpha_p(E)$ and $\beta_p(E)$, respectively.

The imprimitive symmetric square L-function $L(\mathrm{Sym}_i^2 E, s)$ associated to $E/\mathbb{Q}$ is defined as follows: for $Re(s) > 2$,

$$\begin{aligned} L(\mathrm{Sym}_i^2 E, s) &= \frac{\zeta_N(2s-2)}{\zeta_N(s-1)} \sum_{n=1}^{\infty} \frac{a_n^2}{n^s} \\ &= \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - \alpha_p \beta_p p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1}. \\ &= L(f, \tfrac{s}{2}) L(f \otimes \lambda, \tfrac{s}{2}) \zeta_N(s-1), \end{aligned}$$

where $\lambda(n) = \prod_{p^r \| n} (-1)^r$, $L(f \otimes \lambda) = \sum_{n=1}^{\infty} a_n \lambda(n) n^{-s}$ and the subscript N of ζ_N means that we have omitted the Euler factors at the primes dividing N.

By [CS], there exist the symmetric square conductor $B \in \mathbb{Z}$, the primitive symmetric square L-function $L(\mathrm{Sym}_p^2 E, s)$ and the Euler product $U(E, s) = \prod_{p|N} U_p(E, s)$ such that

$$\begin{aligned} \Lambda(\mathrm{Sym}^2 E, s) &:= \Big(\frac{B}{2\pi^{3/2}}\Big)^s \Gamma(s) \Gamma\Big(\frac{s}{2}\Big) L(\mathrm{Sym}_p^2 E, s) \\ &:= \Big(\frac{B}{2\pi^{3/2}}\Big)^s \Gamma(s) \Gamma\Big(\frac{s}{2}\Big) L(\mathrm{Sym}_i^2 E, s) \cdot U(E, s) \end{aligned}$$

satisfies the functional equation

$$\Lambda(\mathrm{Sym}^2 E, s) = \Lambda(\mathrm{Sym}^2 E, 3 - s). \tag{3.3.1}$$

Let

$$F_d(s) = \big(\frac{M_d}{4\pi^2}\big)^s \Gamma^2(s) \frac{L(\mathrm{Sym}_i^2 E, 2s)}{(s-1)\zeta_N(2s-1)}, \tag{3.3.2}$$

$F_d^{(k)}(s)$ its k-th derivative and $\mathcal{F} = \{F_d \mid d \in \mathcal{D}(g)\}$. We note that $\mathcal{F}$ is a finite set.

3.4. **The fundamental equality.** As [Oe], let

$$\Psi(s) = L(f,s)L(f \otimes \lambda, s),$$

$$G(s) = \frac{L(f \otimes \chi_d, s)}{L(f \otimes \lambda, s)},$$

$$G_p(s) = \text{the Euler factor of } G(s) \text{ at a prime } p,$$

$$G(U,s) = \prod_{p<U} G_p(s),$$

$$G(U^*,s) = G(s) - G(U,s) = G(U,s)\Big((\prod_{p \geq U} G_p(s)) - 1 \Big),$$

$$\gamma(s) = (\frac{M_d}{4\pi^2})^s \Gamma(s)^2,$$

$$W_d = W(f)W(f \otimes \chi_d),$$

$$\mu' \in \{1,2\} \text{ such that } W_d = (-1)^{g-\mu'},$$

$$\rho = g - \mu' - 1, \tag{3.4.1}$$

$$J(U) = \int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-1}\gamma(s)\Psi(s)G(U,s)(s-1)^{-\rho-2}\frac{ds}{2\pi i}, \tag{3.4.2}$$

$$J(U^*) = \int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-1}\gamma(s)\Psi(s)G(U^*,s)(s-1)^{-\rho-2}\frac{ds}{2\pi i}. \tag{3.4.3}$$

Let

$$\Lambda(s) = d^{s-1}(\frac{M_d}{4\pi^2})^s\Gamma(s)^2 L(f,s)L(f \otimes \chi_d, s) = d^{s-1}\gamma(s)\Psi(s)G(s).$$

Then we have the functional equation

$$\Lambda(s) = W_d \Lambda(2-s).$$

Since $(-1)^\rho W_d = -1$, we have

$$\begin{aligned}
\int_{1-i\infty}^{1+i\infty} \Lambda(s)(s-1)^{-\rho-2}\frac{ds}{2\pi i} &= W_d(-N)\int_{1-i\infty}^{1+i\infty} \Lambda(2-s)(s-1)^{-\rho-2}\frac{ds}{2\pi i} \\
&= (-1)^\rho W_d \int_{1-i\infty}^{1+i\infty} \Lambda(z)(z-1)^{-\rho-2}\frac{dz}{2\pi i} \\
&= 0.
\end{aligned}$$

Since $\Lambda(s)$ has a zero of order $\geq g$ at $s=1$, by the residue theorem, we have for $\sigma > 1$,

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \Lambda(s)(s-1)^{-\rho-2}\frac{ds}{2\pi i} = \int_{1-i\infty}^{1+i\infty} \Lambda(s)(s-1)^{-\rho-2}\frac{ds}{2\pi i} = 0.$$

Therefore we have

$$J(U) = -J(U^*). \tag{3.4.4}$$

3.5. To prove Theorem 2, we need the following propositions, which are analogues of [Theorem 2, Section 3.5, Oe].

Proposition 4. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and $\mathcal{D}(g)$ the set of fundamental discriminants $d>0$ of real quadratic fields such that the base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$ at $s=1$. Then there are effectively computable constants c_3 and $c_4 > 0$ depending on E and ρ such that for any $d \in \mathcal{D}(g)$ with*

$$d \geq c_3,$$

we have either $h(d)\log\varepsilon_d \geq \frac{L(\mathrm{Sym}_i^2 E,2)\prod_{p|N}\frac{p}{p-1}}{2^{\rho+1}\rho!\sqrt{M}}(\log d)^\rho$ or else, for some positive integer U,

$$|J(U^*)| \leq \frac{c_4 2^\rho M_d^{3/2}}{\pi^2}\prod_{i=1}^{\rho}\Big(\frac{q_i+1}{q_i-1}\Big)\cdot h(d)\log\varepsilon_d \prod_{p\in P(d)}\big(1+\frac{1}{p}\big),$$

where q_i is the i-th prime splitting in $\mathbb{Q}(\sqrt{d})$ (or the i-th prime), $P(d)$ is defined below (1.0.1)*, and M_d, M, ρ, $J(U^*)$ are defined by* (3.2.1), (3.2.2), (3.4.1), (3.4.3)*, respectively.*

Proposition 5. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and $\mathcal{D}(g)$ the set of fundamental discriminants $d>0$ of real quadratic fields such that the base change Hasse-Weil L-function $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq g$ at $s=1$. Then there is an effectively computable constant $c_5 > 1$ depending on E and ρ such that for any $d \in \mathcal{D}(g)$ with*

$$d \geq \max_{1\leq k\leq\rho}\left\{\exp\big(2^{\rho-1}\rho!\sqrt{N}\big),\ \exp\big(L(\mathrm{Sym}_i^2 E,2)\big),\ \exp\Big(2\rho\frac{|F_d^{(k)}(1)|}{|F_d(1)|}\Big)\right\},$$

we have either $h(d)\log\varepsilon_d \geq \frac{L(\mathrm{Sym}_i^2 E,2)}{12}\prod_{i=1}^{\rho}\frac{(q_i-1)(q_i+1-\lfloor 2\sqrt{q_i}\rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i}\rfloor)}\cdot(\log d)^{\rho}\theta(d)$ *or else, for the same* U *in Proposition 4,*

$$|J(U)| \geq \frac{F_d(1)}{c_5\rho!}\prod_{i=1}^{\rho}\frac{q_i+1-\lfloor 2\sqrt{q_i}\rfloor}{q_i+1+\lfloor 2\sqrt{q_i}\rfloor}\cdot(\log d)^{\rho}\prod_{p\in P(d)}\frac{p+1-\lfloor 2\sqrt{p}\rfloor}{p},$$

where q_i *is the* i*-th prime splitting in* $\mathbb{Q}(\sqrt{d})$ *(or the* i*-th prime),* $P(d)$ *is defined below* (1.0.1)*, and* F_d*,* ρ*,* $J(U)$ *are defined by* (3.3.2), (3.4.1), (3.4.2)*, respectively.*

We will prove Proposition 4 in Section 4 and Proposition 5 in Section 5.

Proof of Theorem 2. Let q_i be the i-th prime. Suppose for $d \in \mathcal{D}(g)$ with

$$d \geq \max_{\substack{F_d\in\mathcal{F},\\ 1\leq k\leq\rho}}\left\{c_3,\ \exp(2^{\rho-1}\rho!\sqrt{N}),\ \exp(L(\mathrm{Sym}_i^2 E,2)),\ \exp(2\rho\frac{|F_d^{(k)}(1)|}{|F_d(1)|})\right\},$$

$$h(d)\log\varepsilon_d \leq \frac{L(\mathrm{Sym}_i^2 E,2)\prod_{p|N}\frac{p}{p-1}}{2^{\rho+1}\rho!\sqrt{M}}(\log d)^{\rho}$$

and

$$h(d)\log\varepsilon_d \leq \frac{L(\mathrm{Sym}_i^2 E,2)}{12}\prod_{i=1}^{\rho}\frac{(q_i-1)(q_i+1-\lfloor 2\sqrt{q_i}\rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i}\rfloor)}\cdot(\log d)^{\rho}\theta(d).$$

By (3.4.4), Proposition 4 and Proposition 5, we have

$$\begin{aligned}
&\frac{c_4 2^{\rho}M_d^{3/2}}{\pi^2}\prod_{i=1}^{\rho}\left(\frac{q_i+1}{q_i-1}\right)\cdot h(d)\log\varepsilon_d\prod_{p\in P(d)}\left(1+\frac{1}{p}\right)\\
\geq\ &\frac{F_d(1)}{c_5\rho!}\prod_{i=1}^{\rho}\frac{q_i+1-\lfloor 2\sqrt{q_i}\rfloor}{q_i+1+\lfloor 2\sqrt{q_i}\rfloor}\cdot(\log d)^{\rho}\prod_{p\in P(d)}\frac{p+1-\lfloor 2\sqrt{p}\rfloor}{p}\\
=\ &\frac{M_d L(\mathrm{Sym}_i^2 E,2)}{c_5 2\pi^2\rho!}\prod_{p|N}\frac{p}{p-1}\prod_{i=1}^{\rho}\frac{q_i+1-\lfloor 2\sqrt{q_i}\rfloor}{q_i+1+\lfloor 2\sqrt{q_i}\rfloor}\\
&\cdot(\log d)^{\rho}\prod_{p\in P(d)}\frac{p+1-\lfloor 2\sqrt{p}\rfloor}{p}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&h(d)\log\varepsilon_d\\
\geq\ &\frac{L(\mathrm{Sym}_i^2 E,2)}{c_4c_5 2^{\rho+1}\rho!\sqrt{M_d}}\prod_{p|N}\frac{p}{p-1}\prod_{i=1}^{\rho}\frac{(q_i-1)(q_i+1-\lfloor 2\sqrt{q_i}\rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i}\rfloor)}(\log d)^{\rho}\theta(d)\\
\geq\ &\frac{L(\mathrm{Sym}_i^2 E,2)}{c_4c_5 2^{\rho+1}\rho!2^{n_2/2}3^{n_3/2}\sqrt{N}}\prod_{p|N}\frac{p}{p-1}\prod_{i=1}^{\rho}\frac{(q_i-1)(q_i+1-\lfloor 2\sqrt{q_i}\rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i}\rfloor)}(\log d)^{\rho}\theta(d).
\end{aligned}$$

Finally, if we take

$$c_1 = \max_{\substack{F_d \in \mathcal{F},\\ 1 \le k \le \rho}} \left\{ c_3,\ \exp\left(2^{\rho-1}\rho!\sqrt{N}\right),\ \exp\left(L(\mathrm{Sym}_i^2 E, 2)\right),\ \exp\left(2\rho \frac{|F_d^{(k)}(1)|}{|F_d(1)|}\right) \right\}$$

and

$$c_2 = \frac{L(\mathrm{Sym}_i^2 E, 2)}{c_4 c_5 2^{\rho+1} \rho! 2^{n_2/2} 3^{n_3/2} \sqrt{N}} \prod_{p|N} \frac{p}{p-1} \prod_{i=1}^{\rho} \frac{(q_i-1)(q_i+1-\lfloor 2\sqrt{q_i}\rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i}\rfloor)},$$

Theorem 2 follows. □

4. Proof of Proposition 4

4.1. **Choice of U.** First we choose an appropriate U to prove Proposition 4. Let $\zeta(s)$ be the Riemann zeta function and $\zeta_K(s)$ the Dedekind zeta function of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. Let

$$\tilde{\zeta}_K(s) = \frac{\zeta_K(s)}{\zeta(2s)} = \frac{\zeta(s)L(s,\chi_d)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\nu_n}{n^s}.$$

Then we have $\nu_n \ge 0$ because the Euler product of $\tilde{\zeta}_K(s)$ is

$$\begin{aligned} \frac{\zeta(s)L(s,\chi_d)}{\zeta(2s)} &= \prod_p \frac{1-p^{-2s}}{(1-p^{-s})(1-\chi_d(p)p^{-s})} \\ &= \prod_{p \text{ ramifies in } K} (1+p^{-s}) \prod_{p \text{ splits in } K} \left(\frac{1+p^{-s}}{1-p^{-s}}\right). \end{aligned} \tag{4.1.1}$$

Lemma 6. *For $d > 4$,*

$$\sum_{n < \frac{\sqrt{d}}{4}} \nu_n < \frac{1}{4\log 2} L(1,\chi_d)\sqrt{d}.$$

Proof. See [Lemma 4, Go]. □

We take for U the number $\left(\frac{\sqrt{d}}{4}\right)^{1/m}$, where m is the smallest positive integer such that

$$m^{\rho+1} \ge \frac{(\rho+1)!}{2^{\rho+1}} h(d) \log \varepsilon_d. \tag{4.1.2}$$

The following lemma is an analogue of [Lemma 1, Section 3.5, Oe].

Lemma 7.

(a) *For $d \ge \exp(6\rho^{\rho+1})$, the largest prime divisor of d is greater than U.*

(b) *There are at most ρ prime numbers $q < U$ which split in $K/\mathbb{Q}$.*

Proof. (a) For $d \geq \exp(6\rho^{\rho+1})$, we have

$$\log \varepsilon_d \geq \log\left(\frac{\sqrt{d-4}+\sqrt{d}}{2}\right) > \tfrac{1}{3}\log d > 2\rho^{\rho+1}.$$

Let T be the number of prime divisors of d. Suppose $T \geq 2m+1$, where m is defined in (4.1.2). Let $h^{(+)}(d)$ be the narrow class number of K. Then we have

$$h^{(+)}(d) = \begin{cases} 2h(d) & \text{if } \mathcal{N}(\varepsilon_d) = 1, \\ h(d) & \text{if } \mathcal{N}(\varepsilon_d) = -1, \end{cases}$$

where $\mathcal{N}$ is the field norm $\mathcal{N} : K \to \mathbb{Q}$ such that $\mathcal{N}(x + \sqrt{d}y) = x^2 - dy^2$. By the genus theory, 2^{T-1} divides $h^{(+)}(d)$, so we have

$$m^{\rho+1} > \frac{(\rho+1)!}{2^{\rho+1}} h^{(+)}(d) \cdot \rho^{\rho+1} \geq \frac{(\rho+1)!}{2^{\rho+1}} \rho^{\rho+1} \cdot 2^{2m},$$

which is contradiction. Therefore we have $T \leq 2m$. Since either d or $\frac{d}{4}$ is square-free, at least one of the prime divisors of d is greater than $(\frac{d}{16})^{1/T}$ and so than U.

(b) Suppose $q_1, q_2, \cdots, q_{\rho+1}$ are primes less than U and split in $K/\mathbb{Q}$. By (4.1.1), we have

$$\frac{1+q_1^{-s}}{1-q_1^{-s}} \cdot \frac{1+q_2^{-s}}{1-q_2^{-s}} \cdots \frac{1+q_{\rho+1}^{-s}}{1-q_{\rho+1}^{-s}} \ll \frac{\zeta_K(s)}{\zeta(2s)}$$

and for all pairs $(l_1, l_2, \cdots, l_{\rho+1}) \in \mathbb{N}^{\rho+1}$ such that $l_1 + l_2 + \cdots + l_{\rho+1} \leq m$, we have

$$q_1^{l_1} q_2^{l_2} \cdots q_{\rho+1}^{l_{\rho+1}} < U^m = \frac{\sqrt{d}}{4}.$$

By the Dirichlet class number formula and Lemma 6, we deduce the inequality

$$\sum_{i=0}^{\rho+1} \binom{\rho+1}{i} 2^i \binom{m}{i} < \frac{1}{2\log 2} h(d) \log \varepsilon_d,$$

which contradicts the definition of m. Therefore there are at most ρ prime numbers $q < U$ which split in $K/\mathbb{Q}$. □

4.2. **Some integrals.** Now we introduce some integrals needed to prove Proposition 4.

Lemma 8. *Let $m \geq 1$ be an integer. Let $a < \sigma$ be real numbers. As a function of $x > 0$,*

$$\int_{\sigma-i\infty}^{\sigma+i\infty} x^{-s}(s-a)^{-m}\frac{ds}{2\pi i} = \begin{cases} x^{-a}\frac{|\log x|^{m-1}}{(m-1)!} & 0 < x < 1, \\ 0 & x > 1 \end{cases}$$

is decreasing and convex if $a \geq 0$.

Proof. See [Lemma 1, Section 3.3, Oe] or [p. 95, PK]. □

Lemma 9. *Let $\mu_1, \cdots, \mu_r$ be the positive measures on $\mathbb{R}_+^*$ for which the function $t \mapsto t^\sigma$ is integrable.*
(a) Let

$$\hat{\mu}_j(s) = \int_0^\infty t_j^{-s}\mu_j, \quad (1 \leq j \leq r,\ \mathrm{Re(s)} = \sigma),$$

and

$$J(x) = \int_{\sigma-i\infty}^{\sigma+i\infty} \hat{\mu}_1(s)\cdots\hat{\mu}_r(s)x^{-s}(s-a)^{-m}\frac{ds}{2\pi i}.$$

Then J is the positive function on $\mathbb{R}_+^*$. Further J is convex and decreasing if $a \geq 0$.
(b) Let μ_j' $(1 \leq j \leq r)$ be the positive measures on $\mathbb{R}_+^*$ satisfying the same hypotheses with μ_j and define J' analogously to J. Suppose we have

$$\int_0^x \mu_j([0,t])dt \leq \int_0^x \mu_j'([0,t])dt$$

for all $x \geq 0$ and $a \geq 0$. Then $0 \leq J \leq J'$.

Proof. See [Lemma 2 and Lemma 3, Section 3.3, Oe]. □

The following example will be used in Section 4.3 to get the required upper bound (4.3.9) of $J(U^*)$ from (4.3.8).

Example 10. *Let the measures $\nu_1 = \sum_{n=1}^\infty \delta_n$ and $\nu_1' = \delta_1 + \mathrm{Leb}[1,\infty)$, where δ_n is the Dirac measure centered on n and $\mathrm{Leb}[1,\infty)$ is the standard Lebesgue measure restricted to the interval $[1,\infty)$. Let ν_2 and ν_2' be images of ν_1 and ν_1' by applying $t \mapsto t^2$. We have*

$$\nu_1([0,t]) \leq \nu_1'([0,t]) \quad (\textit{for all } \ t > 0)$$

and

$$\nu_2([0,t]) \le \nu_2'([0,t]) \quad (\textit{for all } \ t>0).$$

For $\mathrm{Re(s)} = \sigma > 1$, *we have*

$$\hat{\nu}_2(s) = \int_0^\infty (t^2)^{-s} \sum_{n=1}^\infty d\delta_n(t) = \sum_{n=1}^\infty \frac{1}{n^{2s}} = \zeta(2s)$$

and

$$\hat{\nu}_2'(s) = 1 + \int_1^\infty (t^2)^{-s} dt = 1 + \frac{1}{2s-1} = \frac{s}{s-\frac{1}{2}}.$$

Lemma 11. *Let* $q(m,n) = am^2 + bmn + cn^2$ *be a positive definite reduced quadratic form with real coefficients and* $D = -(b^2-4ac) > 0$ *be the discriminant of* $q(m,n)$. *Let* $S(x)$ *be the number of* $\{(m,n) \in \mathbb{Z} \times \mathbb{N}^* \mid am^2+bmn+cn^2 \le x\}$. *Then*

(a) $S(x) < \frac{2\pi}{\sqrt{D}} x$.

(b) $S(x) = 0$ *for* $x < \frac{\sqrt{D}}{2}$.

Proof. (a) $S(x)$ is equal to the number of solutions of

$$(2am+bn)^2 + Dn^2 \le 4ax,$$

which is equivalent to

$$\begin{cases} -\sqrt{4ax-Dn^2} - bn \le 2am \le \sqrt{4ax-Dn^2} - bn, \\ 0 < n \le \lambda = \sqrt{\frac{4ax}{D}}. \end{cases}$$

Since $n \neq 0$, $S(x) = 0$ for $x < \frac{D}{4a}$. Since $a \le \frac{\sqrt{D}}{3}$, for $x \ge \frac{D}{4a}$,

$$\begin{aligned} S(x) &\le \sum_{1\le n\le\lambda} \left(\lfloor \frac{\sqrt{4ax-Dn^2}}{a} \rfloor + 1 \right) \\ &\le \frac{\sqrt{D}}{a} \sum_{1\le n \le \lambda} \sqrt{\lambda^2 - n^2} + \lambda \\ &< \frac{\sqrt{D}}{a}\frac{\pi}{4}\lambda^2 + \lambda \\ &= \frac{\pi}{\sqrt{D}} x + \sqrt{\frac{4a}{D}}\sqrt{x} \\ &\le (\frac{\pi}{\sqrt{D}} + \frac{4a}{D}) x \\ &< \frac{2\pi}{\sqrt{D}} x. \end{aligned}$$

(b) Since $a \leq \frac{\sqrt{D}}{3}$ and $n \neq 0$,

$$am^2 + bmn + cn^2 = \frac{(2am+bn)^2 + Dn^2}{4a} \geq \frac{3\sqrt{D}}{4} > \frac{\sqrt{D}}{2}.$$

□

The following example will also be used in Section 4.3 to get the required upper bound (4.3.9) of $J(U^*)$ from (4.3.8).

Example 12. *Let the measures*

$$\nu = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}^{\infty} \delta_{q(m,n)} \quad \textit{and} \quad \nu' = \pi\delta_{\sqrt{D}/2} + \frac{2\pi}{\sqrt{D}}\mathrm{Leb}[\sqrt{\mathrm{D}}/2, \infty),$$

where $\delta_{q(m,n)}$ *is the Dirac measure centered on* $q(m,n)$ *and* $\mathrm{Leb}[\mathrm{x}, \infty)$ *is the standard Lebesgue measure restricted to the interval* $[x, \infty)$*. From Lemma 11, we have*

$$\int_0^x \nu([0,t])dt \leq \int_0^x \nu'([0,t])dt \quad \textit{for all } x \geq 0.$$

For $\mathrm{Re(s)} = \sigma > 1$*, we have*

$$\hat{\nu}(s) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}^{\infty} q(m,n)^{-s}$$

and

$$\begin{aligned}\hat{\nu}'(s) &= \pi\left(\frac{\sqrt{D}}{2}\right)^{-s} + \frac{2\pi}{\sqrt{D}}\int_{\sqrt{D}/2}^{\infty} t^{-s}dt \\ &= \pi\left(\frac{\sqrt{D}}{2}\right)^{-s} + \frac{2\pi}{\sqrt{D}}\frac{1}{s-1}\left(\frac{\sqrt{D}}{2}\right)^{-s+1} \\ &= \pi\frac{s}{s-1}\left(\frac{\sqrt{D}}{2}\right)^{-s}.\end{aligned}$$

4.3. **Upper bound of $J(U^*)$.** By (3.4.3), Lemma 8 and 9, we have

$$J(U^*) \leq \int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-\frac{1}{2}}\gamma(s+\frac{1}{2})\varphi(s)(s-\frac{1}{2})^{-4}\frac{ds}{2\pi i},$$

for all Dirichlet series φ which converges absolutely for $\mathrm{Re}(s) > 1$ and satisfies

$$\Psi(s+\frac{1}{2})G(U^*, s+\frac{1}{2}) \ll \varphi(s). \tag{4.3.1}$$

By (2.0.1) and (2.0.2), we have the following lemma.

Lemma 13. *We can take for φ satisfying* (4.3.1), *the integral of Dirichlet series obtained by expanding*

$$\left(\sum_{(a,b,c)} \frac{2\Gamma(s)}{\Gamma(s/2)^2}\left(\zeta(2s)\cdot\int_{-\log\varepsilon_d}^{\log\varepsilon_d} \check{A}^{-s}d\phi + \int_{-\log\varepsilon_d}^{\log\varepsilon_d}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}(\check{A}m^2+\check{B}mn+\check{C}n^2)^{-s}d\phi\right)\right)^2$$

and removing the terms of the form

$$\left(\frac{2\Gamma(s)}{\Gamma(s/2)^2}\right)^2\cdot\zeta(2s)^2\cdot\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\check{A}_1^{-s}d\phi\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\check{A}_2^{-s}d\phi$$

if

$$\min\{\check{\mathrm{A}}_1(\phi)\mid -\log\varepsilon_{\mathrm{d}}\le\phi\le\log\varepsilon_{\mathrm{d}}\}\cdot\min\{\check{\mathrm{A}}_2(\phi)\mid -\log\varepsilon_{\mathrm{d}}\le\phi\le\log\varepsilon_{\mathrm{d}}\}<4\mathrm{U}.$$

Proof. Let

$$\tilde{\tilde{\zeta}}_K(s)=\tilde{\zeta}_K(s)^2,$$

$$\tilde{\tilde{\zeta}}_{K,p}(s)=\text{the Euler factor of }\tilde{\tilde{\zeta}}_K(s)\text{ at a prime }p,$$

$$\tilde{\tilde{\zeta}}_K(U,s)=\prod_{P<U}\tilde{\tilde{\zeta}}_{K,p}(s),$$

and

$$\tilde{\tilde{\zeta}}_K(U^*,s)=\tilde{\tilde{\zeta}}_K(s)-\tilde{\tilde{\zeta}}_K(U,s).$$

Since

$$\Psi(s+\frac{1}{2})G(s+\frac{1}{2})\ll\zeta_K(s)^2,$$

$$\Psi(s+\frac{1}{2})\ll\zeta(2s)^2,$$

and

$$G(s+\frac{1}{2})\ll\left(\frac{\zeta_K(s)}{\zeta(2s)}\right)^2=\tilde{\tilde{\zeta}}_K(s),$$

we have

$$\Psi(s+\frac{1}{2})G(U^*,s+\frac{1}{2})\ll\zeta(2s)^2\tilde{\tilde{\zeta}}_K(U^*,s).$$

If

$$\min\{\check{\mathrm{A}}_1(\phi)\mid -\log\varepsilon_{\mathrm{d}}\le\phi\le\log\varepsilon_{\mathrm{d}}\}\cdot\min\{\check{\mathrm{A}}_2(\phi)\mid -\log\varepsilon_{\mathrm{d}}\le\phi\le\log\varepsilon_{\mathrm{d}}\}<4\mathrm{U},$$

then

$$\begin{aligned} U &> \frac{1}{4}\cdot \check{A}_1(\phi_1)\cdot \check{A}_2(\phi_2) \\ &= \frac{1}{4}\cdot\Big(A_1(\phi_1)m_1^2+B_1(\phi_1)m_1n_1+C_1(\phi_1)n_1^2\Big) \\ &\quad \cdot\Big(A_2(\phi_2)m_2^2+B_2(\phi_2)m_2n_2+C_2(\phi_2)n_2^2\Big) \\ &= \frac{1}{4}\cdot\frac{v_1^2e^{\phi_1}+{v'}_1^2e^{-\phi_1}}{\mathcal{N}(\mathfrak{b}_1)}\cdot\frac{v_2^2e^{\phi_2}+{v'}_2^2e^{-\phi_2}}{\mathcal{N}(\mathfrak{b}_2)} \\ &\geq \frac{|v_1{v'}_1|}{\mathcal{N}(\mathfrak{b}_1)}\cdot\frac{|v_2{v'}_2|}{\mathcal{N}(\mathfrak{b}_2)} \\ &= \mathcal{N}(\mathfrak{a}_1)\mathcal{N}(\mathfrak{a}_2) \end{aligned}$$

for some $\phi_1, \phi_2 \in [-\log\varepsilon_d, \log\varepsilon_d]$, some rational integers m_1, n_1, m_2, n_2, some $v_1 \in \mathfrak{b}_1$, $v_2 \in \mathfrak{b}_2$, and the corresponding ideals $\mathfrak{a}_1 \in [\mathfrak{b}_1^{-1}]$, $\mathfrak{a}_2 \in [\mathfrak{b}_2^{-1}]$. Note that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are products of prime ideals of norm less than U.

Since the Euler product of $\tilde{\tilde{\zeta}}_K(U,s)$ is

$$\tilde{\tilde{\zeta}}_K(U,s) = \Bigg(\prod_{p<U}(1-p^{-2s})\prod_{\mathcal{N}(\mathfrak{p})<U}\frac{1}{1-\mathcal{N}(\mathfrak{p})^{-s}}\Bigg)^2,$$

by (2.0.1) and (2.0.2), we have $\zeta(2s)^2\tilde{\tilde{\zeta}}_K(U^*,s) \ll$ the integral of Dirichlet series obtained by expanding

$$\Bigg(\sum_{(a,b,c)}\frac{2\Gamma(s)}{\Gamma(s/2)^2}\Bigg(\zeta(2s)\cdot\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\check{A}^{-s}d\phi+\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}(\check{A}m^2+\check{B}mn+\check{C}n^2)^{-s}d\phi\Bigg)\Bigg)^2$$

and removing the terms of the form

$$\left(\frac{2\Gamma(s)}{\Gamma(s/2)^2}\right)^2\cdot\zeta(2s)^2\cdot\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\check{A}_1^{-s}d\phi\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\check{A}_2^{-s}d\phi$$

if

$$\min\{\check{\mathrm{A}}_1(\phi)\mid -\log\varepsilon_{\mathrm{d}}\le\phi\le\log\varepsilon_{\mathrm{d}}\}\cdot\min\{\check{\mathrm{A}}_2(\phi)\mid -\log\varepsilon_{\mathrm{d}}\le\phi\le\log\varepsilon_{\mathrm{d}}\}<4\mathrm{U}.$$

□

Let $u(t)$ be the unit step function

$$u(t)=\begin{cases}1 & t\geq 0,\\ 0 & t<0.\end{cases}$$

We will use the following Mellin transform (see [Table 18.1, Po]).

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \quad \text{for} \quad Re(s) > 0, \tag{4.3.2}$$

$$\frac{\Gamma(s)\Gamma(b)}{\Gamma(s+b)} = \int_0^\infty u(1-t)(1-t)^{b-1} t^{s-1} dt \quad \text{for} \quad Re(s) > 0 \text{ and } Re(b) > 0, \tag{4.3.3}$$

$$-\frac{a^{s+b}}{s+b} = \int_0^\infty u(t-a) t^b t^{s-1} dt \quad \text{for} \quad Re(s) < -Re(b) \text{ and } a > 0. \tag{4.3.4}$$

Lemma 14. *Let $\tilde{\varphi}(s)$ be the Dirichlet series obtained in Lemma 13. Then we have*

$$J(U^*) \le \int_{\sigma - i\infty}^{\sigma + i\infty} d^{s-\frac{1}{2}} \left(\frac{M_d}{4\pi^2} \right)^{s+\frac{1}{2}} 4\Gamma(s+\frac{1}{2})^2 \Gamma(s)^2 \frac{1}{\Gamma(\frac{s}{2})^4} \tilde{\varphi}(s)(s-\frac{1}{2})^{-\rho-2} \frac{ds}{2\pi i}, \tag{4.3.5}$$

and we can apply Lemma 9 to the right side of (4.3.5).

Proof. Recall that $\gamma(s) = \left(\frac{M_d}{4\pi^2}\right)^s \Gamma(s)^2$. To check the conditions of Lemma 9, it suffices to show that the term $\Gamma(s)^2/\Gamma(\frac{s}{2})^4$ can be written by the form in Lemma 9 (a). By the duplication formula of Gamma function,

$$\Gamma(s)^2 \frac{1}{\Gamma(\frac{s}{2})^4} = \frac{\Gamma(\frac{s}{2}+\frac{1}{2})^2}{\Gamma(\frac{s}{2})^2} \frac{2^{2s-2}}{\pi} = \frac{\Gamma(\frac{s}{2}+\frac{1}{2})^2}{\Gamma(\frac{s}{2}+1)^2} \frac{2^{2s-2}}{\pi} (\frac{s}{2})^2. \tag{4.3.6}$$

By (4.3.3), we have

$$\begin{aligned} \frac{\Gamma(\frac{s}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{s}{2}+1)} &= \int_0^\infty u(1-t) \frac{1}{\sqrt{1-t}} t^{\frac{s}{2}-\frac{1}{2}} dt \\ &= \int_0^\infty 2u(1-t) \frac{1}{\sqrt{1-t^2}} t^s dt, \end{aligned}$$

and so we write

$$\Gamma(s)^2 \frac{1}{\Gamma(\frac{s}{2})^4} = \left(\int_0^\infty \frac{2u(1-t)}{\sqrt{1-t^2}} t^s dt \right)^2 \frac{2^{2s-2}}{\pi^2} \cdot \frac{1}{4} \big((s-\frac{1}{2})+\frac{1}{2}\big)^2. \tag{4.3.7}$$

Expanding the term $((s-\frac{1}{2})+\frac{1}{2})^2$ with respect to $(s-\frac{1}{2})$, and applying Lemma 8, the right side of (4.3.5) satisfies the conditions of Lemma 9. $\square$

Applying Lemma 8 and 9 to the right side of (4.3.5), we have

$$
\begin{aligned}
&J(U^*)\\
&\le h(d)^2(2\log\varepsilon_d)^2\int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-\frac{1}{2}}\gamma(s+\frac{1}{2})\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}\zeta(2s)^2\cdot(4U)^{-s}\\
&\quad\cdot(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i}\\
&\quad+2\log\varepsilon_d\int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-\frac{1}{2}}\gamma(s+\frac{1}{2})\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}\zeta(2s)\cdot\Big(\sum_{(a,b,c)}\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\alpha'^{-s}d\phi\Big)\\
&\quad\cdot\Big(\sum_{(a,b,c)}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}\big(\min_{\phi}\{\check{A}m^2+\check{B}mn+\check{C}n^2\}\big)^{-s}\Big)\cdot(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i}\\
&\quad+(2\log\varepsilon_d)^2\int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-\frac{1}{2}}\gamma(s+\frac{1}{2})\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}\\
&\quad\cdot\Big(\sum_{(a,b,c)}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}\big(\min_{\phi}\{\check{A}m^2+\check{B}mn+\check{C}n^2\}\big)^{-s}\Big)^2\cdot(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i},
\end{aligned}
\tag{4.3.8}
$$

where $\min\limits_{\phi}$ means $\min\limits_{-\log\varepsilon_d\le\phi\le\log\varepsilon_d}$.

In the view of Lemma 9, Example 10 and 12, we increase the right side of (4.3.8) by replacing

$$
\begin{aligned}
\zeta(2s) &\quad\text{by}\quad s\big(s-\frac{1}{2}\big)^{-1},\\
\sum_{(m,n)\in\mathbb{Z}\times\mathbb{N}^*}^{\infty} q(m,n)^{-s} &\quad\text{by}\quad \pi\frac{s}{s-1}\Big(\frac{\sqrt{D}}{2}\Big)^{-s},
\end{aligned}
$$

with $D=4d$. Therefore we obtain

$$
J(U^*)\le J_1+J_2+J_3, \tag{4.3.9}
$$

where

$$
\begin{aligned}
J_1 &= 4h(d)^2(\log\varepsilon_d)^2\sqrt{\frac{M_d}{4\pi^2 d}}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}\\
&\quad\cdot\big(\frac{M_d d}{16\pi^2 U}\big)^s s^2(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i},
\end{aligned}
\tag{4.3.10}
$$

$$\begin{aligned} J_2 &= 2\pi h(d)\log\varepsilon_d\sqrt{\frac{M_d}{4\pi^2 d}}\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\cdot\frac{4\Gamma(s)^2}{\Gamma(s/2)^4} \\ &\quad\cdot\Big(\sum_{(a,b,c)}\breve{A}(\phi)^{-s}\Big)\cdot(\frac{M_d\sqrt{d}}{4\pi^2})^s\frac{s^2}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i}d\phi, \end{aligned} \tag{4.3.11}$$

and

$$\begin{aligned} J_3 &= 4\pi^2 h(d)^2(\log\varepsilon_d)^2\sqrt{\frac{M_d}{4\pi^2 d}}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4} \\ &\quad\cdot(\frac{M_d}{4\pi^2})^s(\frac{s}{s-1})^2(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i}. \end{aligned} \tag{4.3.12}$$

4.4. **Estimation of J_1.**

Lemma 15.

(a) *For $x>0$ and $\sigma>1$, we have*

$$\int_{\sigma-i\infty}^{\sigma+i\infty}x^s s^4(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i}\le\begin{cases}\frac{\sqrt{x}}{4\cdot(\rho+3)!}(\log x+4)^{\rho+3} & x>1,\\ 0 & 0<x<1.\end{cases}$$

(b) *For $x>e$ and $\sigma>1$, we have*

$$\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}x^s s^2(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i}\le\frac{8}{\pi}4^\rho(\rho+3)!\sqrt{x}(\log x+4)^{\rho+3}.$$

Proof. (a) By Lemma 8,

$$\begin{aligned} &\int_{\sigma-i\infty}^{\sigma+i\infty}x^s((s-\frac{1}{2})+\frac{1}{2})^4(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i} \\ =&\int_{\sigma-i\infty}^{\sigma+i\infty}x^s(s-\frac{1}{2})^{-\rho}+\frac{4}{2}x^s(s-\frac{1}{2})^{-\rho-1}+\frac{6}{2^2}x^s(s-\frac{1}{2})^{-\rho-2} \\ &\quad+\frac{4}{2^3}x^s(s-\frac{1}{2})^{-\rho-3}+\frac{1}{2^4}x^s(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i} \\ =&\begin{cases}\sqrt{x}\big(\frac{|\log x|^{\rho-1}}{(\rho-1)!}+\frac{2|\log x|^\rho}{\rho!}+\frac{3|\log x|^{\rho+1}}{2\cdot(\rho+1)!}+\frac{|\log x|^{\rho+2}}{2\cdot(\rho+2)!}+\frac{|\log x|^{\rho+3}}{16\cdot(\rho+3)!}\big) & x>1,\\ 0 & 0<x<1\end{cases} \\ \le&\begin{cases}\frac{\sqrt{x}}{4\cdot(\rho+3)!}(\log x+4)^{\rho+3} & x>1,\\ 0 & 0<x<1.\end{cases} \end{aligned}$$

(b) Let I be the integral

$$\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}\cdot x^s s^2(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i}.$$

By (4.3.2), (4.3.7) and (a),

$$\begin{aligned} I &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-t_1}t_1^{-\frac{1}{2}}e^{-t_2}t_2^{-\frac{1}{2}}\frac{2u(1-t_3)}{\sqrt{1-t_3^2}}\frac{2u(1-t_4)}{\sqrt{1-t_4^2}}\cdot\frac{1}{4\pi^2} \\ &\quad\cdot\int_{\sigma-i\infty}^{\sigma+i\infty}\left(4t_1t_2t_3t_4x\right)^s s^4(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i}dT \\ &\le \int\int\int\int_{t_1t_2t_3t_4\ge\frac{1}{4x}} e^{-t_1}t_1^{-\frac{1}{2}}e^{-t_2}t_2^{-\frac{1}{2}}\frac{2u(1-t_3)}{\sqrt{1-t_3^2}}\frac{2u(1-t_4)}{\sqrt{1-t_4^2}}\cdot\frac{1}{4\pi^2} \\ &\quad\cdot\frac{\sqrt{4t_1t_2t_3t_4x}}{4\cdot(\rho+3)!}\big(\log\left(4t_1t_2t_3t_4x\right)+4\big)^{\rho+3}dT, \end{aligned}$$

where $dT = dt_1dt_2dt_3dt_4$. Thus we have

$$\begin{aligned} I &\le \int\int\int\int_{\frac{1}{4x}\le t_1t_2t_3t_4\le\frac{1}{4}} e^{-t_1}t_1^{-\frac{1}{2}}e^{-t_2}t_2^{-\frac{1}{2}}\frac{2u(1-t_3)}{\sqrt{1-t_3^2}}\frac{2u(1-t_4)}{\sqrt{1-t_4^2}} \\ &\quad\cdot\frac{1}{4\pi^2}\cdot\frac{\sqrt{4t_1t_2t_3t_4x}}{4\cdot(\rho+3)!}(\log x+4)^{\rho+3}dT \\ &\quad+\int\int\int\int_{t_1t_2t_3t_4\ge\frac{1}{4}} e^{-t_1}t_1^{-\frac{1}{2}}e^{-t_2}t_2^{-\frac{1}{2}}\frac{2u(1-t_3)}{\sqrt{1-t_3^2}}\frac{2u(1-t_4)}{\sqrt{1-t_4^2}} \\ &\quad\cdot\frac{1}{4\pi^2}\cdot\frac{\sqrt{4t_1t_2t_3t_4x}}{4\cdot(\rho+3)!}\big(4t_1t_2t_3t_4(\log x+4)\big)^{\rho+3}dT \\ &\le \frac{1}{(\rho+3)!}\left(4\Gamma^2(1)\frac{\Gamma^2(\frac{1}{2})}{\Gamma^4(\frac{1}{4})}+\frac{4}{(2\rho+7)^2}\Gamma(\rho+4)^2\frac{\Gamma(\frac{2\rho+7}{2})^2}{\Gamma(\frac{2\rho+7}{4})^4}\right) \\ &\quad\cdot\sqrt{x}(\log x+4)^{\rho+3}. \end{aligned}$$

By (4.3.6), we have $I \le \frac{8}{\pi}4^\rho(\rho+3)!\sqrt{x}(\log x+4)^{\rho+3}$. □

Proposition 16.

$$J_1 \le \frac{4^{\rho+1}(\rho+3)!}{\pi^3}\cdot h(d)^2(\log\varepsilon_d)^2\frac{M_d}{\sqrt{U}}\cdot\left(\log\left(\frac{M_d d}{16\pi^2U}\right)+4\right)^{\rho+3}.$$

Proof. From (4.3.10) and Lemma 15,

$$\begin{aligned} J_1 &= 4h(d)^2(\log\varepsilon_d)^2\sqrt{\frac{M_d}{4\pi^2d}}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4} \\ &\quad\cdot(\frac{M_dd}{16\pi^2U})^s s^2(s-\frac{1}{2})^{-\rho-4}\frac{ds}{2\pi i} \\ &\le \frac{8\cdot4^\rho(\rho+3)!}{\pi\sqrt{4\pi^2}\sqrt{16\pi^2}}\cdot4h(d)^2(\log\varepsilon_d)^2\frac{M_d}{\sqrt{U}}\cdot\left(\log\left(\frac{M_dd}{16\pi^2U}\right)+4\right)^{\rho+3}. \end{aligned}$$

□

4.5. **Estimation of J_2.**

Lemma 17. *For $x > 0$ and $\sigma > 1$, we have*

(a) $\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i} \leq 2^{\rho+3}x,$

(b) $\displaystyle\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}x^s \frac{s^2}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i} \leq \frac{2^{\rho+3}}{\pi}x.$

Proof. (a) Applying the residue theorem to the vertical strip between $\frac{3}{4}$ and σ and using (4.3.4), we have

$$\begin{aligned}
&\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i}\\
=\ & 2^{\rho+3}x + \int_{\frac{3}{4}-i\infty}^{\frac{3}{4}+i\infty} x^s \frac{s^4}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i}\\
=\ & 2^{\rho+3}x - \int_{\frac{3}{4}-i\infty}^{\frac{3}{4}+i\infty}\Big(\int_0^\infty u(t-x)t^{-1}t^{s-1}dt\Big)xs^4(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i}.
\end{aligned}$$

Since $u(t-x)t^{s-2}$ is a nonnegative function of t on $\mathbb{R}_+^*$, by Lemma 9,

$$\int_{\frac{3}{4}-i\infty}^{\frac{3}{4}+i\infty}\Big(\int_0^\infty u(t-x)t^{-1}t^{s-1}dt\Big)x\Big((s-\frac{1}{2})+\frac{1}{2}\Big)^4(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i} \geq 0,$$

and so we have

$$\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i} \leq 2^{\rho+3}x.$$

(b) By (4.3.7) and (a), we have

$$\begin{aligned}
&\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}\cdot x^s \frac{s^2}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i}\\
\leq\ & \Gamma(1+\frac{1}{2})^2\frac{4\Gamma(1)^2}{\Gamma(\frac{1}{2})^4}\cdot 2^{\rho+3}x\\
=\ & \frac{2^{\rho+3}}{\pi}x.
\end{aligned}$$

□

From Lemma 7, we know that there are at most ρ primes $q < U$ which split in $K/\mathbb{Q}$.

Proposition 18. *Let q_i be the i-th prime q which splits in K. Then for $d \geq \exp(6\rho^{\rho+1})$ we have*

$$J_2 \leq \frac{2^\rho}{\pi^2}h(d)\log\varepsilon_d M_d^{3/2}\Big(1+\frac{h(d)}{U}\Big)\prod_{i=1}^{\rho}\Big(\frac{q_i+1}{q_i-1}\Big)\prod_{p\in P(d)}(1+\frac{1}{p}).$$

Proof. We have for $\sigma \geq 1$,

$$\frac{2\Gamma(s)}{\Gamma(s/2)^2}\sum_{(a,b,c)}\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\breve{A}^{-s}d\phi \;<\; \frac{2\Gamma(s)}{\Gamma(s/2)^2}\sum_{(a,b,c)}\int_{-\infty}^{\infty}\breve{A}^{-s}d\phi = \sum_{(a,b,c)}\frac{1}{\mathcal{N}(\mathfrak{b})^s},$$

where $\mathfrak{b}$'s are the corresponding ideals to (a,b,c).

Therefore

$$\frac{2\Gamma(s)}{\Gamma(s/2)^2}\sum_{(a,b,c)}\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\breve{A}^{-s}d\phi < \sum_{(a,b,c)}\frac{1}{\mathcal{N}(\mathfrak{b})^s} \ll \frac{\zeta_K(s)}{\zeta(2s)},$$

and so by (4.1.1) and Lemma 7, for $d \geq \exp(6\rho^{\rho+1})$ we have

$$\frac{2}{\pi}\sum_{(a,b,c)}\int_{-\log\varepsilon_d}^{\log\varepsilon_d}\breve{A}^{-1}d\phi \;<\; \Big(1+\frac{h(d)}{U}\Big)\prod_{i=1}^{\rho}\Big(\frac{1+q_i^{-1}}{1-q_i^{-1}}\Big)\prod_{p\in P(d)}(1+\frac{1}{p}). \tag{4.5.1}$$

By (4.3.11), Lemma 17 and (4.5.1),

$$\begin{aligned} J_2 &= 2\pi h(d)\log\varepsilon_d\sqrt{\frac{M_d}{4\pi^2 d}}\int_{-\log\epsilon_d}^{\log\epsilon_d}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\cdot\frac{4\Gamma(s)^2}{\Gamma(s/2)^4} \\ &\quad\cdot\Big(\sum_{(a,b,c)}\breve{A}(\phi)^{-s}\Big)\cdot(\frac{M_d\sqrt{d}}{4\pi^2})^s\frac{s^2}{s-1}(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i}d\phi \\ &\leq 2^{\rho+3}\pi h(d)\log\varepsilon_d\Big(\frac{M_d}{4\pi^2}\Big)^{3/2}\Big(\frac{2}{\pi}\sum_{(a,b,c)}\int_{-\log\epsilon_d}^{\log\epsilon_d}\breve{A}^{-1}d\phi\Big) \\ &< \frac{2^\rho}{\pi^2}h(d)\log\varepsilon_d M_d^{3/2}\Big(1+\frac{h(d)}{U}\Big)\prod_{i=1}^{\rho}\Big(\frac{1+q_i^{-1}}{1-q_i^{-1}}\Big)\prod_{p\in P(d)}(1+\frac{1}{p}). \end{aligned}$$

□

4.6. **Estimation of J_3.**

Lemma 19. *For $x > 0$ and $\sigma > 1$, we have*

(a) $$\int_{\sigma-i\infty}^{\sigma+i\infty}x^s\frac{s^4}{(s-1)^2}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \leq \frac{81}{16}x^{\frac{3}{2}},$$

(b) $$\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}x^s\frac{s^2}{(s-1)^2}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \leq 4x^{\frac{3}{2}}.$$

Proof. (a) For $\alpha > 0$, let μ'_α be the images of $\delta_1 + \mathrm{Leb}[1,\infty)$ by applying $t \mapsto t^{1/\alpha}$. By the same way as in Example 10, we have

$$\hat{\mu}'_\alpha(s) = 1 + \int_1^\infty (t^{1/\alpha})^{-s}dt = 1 + \frac{1}{s/\alpha - 1} = \frac{s}{s-\alpha}.$$

For $0 < \epsilon < 1$, we have

$$\begin{aligned}\frac{s^4}{(s-1-\epsilon)(s-1+\epsilon)} - \frac{s^4}{(s-1)^2} &= \epsilon^2 \cdot \Big(\frac{s}{s-1}\Big)^2 \cdot \frac{s}{s-1-\epsilon} \cdot \frac{s}{s-1+\epsilon} \\ &= \epsilon^2 \cdot \hat{\mu}_1'(s)^2 \cdot \hat{\mu}_{1+\epsilon}'(s) \cdot \hat{\mu}_{1-\epsilon}'(s),\end{aligned}$$

and so by Lemma 9, we have the following inequality.

$$\begin{aligned}&\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{(s-1)^2}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \\ \leq\ &\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{(s-1-\epsilon)(s-1+\epsilon)}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i}.\end{aligned}$$

Translating the vertical line to the right such that $\sigma > \frac{3}{2}$, substituting $\frac{1}{2}$ for ϵ, applying the residue theorem to the vertical strip between 1 and $\sigma > \frac{3}{2}$, and using (4.3.4), we have

$$\begin{aligned}&\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{(s-1)^2}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \\ \leq\ &\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{(s-\frac{3}{2})(s-\frac{1}{2})}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \\ =\ &(\frac{3}{2})^4 x^{\frac{3}{2}} + \int_{1-i\infty}^{1+i\infty} x^s \frac{s^4}{(s-\frac{3}{2})(s-\frac{1}{2})}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \\ =\ &(\frac{3}{2})^4 x^{\frac{3}{2}} - \int_{1-i\infty}^{1+i\infty}\Big(\int_0^\infty u(t-x)t^{-\frac{3}{2}}t^{s-1}dt\Big)x^{\frac{3}{2}}s^4(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i}.\end{aligned}$$

Since $u(t-x)t^{s-\frac{5}{2}}$ is a nonnegative function of t on $\mathbb{R}_+^*$, by Lemma 9,

$$\int_{1-i\infty}^{1+i\infty}\Big(\int_0^\infty u(t-x)t^{-\frac{3}{2}}t^{s-1}dt\Big)x^{\frac{3}{2}}\Big((s-\frac{1}{2})+\frac{1}{2}\Big)^4(s-\frac{1}{2})^{-\rho-3}\frac{ds}{2\pi i} \geq 0,$$

and so we have

$$\int_{\sigma-i\infty}^{\sigma+i\infty} x^s \frac{s^4}{(s-1)^2}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \leq (\frac{3}{2})^4 x^{\frac{3}{2}}.$$

(b) By (4.3.7) and (a), we have

$$\begin{aligned}&\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s+\frac{1}{2})^2\frac{4\Gamma(s)^2}{\Gamma(s/2)^4}\cdot x^s\frac{s^2}{(s-1)^2}(s-\frac{1}{2})^{-\rho-2}\frac{ds}{2\pi i} \\ \leq\ &\Gamma(\frac{3}{2}+\frac{1}{2})^2\frac{4\Gamma(\frac{3}{2})^2}{\Gamma(\frac{3}{4})^4}\cdot(\frac{3}{2})^2 x^{\frac{3}{2}} \\ \leq\ &4x^{\frac{3}{2}}.\end{aligned}$$

□

Proposition 20.

$$J_3 \le \frac{1}{\pi^2} h(d)^2 (\log \varepsilon_d)^2 \frac{1}{\sqrt{d}} M_d^2.$$

Proof. By (4.3.12) and Lemma 19,

$$\begin{aligned} J_3 &= 4\pi^2 h(d)^2 (\log \varepsilon_d)^2 \sqrt{\frac{M_d}{4\pi^2 d}} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma(s+\frac{1}{2})^2 \frac{4\Gamma(s)^2}{\Gamma(s/2)^4} \\ &\quad \cdot (\frac{M_d}{4\pi^2})^s (\frac{s}{s-1})^2 (s-\frac{1}{2})^{-\rho-2} \frac{ds}{2\pi i} \\ &\le 16\pi^2 h(d)^2 (\log \varepsilon_d)^2 \frac{1}{\sqrt{d}} (\frac{M_d}{4\pi^2})^2. \end{aligned}$$

□

4.7. Now we can prove Proposition 4.

Proof of Proposition 4. Let

$$J_0 = \frac{2^\rho}{\pi^2} h(d) \log \varepsilon_d M_d^{3/2} \prod_{i=1}^{\rho} \left(\frac{q_i+1}{q_i-1}\right) \prod_{p \in P(d)} (1+\frac{1}{p}).$$

Assume $h(d)\log\varepsilon_d \le \frac{L(\mathrm{Sym}_i^2 E, 2) \prod_{p|N} \frac{p}{p-1}}{2^{\rho+1}\rho! \sqrt{M}} (\log d)^\rho$. By (4.1.2), we have

$$\frac{(\rho+1) L(\mathrm{Sym}_i^2 E, 2) \prod_{p|N} \frac{p}{p-1}}{2^{2\rho+2}\sqrt{M}} (\log d)^\rho > (m-1)^{\rho+1}$$

and so we have

$$\begin{aligned} \log U = \frac{1}{m} \log \frac{\sqrt{d}}{4} &> \frac{\frac{1}{2}(\log d - \log 16)}{1 + \frac{1}{4}\left(\frac{(\rho+1)L(\mathrm{Sym}_i^2 E,2)\prod_{p|N}\frac{p}{p-1}}{\sqrt{M}}\right)^{\frac{1}{\rho+1}} (\log d)^{\frac{\rho}{\rho+1}}} \\ &> 2\left(\frac{\sqrt{M}\log d}{2(\rho+1)L(\mathrm{Sym}_i^2 E, 2)\prod_{p|N}\frac{p}{p-1}}\right)^{\frac{1}{\rho+1}} \qquad (4.7.1) \end{aligned}$$

for sufficiently large d, which depends on N, ρ, and $L(\mathrm{Sym}_i^2 E, 2)$. By Proposition 16, we have

$$\begin{aligned} &\log\left(\frac{J_1}{J_0}\right) \\ \le\ &\log \frac{12\binom{\rho+3}{3} L(\mathrm{Sym}_i^2 E, 2) \prod_{p|N}\frac{p}{p-1}}{\pi} - \frac{\log M + \log M_d}{2} + \rho \log\log d \\ &+(\rho+3)\log\left(\log\left(\frac{Md}{16\pi^2 U}\right) + 4\right) - \left(\frac{\sqrt{M}\log d}{2(\rho+1)L(\mathrm{Sym}_i^2 E, 2)\prod_{p|N}\frac{p}{p-1}}\right)^{\frac{1}{\rho+1}} \\ =:\ & c_6. \qquad (4.7.2) \end{aligned}$$

By Proposition 18, for $d \geq \exp(6\rho^{\rho+1})$ we have

$$\begin{aligned} & \log\left(\frac{J_2}{J_0} - 1\right) \\ \leq\ & \log \frac{L(\mathrm{Sym}_i^2 E, 2) \prod_{p|N} \frac{p}{p-1}}{2^{\rho+1}\rho!} - \frac{\log M}{2} + \rho \log\log d \\ & -2\left(\frac{\sqrt{M}\log d}{2(\rho+1)L(\mathrm{Sym}_i^2 E, 2)\prod_{p|N}\frac{p}{p-1}}\right)^{\frac{1}{\rho+1}} \\ =:\ & c_7. \end{aligned}$$

By Proposition 20, we have

$$\begin{aligned} \log\left(\frac{J_3}{J_0}\right) &\leq \log \frac{L(\mathrm{Sym}_i^2 E, 2) \prod_{p|N} \frac{p}{p-1}}{2^{2\rho+1}\rho!} + \rho \log\log d - \frac{\log d}{2} \\ &=: c_8. \end{aligned}$$

Since c_6, c_7, and c_8 are decreasing with respect to sufficiently large d with $d \geq \exp(6\rho^{\rho+1})$, we can take their maximum values. Thus if we take a constant $c_3 \geq \exp(6\rho^{\rho+1})$ satisfying (4.7.1), and a constant

$$c_4 > (1 + e^{c_6} + e^{c_7} + e^{c_8}),$$

Proposition 4 follows from (4.3.9). □

5. Proof of Proposition 5

5.1. **Lower bound of $|J(U)|$.** Applying the residue theorem to (3.4.2), that is,

$$J(U) = \int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-1}\left(\frac{M_d}{4\pi^2}\right)^s \Gamma^2(s)\Psi(s)G(U,s)(s-1)^{-\rho-2}\frac{ds}{2\pi i},$$

we have

$$\begin{aligned} J(U) \quad = \quad & \left[\text{the residue of } d^{s-1}\left(\frac{M_d}{4\pi^2}\right)^s \Gamma^2(s)\Psi(s)G(U,s)(s-1)^{-\rho-2} \ \text{ at } \ s=1\right] \\ & + J_{-1}(U), \end{aligned}$$

where

$$J_{-1}(U) = \int_{\mathcal{C}} d^{s-1}\left(\frac{M_d}{4\pi^2}\right)^s \Gamma^2(s)\Psi(s)G(U,s)(s-1)^{-\rho-2}\frac{ds}{2\pi i}$$

with the directed path $\mathcal{C} : 1-i\infty \rightarrow 1-i\eta' \rightarrow 1-\eta-i\eta' \rightarrow 1-\eta+i\eta' \rightarrow 1+i\eta' \rightarrow 1+i\infty$. η and η' are to be determined later. Also we have

$$|J_{-1}(U)| \leq \sup_{s\in\mathcal{C}} |G(U,s)| \cdot \sum_{r=1}^{5} I_r(s)ds, \tag{5.1.1}$$

where

$$I_1 = \int_{1+i\eta'}^{1+i\infty}, \; I_2 = \int_{1-i\infty}^{1-i\eta'}, \; I_3 = \int_{1-\eta+i\eta'}^{1+i\eta'}, \; I_4 = \int_{1-\eta-i\eta'}^{1-i\eta'}, \; I_5 = \int_{1-\eta-i\eta'}^{1-\eta+i\eta'}$$

of which the integrands are

$$|d^{s-1}(\frac{M_d}{4\pi^2})^s\Gamma^2(s)\Psi(s)(s-1)^{-\rho-2}|\frac{|ds|}{2\pi}.$$

We note that $F_d(s) = \left(\frac{M_d}{4\pi^2}\right)^s\Gamma^2(s)\Psi(s)(s-1)^{-1}$ (cf. (3.3.2)). Since $\Psi(s)$ has a zero at $s=1$, $F_d(s)$ is a holomorphic function. Then we have

$$J(U) = \int_{\sigma-i\infty}^{\sigma+i\infty} d^{s-1}F_d(s)G(U,s)(s-1)^{-\rho-1}\frac{ds}{2\pi i},$$

and

$$\begin{aligned}
& J(U) - J_{-1}(U) \\
= \; & \frac{1}{\rho!}(\frac{d}{ds})^\rho\Big[d^{s-1}F_d(s)G(U,s)\Big]_{s=1} \\
= \; & \frac{1}{\rho!}\sum_{i=0}^{\rho}\left\{\binom{\rho}{i}F_d^{(\rho-i)}(1)\cdot\sum_{j=0}^{i}\binom{i}{j}(\log d)^{i-j}G^{(j)}(U,1)\right\} \\
= \; & \frac{1}{\rho!}(\log d)^\rho F_d(1)G(U,1) \\
\cdot \; & \sum_{i=0}^{\rho}\left\{\binom{\rho}{i}(\log d)^{-(\rho-i)}\cdot\frac{F_d^{(\rho-i)}(1)}{F_d(1)}\cdot\sum_{j=0}^{i}\binom{i}{j}(\log d)^{-j}\frac{G^{(j)}(U,1)}{G(U,1)}\right\}.
\end{aligned}$$

Let

$$H = \sum_{i=0}^{\rho}\left\{\binom{\rho}{i}(\log d)^{-(\rho-i)}\cdot\frac{F_d^{(\rho-i)}(1)}{F_d(1)}\cdot\sum_{j=0}^{i}\binom{i}{j}(\log d)^{-j}\frac{G^{(j)}(U,1)}{G(U,1)}\right\},$$

$$H_{(\rho)} = 1 - \sum_{j=1}^{\rho}\binom{\rho}{j}(\log d)^{-j}\cdot|\frac{G^{(j)}(U,1)}{G(U,1)}|,$$

and for $0 \leq i \leq \rho-1$

$$H_{(i)} = \binom{\rho}{i}(\log d)^{-(\rho-i)}\cdot|\frac{F_d^{(\rho-i)}(1)}{F_d(1)}|\cdot\sum_{j=0}^{i}\binom{i}{j}(\log d)^{-j}\cdot|\frac{G^{(j)}(U,1)}{G(U,1)}|.$$

Then we have

$$|H| \geq H_{(\rho)} - \sum_{i=0}^{\rho-1} H_{(i)}.$$

Let

$$H_{-1} = \left| \frac{J_{-1}(U)}{\frac{1}{\rho!}(\log d)^\rho F_d(1)G(U,1)} \right|.$$

Then we have

$$\begin{aligned} |J(U)| &\geq |J(U) - J_{-1}(U)| - |J_{-1}(U)| \\ &\geq \frac{1}{\rho!}(\log d)^\rho |F_d(1)G(U,1)| \cdot (H_{(\rho)} - \sum_{i=0}^{\rho-1} H_{(i)} - H_{-1}). \qquad (5.1.2) \end{aligned}$$

5.2. **Estimation of $H_{(\rho)}$ and $\sum_{i=0}^{\rho-1} H_{(i)}$.** By Lemma 7, we have

$$\begin{aligned} G(U,s) &= \prod_{\substack{p\in P(d)\\ p<U}} G_p(s) \prod_{\substack{q\in\{q_1,\cdots,q_\rho\}\\ q<U}} G_q(s) \\ &= \prod_{\substack{p\in P(d)\\ p<U}} (1+\alpha_p p^{-s})(1+\beta_p p^{-s}) \prod_{\substack{q\in\{q_1,\cdots,q_\rho\}\\ q<U}} \frac{(1+\alpha_q q^{-s})(1+\beta_q q^{-s})}{(1-\alpha_q q^{-s})(1-\beta_q q^{-s})}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{G(U,s)}{G(U,1)} &= \prod_{\substack{p\in P(d)\\ p<U}} \frac{(1+\alpha_p p^{-s})(1+\beta_p p^{-s})}{(1+\alpha_p p^{-1})(1+\beta_p p^{-1})} \\ &\quad \cdot \prod_{\substack{q\in\{q_1,\cdots,q_\rho\}\\ q<U}} \frac{(1+\alpha_q q^{-s})(1+\beta_q q^{-s})}{(1+\alpha_q q^{-1})(1+\beta_q q^{-1})} \frac{(1-\alpha_q q^{-1})(1-\beta_q q^{-1})}{(1-\alpha_q q^{-s})(1-\beta_q q^{-s})}. \end{aligned}$$

For $k \geq 1$, we have

$$\begin{aligned} \frac{G_p^{(k)}(U,s)}{G_p(U,s)} &= (-\log p)^k \frac{\alpha_p p^{-s} + \beta_p p^{-s} + \sum_{i=0}^k \binom{k}{i} \alpha_p \beta_p p^{-2s}}{(1+\alpha_p p^{-s})(1+\beta_p p^{-s})} \\ &= (-\log p)^k \frac{1_N(p)\cdot 2^k p^{1-2s} + (\alpha_p+\beta_p)p^{-s}}{1+(\alpha_p+\beta_p)p^{-s} + 1_N(p)\cdot p^{1-2s}} \end{aligned}$$

and

$$\frac{G_p^{(k)}(U,1)}{G_p(U,1)} = (-\log p)^k \frac{1_N(p)\cdot 2^k + \alpha_p + \beta_p}{p + \alpha_p + \beta_p + 1_N(p)}. \qquad (5.2.1)$$

Lemma 21.

(a) *For* $-2 \le t \le 2$, $k \ge 1$ *and any prime* p, *we have*

$$(\log p)^k \cdot |\frac{2^k + t\sqrt{p}}{p+1+t\sqrt{p}}| \le (2k)^k.$$

(b) *For* $t \in \{-1, 0, 1\}$, $k \ge 1$ *and any prime* p, *we have*

$$(\log p)^k \cdot |\frac{t}{p+t}| \le (2k)^k.$$

Proof. (a) We note that $\frac{2^k+t\sqrt{p}}{p+1+t\sqrt{p}} = 1 + \frac{2^k-p-1}{p+1+t\sqrt{p}}$ attains its maximum value and minimum value at $t = \pm 2$. For any $k \ge 1$ and any prime p,

$$\frac{2^k + 2\sqrt{p}}{(\sqrt{p}+1)^2} \le \frac{2^k}{\sqrt{p}}.$$

For any $k \ge 2$ and any prime p,

$$\frac{2\sqrt{p} - 2^k}{(\sqrt{p}-1)^2} \le \frac{2^k}{\sqrt{p}}$$

and for $k \ge 2$ and any prime $p \ge 11$,

$$\frac{2^k - 2\sqrt{p}}{(\sqrt{p}-1)^2} \le \frac{2^k}{\sqrt{p}}.$$

Let $l(x) = (\log x)^k \cdot \frac{2^k}{\sqrt{x}}$. Since $l^{(1)}(x) = \frac{(2\log x)^{k-1}}{x\sqrt{x}}(2k - \log x)$, we have for any $x > 1$,

$$l(x) \le l(e^{2k}) = \Big(\frac{4k}{e}\Big)^k < (2k)^k.$$

If $k = 1$ and $t = -2$, we have for any prime p

$$\log p \cdot \frac{|2 - 2\sqrt{p}|}{p+1-2\sqrt{p}} = \log p \cdot \frac{2}{\sqrt{p}-1} \le 2.$$

Therefore it suffices to show that for $k \ge 2$ and $p \in \{2, 3, 5, 7\}$,

$$(\log p)^k \cdot \frac{2^k - 2\sqrt{p}}{(\sqrt{p}-1)^2} \le (2k)^k. \tag{5.2.2}$$

For $k \ge 5$ and $p \in \{2, 3, 5, 7\}$,

$$(\log p)^k \cdot \frac{2^k - 2\sqrt{p}}{(\sqrt{p}-1)^2} \le (\log 7)^k \cdot \frac{2^k}{(\sqrt{2}-1)^2} \le (2k)^k.$$

By simple calculation, (5.2.2) holds for $k \in \{2, 3, 4\}$ and $p \in \{2, 3, 5, 7\}$.
(b) Let $l(x) = 2(\log x)^k \cdot \frac{1}{x}$. Since $l^{(1)}(x) = \frac{2(\log x)^{k-1}}{x^2}(k - \log x)$, we have for any $x > 1$,

$$l(x) \le l(e^k) = 2\Big(\frac{k}{e}\Big)^k < (2k)^k.$$

Therefore we have for $t \in \{-1, 0, 1\}$ and any prime p,

$$(\log p)^k \cdot |\frac{t}{p+t}| \le l(p) < (2k)^k.$$

□

Proposition 22. *Recall that T is the number of prime divisors of d. Let $V = \max_{\substack{F_d \in \mathcal{F} \\ 1 \le k \le \rho}} \left\{ \frac{|F_d^{(k)}(1)|}{|F_d(1)|} \right\}$. Then we have*

(a) $H_{(\rho)} \ge 2 - \exp\left(\frac{2(T+2\rho)}{\log d}\right)$.

(b) $\sum_{i=0}^{\rho-1} H_{(i)} \le \frac{\rho V}{\log d} \cdot \exp\left(\frac{4(T+2\rho)}{\log d}\right)$.

Proof. (a) By (5.2.1) and Lemma 21, for $j \ge 1$,

$$\begin{aligned} & (\log d)^{-j} \cdot |\frac{G^{(j)}(U,1)}{G(U,1)}| \\ \le\ & (\log d)^{-j} \sum_{k=1}^{j} \binom{T-1+2\rho}{k} \binom{j-1}{k-1} |\frac{G_{p_1}^{(j_1)}(U,1) \cdots G_{p_k}^{(j_k)}(U,1)}{G_{p_1}(U,1) \cdots G_{p_k}(U,1)}| \\ \le\ & (\log d)^{-j} \sum_{k=1}^{j} (T-1+2\rho)^k \binom{j-1}{k-1} (2j)^j \\ \le\ & \Big(\frac{2(T+2\rho)j}{\log d}\Big)^j. \end{aligned}$$

Thus we have

$$\begin{aligned} H_{(\rho)} &\ge 1 - \sum_{j=1}^{\rho} \binom{\rho}{j} \Big(\frac{2(T+2\rho)j}{\log d}\Big)^j \\ &\ge 2 - \Big(1 + \frac{2(T+2\rho)\rho}{\log d}\Big)^\rho \\ &\ge 2 - \exp\Big(\frac{2(T+2\rho)}{\log d}\Big). \end{aligned}$$

(b) For $0 \le i \le \rho - 1$, we have

$$\begin{aligned} \sum_{j=0}^{i} \binom{i}{j} (\log d)^{-j} \cdot |\frac{G^{(j)}(U,1)}{G(U,1)}| &\le 1 + \sum_{j=1}^{i} \binom{i}{j} \Big(\frac{2(T+2\rho)j}{\log d}\Big)^j \\ &\le \Big(1 + \frac{2(T+2\rho)i}{\log d}\Big)^i. \end{aligned}$$

Thus

$$\sum_{i=0}^{\rho-1} H_{(i)} \le \sum_{i=0}^{\rho-1} \left\{ \binom{\rho}{i} (\log d)^{-(\rho-i)} \cdot |\frac{F_d^{(\rho-i)}(1)}{F_d(1)}| \cdot \Big(1 + \frac{2(T+2\rho)i}{\log d}\Big)^i \right\}$$

$$\begin{aligned}
&\le \frac{V}{(\log d)^\rho}\sum_{i=0}^{\rho-1}\left\{\binom{\rho}{i}\cdot\Big(\log d+2(T+2\rho)(\rho-1)\Big)^i\right\}\\
&\le \frac{\rho V}{(\log d)^\rho}\cdot\Big(\log d+2(T+2\rho)\rho\Big)^{\rho-1}\\
&\le \frac{\rho V}{\log d}\cdot\Big(1+\frac{2(T+2\rho)\rho}{\log d}\Big)^{\rho-1}\\
&\le \frac{\rho V}{\log d}\cdot\exp\Big(\frac{4(T+2\rho)}{\log d}\Big).
\end{aligned}$$

□

5.3. **Estimation of H_{-1}.**

Lemma 23. *For $\frac{1}{2}\le\sigma=\mathrm{Re(s)}\le\frac{5}{2}$, we have*

$$\left|\frac{L(\mathrm{Sym}_p^2E,s)}{\zeta(s-1)}\right|\le\frac{7B^2}{4\pi^{3/2}}\left|s(s+1)\right|,$$

where B is the symmetric conductor of E.

Proof. We have the following functional equation.

$$(s-1)\pi^{-s/2}\Gamma\Big(\frac{s+2}{2}\Big)\zeta(s)=-s\pi^{(s-1)/2}\Gamma\Big(\frac{3-s}{2}\Big)\zeta(1-s).\tag{5.3.1}$$

By (3.3.1), (5.3.1) and the duplication formula for the Gamma function, we have

$$\frac{\big(\frac{B}{\pi}\big)^s\Gamma\big(\frac{s}{2}\big)^2}{(s-2)}\frac{L(\mathrm{Sym}_p^2E,s)}{\zeta(s-1)}=\frac{\sqrt{\pi}\big(\frac{B}{\pi}\big)^{3-s}\Gamma\big(\frac{3-s}{2}\big)^2}{-(s-1)}\frac{L(\mathrm{Sym}_p^2E,3-s)}{\zeta(2-s)}.\tag{5.3.2}$$

By the Euler product of $L(\mathrm{Sym}_p^2E,s)/\zeta(s-1)$, we have

$$\left|\frac{L(\mathrm{Sym}_p^2E,\frac{5}{2}-it)}{\zeta(\frac{3}{2}-it)}\right|\le\zeta(\tfrac{3}{2})^2<7.$$

By (5.3.2), we have

$$\begin{aligned}
\left|\frac{L(\mathrm{Sym}_p^2E,\frac{1}{2}+it)}{\zeta(-\frac{1}{2}+it)}\right| &= \frac{B^2}{\pi^{3/2}}\left|\frac{\Gamma(\frac{5}{4}-i\frac{t}{2})^2}{\Gamma(\frac{1}{4}+i\frac{t}{2})^2}\right|\cdot\left|\frac{-\frac{3}{2}+it}{\frac{1}{2}-it}\right|\cdot\left|\frac{L(\mathrm{Sym}_p^2E,\frac{5}{2}-it)}{\zeta(\frac{3}{2}-it)}\right|\\
&< \frac{7B^2}{4\pi^{3/2}}\left|\frac{1}{2}+it\right|\cdot\left|\frac{3}{2}+it\right|.
\end{aligned}$$

Hence, the function

$$\frac{L(\mathrm{Sym}_p^2E,s)}{\zeta(s-1)}s^{-1}(s+1)^{-1}$$

is bounded by

$$\frac{7B^2}{4\pi^{3/2}}$$

on the lines $\sigma = \frac{1}{2}$ and $\sigma = \frac{5}{2}$. By Lindelöf theorem (cf. [p. 15, HR]), this implies that

$$\left|\frac{L(\mathrm{Sym}_p^2 E, s)}{\zeta(s-1)}\right| \le \frac{7B^2}{4\pi^{3/2}} |s(s+1)| \quad (\tfrac{1}{2} \le \sigma \le \tfrac{5}{2}).$$

□

Let E' be a quadratic twist of E by a square free integer D such that the conductor N' of E' satisfies $\mathrm{ord}_p(N') \le \mathrm{ord}_p(N'')$ for all primes p and all quadratic twists E'' of E with conductor N''. Then we have

$$L(\mathrm{Sym}_p^2 E, s) = L(\mathrm{Sym}_p^2 E', s)$$

and the symmetric square conductor B' of E' is equal to B.

Let

$$\begin{aligned} S_1 &= \{p,\ \text{prime} : p \mid D,\ p \nmid N'\}, \\ S_2 &= \{p,\ \text{prime} : p \mid D,\ p \,||\, N'\}. \end{aligned}$$

Then we see that for odd prime p, if $p \in S_1$ or $p \in S_2$, then $\mathrm{ord}_p(N) = 2$ and if $p^2 \mid N'$, then $\mathrm{ord}_p(N) = \mathrm{ord}_p(N')$. Also we can write

$$\begin{aligned} N &= M D_1^2 D_2^2 2^{\lambda_E}, \\ N' &= M D_2 2^{\lambda_{E'}}, \end{aligned}$$

where M is odd, D_1 is the product of the odd primes in S_1, D_2 is the product of the odd primes in S_2, and $\lambda_E = \mathrm{ord}_2(N) \ge \lambda_{E'} = \mathrm{ord}_2(N')$. From the definition of the imprimitive symmetric square L-functions,

$$\begin{aligned} L(\mathrm{Sym}_i^2 E, s) &= L(\mathrm{Sym}_i^2 E', s) \\ &\quad \times \prod_{p \in S_1} (1 - \alpha_p^2(E')p^{-s})(1 - p^{1-s})(1 - \beta_p^2(E')p^{-s}) \\ &\quad \times \prod_{p \in S_2} (1 - p^{-s}). \end{aligned} \tag{5.3.3}$$

Let $B = B' = \prod_p p^{\delta_p}$. Then we have

$$\begin{cases} \text{for } p \nmid N', & \delta_p = 0,\ U_p(E', s) = 1, \\ \text{for } p \parallel N', & \delta_p = 1,\ U_p(E', s) = 1, \\ \text{for } p^2 \mid N', & \delta_p \ge 1, \text{ there are three possibilities for} \\ & U_p(E', s) : 1,\ (1 \pm p^{1-s})^{-1} \end{cases} \tag{5.3.4}$$

(cf. [CS], [Del] and [Wa]).

Lemma 24. *For* $\frac{3}{4} \le \sigma = \mathrm{Re(s)} \le \frac{5}{4}$*,*

$$|\Psi(s)| \le \frac{7}{4\pi^{3/2}} B^2 R(N) |2s(2s+1)|$$

where B *is the symmetric conductor of* E*,* N *is the conductor of* E *and*

$$R(N) = \prod_{p \| N} \frac{\sqrt{p}}{\sqrt{p}-1} \prod_{p^2|N} \Big(\frac{\sqrt{p}+1}{\sqrt{p}}\Big)^2 \frac{\sqrt{p}+1}{\sqrt{p}-1}.$$

Proof. By (5.3.3), we have

$$\begin{aligned}
\Psi(s) &= L(f,s)L(f\otimes\lambda,s) \\
&= \frac{L(\mathrm{Sym}_i^2 E, 2s)}{\zeta_N(2s-1)} \\
&= \frac{L(\mathrm{Sym}_i^2 E', 2s)}{\zeta(2s-1)} \times \prod_{p|N} (1-p^{1-2s})^{-1} \\
&\quad \times \prod_{p\in S_1} (1-\alpha_p^2(E')p^{-2s})(1-p^{1-2s})(1-\beta_p^2(E')p^{-2s}) \times \prod_{p\in S_2} (1-p^{-2s}) \\
&= \frac{L(\mathrm{Sym}_p^2 E', 2s)}{\zeta(2s-1)} \times \prod_{p|N} (1-p^{1-2s})^{-1} \times \prod_{p^2|N'} U_p(E',2s)^{-1} \\
&\quad \times \prod_{p\in S_1} (1-\alpha_p^2(E')p^{-2s})(1-p^{1-2s})(1-\beta_p^2(E')p^{-2s}) \times \prod_{p\in S_2} (1-p^{-2s}).
\end{aligned}$$

By (5.3.4), we have for $2\sigma \ge 3/2$,

$$\begin{aligned}
&\left|\prod_{p|N} (1-p^{1-2s})^{-1}\right| \times \left|\prod_{p^2|N'} U_p(E',2s)^{-1}\right| \\
&\times \left|\prod_{p\in S_1} \Big\{(1-\alpha_p^2(E')p^{-2s})(1-p^{1-2s})(1-\beta_p^2(E')p^{-2s})\Big\}\right| \times \left|\prod_{p\in S_2} (1-p^{-2s})\right| \\
\le\ & \prod_{p|N} \frac{1}{1-|p^{1-2s}|} \prod_{p^2|N} (1+|p^{1-2s}|)^3 \\
\le\ & \prod_{p\|N} \frac{1}{1-|p^{1-2s}|} \prod_{p^2|N} (1+|p^{1-2s}|)^2 \frac{1+|p^{1-2s}|}{1-|p^{1-2s}|} \\
\le\ & \prod_{p\|N} \frac{\sqrt{p}}{\sqrt{p}-1} \prod_{p^2|N} \Big(\frac{\sqrt{p}+1}{\sqrt{p}}\Big)^2 \frac{\sqrt{p}+1}{\sqrt{p}-1}.
\end{aligned}$$

Thus Lemma 24 follows from Lemma 23. □

Lemma 25.

$$\frac{\sup_{s\in\mathcal{C}}|G(U,s)|}{|G(U,1)|}\le 108\cdot\frac{2^{T+3\rho}}{\theta(d)}\cdot\prod_{q\in\{q_1,\cdots,q_\rho\}}\frac{(q+1)(q+\lfloor 2\sqrt{q}\rfloor+1)}{(q-1)(q-\lfloor 2\sqrt{q}\rfloor+1)}.$$

Proof. For $0\le\epsilon\le\frac{1}{4}$ and $s=1-\epsilon+it$, let

$$D_p(\epsilon)=\frac{1+\frac{\alpha_p+\beta_p}{p^{1-\epsilon+it}}+\frac{1}{p^{1-2\epsilon+i2t}}}{1+\frac{\alpha_p+\beta_p}{p}+\frac{1}{p}}=\frac{p+(\alpha_p+\beta_p)p^{\epsilon-it}+p^{2\epsilon-i2t}}{p+(\alpha_p+\beta_p)+1},$$

and

$$D_q(\epsilon)=\frac{q+(\alpha_q+\beta_q)q^{\epsilon-it}+q^{2\epsilon-i2t}}{q+(\alpha_q+\beta_q)+1}\cdot\frac{q-(\alpha_q+\beta_q)+1}{q-(\alpha_q+\beta_q)q^{\epsilon-it}+q^{2\epsilon-i2t}}.$$

Since $-2\sqrt{p}\le\alpha_p+\beta_p\le 2\sqrt{p}$ and $\alpha_p+\beta_p\in\mathbb{Z}$ for all primes p, we have

$$\begin{aligned}&\frac{\sup_{s\in\mathcal{C}}|G(U,s)|}{|G(U,1)|}\\ &=\prod_{\substack{p\in P(d)\\ p<U}}|D_p(\epsilon)|\cdot\prod_{\substack{q\in\{q_1,\cdots,q_\rho\}\\ q<U}}|D_q(\epsilon)|\\ &\le\prod_{p\in P(d)}\frac{(\sqrt{p}+\sqrt[4]{p})^2}{p-\lfloor 2\sqrt{p}\rfloor+1}\cdot\prod_{q\in\{q_1,\cdots,q_\rho\}}\left(\frac{\sqrt{q}+\sqrt[4]{q}}{\sqrt{q}-\sqrt[4]{q}}\right)^2\frac{q+\lfloor 2\sqrt{q}\rfloor+1}{q-\lfloor 2\sqrt{q}\rfloor+1}.\end{aligned}$$

Since $(\sqrt{p}+\sqrt[4]{p})^2<2(p+1)$ for $p\ge 29$, we have

$$\begin{aligned}\prod_{p\in P(d)}(\sqrt{p}+\sqrt[4]{p})^2&\le\prod_{\substack{2\le n<29\\ \text{prime } n}}\frac{(\sqrt{n}+\sqrt[4]{n})^2}{2(n+1)}\cdot 2^{T-1}\prod_{p\in P(d)}(p+1)\\ &\le 1.2\cdot 2^T\prod_{p\in P(d)}(p+1).\end{aligned}$$

Since $\left(\frac{x+1}{x-1}\right)^2<\frac{8(x^4+1)}{x^4-1}$ for $x\ge 2$, we have

$$\begin{aligned}\prod_{q\in\{q_1,\cdots,q_\rho\}}\left(\frac{\sqrt[4]{q}+1}{\sqrt[4]{q}-1}\right)^2&\le\prod_{\substack{2\le n<16\\ \text{prime } n}}\frac{(n-1)(\sqrt[4]{n}+1)^2}{8(n+1)(\sqrt[4]{n}-1)^2}\cdot 8^\rho\prod_{q\in\{q_1,\cdots,q_\rho\}}\frac{q+1}{q-1}\\ &\le 90\cdot 8^\rho\prod_{q\in\{q_1,\cdots,q_\rho\}}\frac{q+1}{q-1}.\end{aligned}$$

Therefore

$$\frac{\sup_{s\in\mathcal{C}}|G(U,s)|}{|G(U,1)|}\le 108\cdot\frac{2^{T+3\rho}}{\theta(d)}\cdot\prod_{q\in\{q_1,\cdots,q_\rho\}}\frac{(q+1)(q+\lfloor 2\sqrt{q}\rfloor+1)}{(q-1)(q-\lfloor 2\sqrt{q}\rfloor+1)}.$$

□

Proposition 26. *We put* $\eta = \frac{1}{4}$ *and* $\eta' = \frac{1}{4}(M_d d)^{\frac{1}{4(\rho+2)}}$ *into* (5.1.1)*. Then we have*

$$H_{-1} \le \frac{0.08 \cdot 4^{\rho+2} \cdot M_d^{\frac{3}{4}} R(N) B^2}{\frac{1}{\rho!} d^{\frac{1}{4}} (\log d)^{\rho} F_d(1)} \cdot 108 \cdot \frac{2^{T+3\rho}}{\theta(d)} \cdot \prod_{q \in \{q_1, \cdots, q_\rho\}} \frac{(q+1)(q + \lfloor 2\sqrt{q} \rfloor + 1)}{(q-1)(q - \lfloor 2\sqrt{q} \rfloor + 1)},$$

where

$$R(N) = \prod_{p \| N} \frac{\sqrt{p}}{\sqrt{p}-1} \prod_{p^2 | N} \Big(\frac{\sqrt{p}+1}{\sqrt{p}}\Big)^2 \frac{\sqrt{p}+1}{\sqrt{p}-1}.$$

Proof. We note that for $\sigma > 0$,

$$|\Gamma(s)| \le \sqrt{2\pi} \exp\big(\tfrac{1}{12\sigma}\big) |s|^{\sigma - \frac{1}{2}} \begin{cases} \exp(-\sigma) & \text{if } |\frac{\sigma}{t}| \ge \frac{\pi}{2}, \\ \exp(-\frac{\pi}{2}|t|) & \text{if } |\frac{\sigma}{t}| \le \frac{\pi}{2} \end{cases}$$

(cf. [(46), Go]) and the upper bound of $|\Psi(s)|$ is given in Lemma 24.

Firstly, we consider the integral I_1. For $s = 1 + iy$, $\eta' \le y < \infty$,

$$\begin{aligned} I_1 &= \int_{1+i\eta'}^{1+i\infty} \Big| d^{s-1} \Big(\frac{M_d}{4\pi^2}\Big)^s \Gamma^2(s) \Psi(s) (s-1)^{-\rho-2} \Big| \frac{|ds|}{2\pi} \\ &\le \frac{7 M_d R(N) B^2}{4\pi^2 \cdot 4\pi^{3/2}} \int_{\frac{2}{\pi}}^{\infty} e^{1/6} |1+iy| e^{-\pi y} \cdot |(2+i2y)(3+i2y)| \cdot \eta'^{-\rho-2} dy \\ &\le 0.007 \cdot \Big(\frac{1}{\eta'}\Big)^{\rho+2} \cdot M_d R(N) B^2. \end{aligned}$$

Similarly we have

$$I_2 \le 0.007 \cdot \Big(\frac{1}{\eta'}\Big)^{\rho+2} \cdot M_d R(N) B^2.$$

Secondly, we consider the integral I_3. For $s = x + i\eta'$, $\frac{3}{4} \le x < 1$,

$$\begin{aligned} I_3 &= \int_{\frac{3}{4}+i\eta'}^{1+i\eta'} \Big| d^{s-1} \big(\frac{M_d}{4\pi^2}\big)^s \Gamma^2(s) \Psi(s) (s-1)^{-\rho-2} \Big| \frac{|ds|}{2\pi} \\ &\le \frac{7 M_d R(N) B^2}{4\pi^2 \cdot 4\pi^{3/2}} \int_{\frac{3}{4}}^{1} e^{2/9} |x + i\eta'| e^{-\pi\eta'} \cdot |(2x + i2\eta')(2x+1+i2\eta')| \\ &\quad \cdot \eta'^{-\rho-2} dx \\ &\le \frac{7 M_d R(N) B^2}{4\pi^2 \cdot 4\pi^{3/2}} \int_{\frac{3}{4}}^{1} \eta'^{-\rho-2} dx \\ &\le 0.002 \cdot \Big(\frac{1}{\eta'}\Big)^{\rho+2} \cdot M_d R(N) B^2. \end{aligned}$$

Similarly we have

$$I_4 \le 0.002 \cdot \Big(\frac{1}{\eta'}\Big)^{\rho+2} \cdot M_d R(N) B^2.$$

Finally, we estimate the integral I_5. For $s=\frac{3}{4}+iy,\ -\eta'\le y\le\eta'$,

$$\begin{aligned} I_5 &= \int_{\frac{3}{4}-i\eta'}^{\frac{3}{4}+i\eta'} \left| d^{s-1}\left(\frac{M_d}{4\pi^2}\right)^s\Gamma^2(s)\Psi(s)(s-1)^{-\rho-2}\right| \frac{|ds|}{2\pi} \\ &= 2d^{-\frac{1}{4}}\left(\frac{M_d}{4\pi^2}\right)^{\frac{3}{4}}\left\{\int_0^{\frac{3}{2\pi}} \frac{\left|\Gamma^2(s)\Psi(s)\right|}{|s-1|^{\rho+2}}\frac{|dy|}{2\pi} + \int_{\frac{3}{2\pi}}^{\eta'} \frac{\left|\Gamma^2(s)\Psi(s)\right|}{|s-1|^{\rho+2}}\frac{|dy|}{2\pi}\right\} \\ &=: 2d^{-\frac{1}{4}}\{I_{5,1}+I_{5,2}\}. \end{aligned}$$

Further we have

$$\begin{aligned} I_{5,1} &\le \left(\frac{M_d}{4\pi^2}\right)^{\frac{3}{4}}\frac{7R(N)B^2}{4\pi^{3/2}}\int_0^{\frac{3}{2\pi}} e^{2/9}\left|\tfrac{3}{4}+iy\right|^{1/2}e^{-2\cdot 7/8}\cdot\left|\left(\tfrac{3}{2}+i2y\right)\left(\tfrac{5}{2}+i2y\right)\right| \\ &\quad \cdot 4^{\rho+2}dy \\ &\le 0.022\cdot 4^{\rho+2}\cdot M_d^{\frac{3}{4}}R(N)B^2. \end{aligned}$$

and

$$\begin{aligned} I_{5,2} &\le \left(\frac{M_d}{4\pi^2}\right)^{\frac{3}{4}}\frac{7R(N)B^2}{4\pi^{3/2}}\int_{\frac{3}{2\pi}}^{\eta'} e^{2/9}\left|\tfrac{3}{4}+iy\right|^{1/2}e^{-\pi y}\cdot\left|\left(\tfrac{3}{2}+i2y\right)\left(\tfrac{5}{2}+i2y\right)\right| \\ &\quad \cdot\left|\tfrac{1}{4}+iy\right|^{-\rho-2}dy \\ &\le \left(\frac{M_d}{4\pi^2}\right)^{\frac{3}{4}}\frac{7R(N)B^2}{4\pi^{3/2}}\int_{\frac{3}{2\pi}}^{\infty} e^{2/9}\left(\tfrac{3}{4}\right)^{1/2}e^{-\pi y}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot 4^{\rho+2}dy \\ &\le 0.006\cdot 4^{\rho+2}\cdot M_d^{\frac{3}{4}}R(N)B^2. \end{aligned}$$

Therefore we have

$$I_5 \quad\le\quad 0.056\cdot 4^{\rho+2}\cdot M_d^{\frac{3}{4}}R(N)B^2d^{-\frac{1}{4}}$$

By (5.1.1), we have

$$\begin{aligned} |J_{-1}(U)| &\le \left\{0.018\cdot\left(\frac{1}{\eta'}\right)^{\rho+2}+0.056\cdot 4^{\rho+2}\cdot M_d^{-\frac{1}{4}}d^{-\frac{1}{4}}\right\}\cdot M_dR(N)B^2 \\ &\quad \cdot\sup_{s\in C}|G(U,s)| \\ &\le 0.08\cdot 4^{\rho+2}\cdot d^{-\frac{1}{4}}\cdot M_d^{\frac{3}{4}}R(N)B^2\cdot\sup_{s\in C}|G(U,s)|. \end{aligned}$$

By Lemma 25, we have

$$H_{-1} \quad=\quad \left|\frac{J_{-1}(U)}{\frac{1}{\rho!}(\log d)^\rho F_d(1)G(U,1)}\right|$$

$$\leq \quad \frac{0.08 \cdot 4^{\rho+2} \cdot M_d^{\frac{3}{4}} R(N) B^2}{\frac{1}{\rho!} d^{\frac{1}{4}} (\log d)^{\rho} F_d(1)} \cdot 108 \cdot \frac{2^{T+3\rho}}{\theta(d)} \cdot \prod_{q \in \{q_1, \cdots, q_\rho\}} \frac{(q+1)(q+\lfloor 2\sqrt{q} \rfloor + 1)}{(q-1)(q-\lfloor 2\sqrt{q} \rfloor + 1)}.$$

□

5.4. Now we can prove Proposition 5.

Proof of Proposition 5. Suppose that for $d \in \mathcal{D}(g)$ with

$$d \geq \max_{1 \leq k \leq \rho} \left\{ \exp\left(2^{\rho-1}\rho!\sqrt{N}\right), \exp\left(L(\mathrm{Sym}_i^2 E, 2)\right), \exp\left(2\rho \frac{|F_d^{(k)}(1)|}{|F_d(1)|}\right) \right\},$$

$$h(d) \log \varepsilon_d \leq \frac{L(\mathrm{Sym}_i^2 E, 2)}{12} (\log d)^{\rho} \theta(d) \prod_{i=1}^{\rho} \frac{(q_i - 1)(q_i + 1 - \lfloor 2\sqrt{q_i} \rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i} \rfloor)}.$$

Since $\log \varepsilon_d > \log\left(\frac{\sqrt{d}}{4}\right) \geq \frac{\log d}{3}$ and $2^{T-2} \mid h(d)$, where T is the number of prime divisors of d, we have

$$\begin{aligned} & 2^T \log d \\ \leq\ & 12 h(d) \log \varepsilon_d \\ \leq\ & L(\mathrm{Sym}_i^2 E, 2)(\log d)^{\rho} \theta(d) \prod_{i=1}^{\rho} \frac{(q_i - 1)(q_i + 1 - \lfloor 2\sqrt{q_i} \rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i} \rfloor)}. \end{aligned} \tag{5.4.1}$$

Since $d > \exp(2^{\rho-1}\rho!\sqrt{N}) > \exp(2^{\rho})$, we have

$$\rho < \frac{\log\log d}{\log 2}. \tag{5.4.2}$$

Since $d > \exp\left(L(\mathrm{Sym}_i^2 E, 2)\right)$, by (5.4.1) and (5.4.2), we have

$$\begin{aligned} T \quad & \leq \quad \frac{\log L(\mathrm{Sym}_i^2 E, 2)}{\log 2} + \frac{\rho - 1}{\log 2} \log\log d \\ & \leq \quad \left(\frac{\log\log d}{\log 2}\right)^2. \end{aligned}$$

By Proposition 22, we have

$$H_{(\rho)} \geq 2 - \exp\left(\frac{2(T+2\rho)}{\log d}\right).$$

and

$$\sum_{i=0}^{\rho-1} H_{(i)} \leq \frac{\rho V}{\log d} \cdot \exp\left(\frac{4(T+2\rho)}{\log d}\right).$$

By Proposition 26 and (5.4.1), we have

$$\begin{aligned} H_{-1} \quad \leq \quad & \frac{0.08 \cdot 4^{\rho+2} \cdot M_d^{\frac{3}{4}} R(N) B^2}{\frac{1}{\rho!} d^{\frac{1}{4}} (\log d)^{\rho} F_d(1)} \cdot 108 \cdot \frac{2^{3\rho}}{\theta(d)} \cdot \prod_{i=1}^{\rho} \frac{(q_i+1)(q_i+\lfloor 2\sqrt{q_i}\rfloor+1)}{(q_i-1)(q_i-\lfloor 2\sqrt{q_i}\rfloor+1)} \\ & \cdot L(\mathrm{Sym}_i^2 E, 2)(\log d)^{\rho-1}\theta(d) \prod_{i=1}^{\rho} \frac{(q_i-1)(q_i+1-\lfloor 2\sqrt{q_i}\rfloor)}{(q_i+1)(q_i+1+\lfloor 2\sqrt{q_i}\rfloor)} \\ \leq \quad & \frac{280\pi^2 \cdot 2^{5\rho} \cdot R(N) B^2 \prod_{p|N}(\frac{p-1}{p})}{\sqrt[4]{M_d}} \frac{1}{d^{\frac{1}{4}}(\log d)}. \end{aligned}$$

Thus we have

$$\begin{aligned} & H_{(\rho)} - \sum_{i=0}^{\rho-1} H_{(i)} - H_{-1} \\ \geq \quad & 2 - \exp\Big(\frac{2(T+2\rho)}{\log d}\Big) - \frac{\rho V}{\log d} \cdot \exp\Big(\frac{4(T+2\rho)}{\log d}\Big) \\ & - \frac{280\pi \cdot 2^{5\rho} \cdot B^2 R(N) \prod_{p|N}(\frac{p-1}{p})}{\sqrt[4]{M_d}} \frac{1}{d^{\frac{1}{4}}(\log d)} \\ \geq \quad & 2 - \exp\left(\frac{2}{\log d}\Big(\frac{\log\log d}{\log 2}+1\Big)^2\right) \\ & - \frac{\rho V}{\log d} \cdot \exp\left(\frac{4}{\log d}\Big(\frac{\log\log d}{\log 2}+1\Big)^2\right) \\ & - \frac{280\pi \cdot 2^{5\rho} \cdot B^2 R(N) \prod_{p|N}(\frac{p-1}{p})}{\sqrt[4]{M_d}} \frac{1}{d^{\frac{1}{4}}(\log d)}. \qquad (5.4.3) \end{aligned}$$

Since $d > \max\{\exp(2^{\rho-1}\rho!\sqrt{N}), \exp(L(\mathrm{Sym}_i^2 E, 2)), \exp(2\rho V)\}$, by (5.4.3), we can take $c_5 > 1$ such that

$$H_{(\rho)} - \sum_{i=0}^{\rho-1} H_{(i)} - H_{-1} > \frac{1}{c_5}.$$

For $d > \max\{\exp(2^{\rho-1}\rho!\sqrt{N}), \exp(L(\mathrm{Sym}_i^2 E, 2)), \exp(2\rho V)\}$, by (5.1.2), we have

$$\begin{aligned} |J(U)| \quad \geq \quad & \frac{1}{c_5 \rho!} (\log d)^{\rho} F_d(1) G(U,1) \\ \geq \quad & \frac{F_d(1)}{c_5 \rho!} \cdot \prod_{i=1}^{\rho} \frac{q_i+1-\lfloor 2\sqrt{q_i}\rfloor}{q_i+1+\lfloor 2\sqrt{q_i}\rfloor} \cdot (\log d)^{\rho} \prod_{p\in P(d)} \frac{p+1-\lfloor 2\sqrt{p}\rfloor}{p} \end{aligned}$$

as desired. □

6. Proof of Theorem 3

Consider the elliptic curve $E : y^2+y = x^3+x^2-72x+210$ which has prime conductor $N = 501029$. We remind that the Mordell-Weil group $E(\mathbb{Q})$ has rank 4. Suppose that the conjecture of Birch and Swinnerton-Dyer is true for E, that is, the Hasse-Weil L-function $L_{E/\mathbb{Q}}(s)$ associated to E has a zero of order 4 at $s = 1$. Then the root number of E is equal to 1.

Let Δ be a square free integer such that $\Delta = n^2+r$ with $r \in \{\pm 1, \pm 4\}$ and d be the fundamental discriminant of the real quadratic $\mathbb{Q}(\sqrt{\Delta})$ of narrow Richaud-Degert type. If a prime p splits in $\mathbb{Q}(\sqrt{d})$, then we have

$$h(d) \geq \frac{1}{\log p} \log \frac{\sqrt{d}-2}{2} \tag{6.0.1}$$

(cf. [p. 86, Mo]). Thus if $\chi_d(-N) = \chi_d(N) = 1$, we have

$$h(d) \geq \frac{1}{3\log N} \log d \geq \frac{1}{40} \log d.$$

Now we assume $\chi_d(-N) = \chi_d(N) = -1$ and $E(d)$ be the quadratic twist of E. Then the root number of $E(d)$ is equal to -1. Thus $L_{E/\mathbb{Q}(\sqrt{d})}(s)$ has a zero of order $\geq 4+1$ at $s = 1$. Since $\log \varepsilon_d \leq \log(2\sqrt{d})$, by Theorem 1.2, we have for $d > c_1$,

$$h(d) \geq c_2'(\log d)\theta(d) \geq c_2'(\log d) \prod_{p|d,\, p\neq d} \left(1 - \tfrac{\lfloor 2\sqrt{p} \rfloor}{p+1}\right), \tag{6.0.2}$$

where $c_2' = 2c_2 \frac{\log c_1}{\log c_1 + \log 4}$.

Proof of Theorem 3. Here we calculate the constants c_1 and c_2. Let $E(D)$ be the quadratic twist of E by a square free integer D and $N_{E(D)}$ the conductor of $E(D)$. Then we have

$$\begin{aligned}
&E : y^2 = x^3 + x^2 - 72x + \tfrac{841}{4}, && N = 501029\\
&E(2) : y^2 = x^3 + 2x^2 - 2^2 \cdot 72x + 2^3 \cdot \tfrac{841}{4}, && N_{E(2)} = 2^6 \cdot 501029\\
&E(3) : y^2 = x^3 + 3x^2 - 3^2 \cdot 72x + 3^3 \cdot \tfrac{841}{4}, && N_{E(3)} = 2^4 \cdot 3^2 \cdot 501029\\
&E(6) : y^2 = x^3 + 6x^2 - 6^2 \cdot 72x + 6^3 \cdot \tfrac{841}{4}, && N_{E(6)} = 2^6 \cdot 3^2 \cdot 501029\\
&E(-1) : y^2 = x^3 - x^2 - 72x - \tfrac{841}{4}, && N_{E(-1)} = 2^4 \cdot 501029\\
&E(-2) : y^2 = x^3 - 2x^2 - 2^2 \cdot 72x - 2^3 \cdot \tfrac{841}{4}, && N_{E(-2)} = 2^6 \cdot 501029\\
&E(-3) : y^2 = x^3 - 3x^2 - 3^2 \cdot 72x - 3^3 \cdot \tfrac{841}{4}, && N_{E(-3)} = 3^2 \cdot 501029\\
&E(-6) : y^2 = x^3 - 6x^2 - 6^2 \cdot 72x - 6^3 \cdot \tfrac{841}{4}, && N_{E(-6)} = 2^4 \cdot 3^2 \cdot 501029
\end{aligned}$$

Since any quadratic twits of E is a quadratic twist of one of the above elliptic curves by an integer coprime to 6, we have $n_2 = 0$, $n_3 = 0$ and $M = M_d \in \{501029,\ 501029^{3/2}\}$. Since $(d, 501029) = 1$, we have $M = M_d = N = 501029$ and

$$F_d(s) = L(\mathrm{Sym}_i^2 E, 2s)(\frac{N}{4\pi^2})^s \Gamma(s)^2 \frac{1}{(s-1)\zeta(2s-1)} \frac{1}{1-N^{-2s+1}}.$$

Let $c(E)$ be the Manin's constant of E, vol(E) the volume of a minimal period lattice Λ with $E \simeq \mathbb{C}/\Lambda$ and deg(E) the modular degree of E. These invariants can be calculated by Sage and we have

$$\begin{aligned} L(\mathrm{Sym}_i^2 E, 2) &= \frac{2\pi c(E)^2 \mathrm{vol(E)}\deg(\mathrm{E})}{N} \\ &= 4.12289... \end{aligned}$$

(cf. [p. 490, Wa]). The Laurent expansion of the Riemann zeta function can be written in the form,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n$$

where γ_n are the so-called Stieltjes constants. Then we have

$$(s-1)\zeta(2s-1) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-2)^n \gamma_n}{n!} (s-1)^{n+1}.$$

It is well known that $\Gamma^{(1)}(1) = -\gamma_0$ and $\Gamma^{(2)}(1) = \gamma_0^2 + \frac{\pi^2}{6}$. Thus we have

$$\begin{aligned} \left| \frac{F_d^{(1)}(1)}{F_d(1)} \right| &= \left| 2\frac{L^{(1)}(\mathrm{Sym}_i^2 E, 2)}{L(\mathrm{Sym}_i^2 E, 2)} + \log(\frac{N}{4\pi^2}) + 2\frac{\Gamma^{(1)}(1)}{\Gamma(1)} - 2\gamma_0 - \frac{2\log N}{N-1} \right| \\ &< 2\frac{|L^{(1)}(\mathrm{Sym}_i^2 E, 2)|}{L(\mathrm{Sym}_i^2 E, 2)} + 7.3 \end{aligned}$$

and

$$\begin{aligned} \left| \frac{F_d^{(2)}(1)}{F_d(1)} \right| &= \Big| 4\frac{L^{(2)}(\mathrm{Sym}_i^2 E, 2)}{L(\mathrm{Sym}_i^2 E, 2)} + \Big(\log(\frac{N}{4\pi^2})\Big)^2 + 2\frac{\Gamma(1)\Gamma^{(2)}(1) + \Gamma^{(1)}(1)^2}{\Gamma(1)^2} \\ &\quad + 8(\gamma_0^2 + \gamma_1) + \frac{8(\log N)^2 + 4(N-1)(\log N)^2}{(N-1)^2} \\ &\quad + 4\frac{L^{(1)}(\mathrm{Sym}_i^2 E, 2)}{L(\mathrm{Sym}_i^2 E, 2)} \Big(\log(\frac{N}{4\pi^2}) + 2\frac{\Gamma^{(1)}(1)}{\Gamma(1)} - 2\gamma_0 - \frac{2\log N}{N-1}\Big) \\ &\quad + 2\log(\frac{N}{4\pi^2}) \Big(2\frac{\Gamma^{(1)}(1)}{\Gamma(1)} - 2\gamma_0 - \frac{2\log N}{N-1}\Big) \end{aligned}$$

$$+4\frac{\Gamma^{(1)}(1)}{\Gamma(1)}\Big(-2\gamma_0-\frac{2\log N}{N-1}\Big)+8\gamma_0\frac{\log N}{N-1}\Bigg|$$

$$< \quad 4\frac{|L^{(2)}(\mathrm{Sym}_i^2E,2)|}{L(\mathrm{Sym}_i^2E,2)}+7.3\cdot 4\frac{|L^{(1)}(\mathrm{Sym}_i^2E,2)|}{L(\mathrm{Sym}_i^2E,2)}+55.1.$$

By numerical computations with Magma, we have the following rough upper bounds

$$|L^{(1)}(\mathrm{Sym}_i^2E,2)|\le 20$$

and

$$|L^{(2)}(\mathrm{Sym}_i^2E,2)|\le 1000.$$

We substitute N for both M and M_d in (4.7.2) in the proof of Proposition 4. Since

$$\frac{\partial}{\partial X}\left((2\rho+3)\log X-\left(\frac{\sqrt{N}}{2(\rho+1)L(\mathrm{Sym}_i^2E,2)\prod_{p|N}\frac{p}{p-1}}\right)^{\frac{1}{\rho+1}}X^{\frac{1}{\rho+1}}\right)$$

$$=\frac{1}{X}\left((2\rho+3)-\frac{1}{\rho+1}\left(\frac{\sqrt{N}}{2(\rho+1)L(\mathrm{Sym}_i^2E,2)\prod_{p|N}\frac{p}{p-1}}\right)^{\frac{1}{\rho+1}}X^{\frac{1}{\rho+1}}\right)$$

and

$$X=\log d\ge 5180\ge\frac{2(\rho+1)^{\rho+2}(2\rho+3)^{\rho+1}L(\mathrm{Sym}_i^2E,2)\prod_{p|N}\frac{p}{p-1}}{\sqrt{N}},$$

its primitive function of $X=\log d$ attains the maximum value at $\log d=5180$. Also, $d\ge\exp(5180)$ satisfies (4.7.1). Note that $\exp(2^{\rho-1}\rho!\sqrt{N})=2831.3....$ So we have

$$c_1=\max\Bigg\{\exp(5180),\ \exp(2^{\rho-1}\rho!\sqrt{N}),\ \exp\Big(4\big(\frac{2|L^{(1)}(\mathrm{Sym}_i^2E,2)|}{4.1}+7.3\big)\Big),$$
$$\exp\Big(4\big(\frac{4|L^{(2)}(\mathrm{Sym}_i^2E,2)|}{4.1}+\frac{29.2|L^{(1)}(\mathrm{Sym}_i^2E,2)|}{4.1}+55.1\big)\Big)\Bigg\}$$
$$=\exp(5180).$$

Further, we can take

$$c_6$$
$$\le\ \log\frac{12\binom{\rho+3}{3}L(\mathrm{Sym}_i^2E,2)\prod_{p|N}\frac{p}{p-1}}{\pi\sqrt{N}\sqrt{M_d}}+(2\rho+3)\log\log d$$
$$-\left(\frac{\sqrt{N}\log d}{2(\rho+1)L(\mathrm{Sym}_i^2E,2)\prod_{p|N}\frac{p}{p-1}}\right)^{\frac{1}{\rho+1}}$$

$$
\begin{aligned}
&\leq \log \frac{12\binom{\rho+3}{3} L(\mathrm{Sym}_i^2 E, 2) \prod_{p|N} \frac{p}{p-1}}{\pi N} + (2\rho+3)\log(5180) \\
&\quad - \left(\frac{\sqrt{N}(5180)}{2(\rho+1)L(\mathrm{Sym}_i^2 E, 2)\prod_{p|N}\frac{p}{p-1}} \right)^{\frac{1}{\rho+1}} \\
&= \log(0.336...).
\end{aligned}
$$

Similarly, we can calculate c_7 and c_8 and so we can take

$$c_4 = 1 + e^{c_6} + e^{c_7} + e^{c_8} < 1.34.$$

Since $\frac{\rho V}{\log d} < \frac{1}{2}$, from (5.4.3) we have

$$c_5 < 2.82.$$

Thus we have

$$2^{n_2/2} \cdot 3^{n_3/2} \cdot c_4 \cdot c_5 < 3.78.$$

By (6.0.1), we may assume that for $1 \leq i \leq \rho$,

$$q_i > \exp\left(\frac{2^{\rho+1}\rho!\sqrt{N}\prod_{p|N}\frac{p-1}{p}}{3L(\mathrm{Sym}_i^2 E, 2)} \right) > e^{890},$$

and so we can take

$$
\begin{aligned}
c_2 &= \frac{1}{3.78}\frac{L(\mathrm{Sym}_i^2, 2)}{2^{\rho+1}\rho!\sqrt{N}} \prod_{p|N}\frac{p}{p-1}\prod_{i=1}^{\rho}\frac{(q_i-1)(q_i+1-\lfloor\sqrt{2q_i}\rfloor)}{(q_i+1)(q_i+1+\lfloor\sqrt{2q_i}\rfloor)} \\
&> \frac{1}{10390}.
\end{aligned}
$$

Thus we have $c_2' = \frac{1}{5200}$. Finally, we note that (6.0.2) holds for $d \leq \exp(5180)$, because $h(d) \geq 1$ and $\theta(d) \leq 1$. □

Acknowledgment. The authors thank the referees for their careful readings and many valuable suggestions.

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